King Saud University **Department of Mathematics**

Final Exam **280-Math** **(1441/1442)**

Note. The inequality: $\ln x < x\,,\ \ x \in [1,\infty)$ may be of help to you during the solution.

Question1(5)**.** Find the following limit or prove that it does not exist

(a) $\lim_{n\to\infty} \frac{2^n n!}{(2n+1)!}$ (b) $\lim_{n\to\infty} \sum_{k=1}^{n} (\frac{1}{n} + \frac{k^3}{n^4})$

Question2(5)**.** Use appropriate method to find the sup and inf of the set:

$$E = \left\{ \frac{(-1)^n 2^n n!}{(2n+1)!} \ \ ; n \in \mathbb{N} \right\}$$

Question3 (5). Determine whether the following series are convergent or divergent:

(a) $\sum_{n=1}^{\infty} \sqrt[n]{2^n n^{n+1}}\ \sin\frac{1}{n}$ (b) $\sum_{n=1}^{\infty} \int_1^2 e^{-nx^2} dx$

Question4 (5). (a) Calculate the following limit or show that it does not exist:

$$\lim_{x\to 0} x^2 \sin\frac{1}{x^2}$$

(b) Let $f(x) = x(x+1)(x+2)(x+3)$. Prove that all solutions of the equation $f'(x) = 0$ are real.

Question5 (5). Decide whether the following function is uniformly continuous:

$$f(x) = \frac{\tan 3x}{x\cos 3x} \ \ on\ (0,1)$$

Question6 (5). Determine whether the integral $\int_3^{\infty} \frac{1}{3+\sin x+\ln x} dx$ converges or not.

Question7 (5). Study the U-convergence of the function sequence $f_n(x) = \frac{nx}{1+nx}$ on the following intervals: (a) $[0,\infty)$ (b) $[1,\infty)$.

Question8 (5). (a) Find the sum of the power series $\sum_{n=0}^{\infty} (n+1)x^n$ over the interval of convergence. (b) Find the sum of the number series $\sum_{n=1}^{\infty} \frac{n}{3^n}$.

Solutions

Question1(5)**.** Find the following limit or prove that it does not exist

(a) $\lim\limits_{n\to\infty} \dfrac{2^n n!}{(2n+1)!}$ (b) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (\dfrac{1}{n} + \dfrac{k^3}{n^4})$

Solution. (a) First solution. Consider the series $\sum\limits_{n=1}^{\infty} \dfrac{2^n n!}{(2n+1)!}$ (*) . This is a positive term series.

Let's find $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{2^{n+1}(n+1)!}{(2n+3)!} \dfrac{(2n+1)!}{2^n n!} = \lim\limits_{n\to\infty} \dfrac{2(n+1)}{(2n+3)(2n+2)} = \lim\limits_{n\to\infty} \dfrac{1}{(2n+3)} = \dfrac{1}{2} < 1$.

By Ratio test The series (*) converges, therefore $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{2^n n!}{(2n+1)!} = 0$.

Another solution. $0 \le \dfrac{2^n n!}{(2n+1)!} = \dfrac{2^n n!}{(n!)(n+1)(n+2)(n+3)\cdots(n+n)(2n+1)} =$

$$= \frac{1}{(n+1)} \frac{2}{(n+2)} \frac{2}{(n+3)} \cdots \frac{2}{(n+n)} \frac{2}{(2n+1)} < \frac{1}{(n+1)} \underset{as\, n\to\infty}{\to} 0$$

It follows by squeezing rule that $\lim\limits_{n\to\infty} \dfrac{2^n n!}{(2n+1)!} = 0$.

Third solution. Since $\dfrac{a_{n+1}}{a_n} = \dfrac{2^{n+1}(n+1)!}{(2n+3)!} \dfrac{(2n+1)!}{2^n n!} = \dfrac{2(n+1)}{(2n+3)(2n+2)} = \dfrac{1}{(2n+3)} < 1$, the sequence $a_n = \dfrac{2^n n!}{(2n+1)!}$ is decreasing . In addition, the sequence a_n is bounded below (by 0). Hence a_n is convergent. Let $\lim\limits_{n\to\infty} a_n = l$. By properties $\lim\limits_{n\to\infty} a_{n+1} = l$ **.**

Since $a_{n+1} = \dfrac{1}{(2n+3)} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{1}{(2n+3)} a_n = \lim\limits_{n\to\infty} \dfrac{1}{(2n+3)} \lim\limits_{n\to\infty} a_n \Rightarrow l = 0l = 0$

(b) First solution. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (\dfrac{1}{n} + \dfrac{k^3}{n^4}) = 1 + \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \left(\dfrac{k}{n}\right)^3 = 1 + \int\limits_0^1 x^3 dx = \dfrac{5}{4}$

or $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (\dfrac{1}{n} + \dfrac{k^3}{n^4}) = \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \left(1 + (\dfrac{k}{n})^3\right) = \int\limits_0^1 (1 + x^3)\, dx = \dfrac{5}{4}$

Another solution. $\lim_{n\to\infty}\sum_{k=1}^{n}(\frac{1}{n}+\frac{k^3}{n^4})=\lim_{n\to\infty}\left(1+\frac{\left(\frac{n(n+1)}{2}\right)^2}{n^4}\right)=1+\frac{1}{4}=\frac{5}{4}$

Question2(5)**.** Use appropriate method to find the sup and inf of the set:

$$E=\left\{\frac{(-1)^n 2^n n!}{(2n+1)!}\ ;\ n\in\mathbb{N}\right\}$$

Solution. If $x_n=\frac{(-1)^n 2^n n!}{(2n+1)!}=(-1)^n a_n$, then

$$\frac{a_{n+1}}{a_n}=\frac{2^{n+1}(n+1)!}{(2n+3)!}\frac{(2n+1)!}{2^n n!}=\frac{2(n+1)}{(2n+3)(2n+2)}=\frac{1}{2n+3}<1$$

It means that the sequences a_n is decreasing . That is

$$a_1\geq a_2\geq a_3\geq a_4\geq\cdots\geq a_n\geq\cdots>0 \qquad (*)$$

From (*) we conclude that

$$a_2\geq a_4\geq a_6\cdots>0>\cdots\geq -a_5\geq -a_3\geq -a_1$$

Hence $x_2=a_2=\sup E$ *and* $x_1=-a_1=\inf E$

Thus $\sup E=x_2=\frac{1}{15}$ *and* $\inf E=x_1=-\frac{1}{3}$

Question3 (5). Determine whether the following series are convergent or divergent:

(a) $\sum_{n=1}^{\infty}\sqrt[n]{2^n n^{n+1}}\ \sin\frac{1}{n}$ (b) $\sum_{n=1}^{\infty}\int_1^2 e^{-nx^2}dx$

Solution. (a) $\lim_{n\to\infty}\sqrt[n]{2^n n^{n+1}}\ \sin\frac{1}{n}=\lim_{n\to\infty}2n\sqrt[n]{n}\ \sin\frac{1}{n}=\lim_{n\to\infty}2\sqrt[n]{n}\ \frac{\sin\frac{1}{n}}{\frac{1}{n}}=2(1)(1)=2$

Hence the series (a) diverges by nth term test.

Another solution. It is clear that $0 \le a_n = \sqrt[n]{2^n n^{n+1}}\ \sin\frac{1}{n}\ \ \forall n \in \mathbb{N}$ and $0 \le b_n = \sin\frac{1}{n}\ \ \forall n \in \mathbb{N}$.

Since $1 < 2n\sqrt[n]{n} = \sqrt[n]{2^n n^{n+1}}\ \ \forall n \in \mathbb{N}$, we have $b_n \le a_n\ \ \forall n \in \mathbb{N}$. The series $\sum_{n=1}^{\infty} b_n$ diverges by LCT with $\sum_{n=1}^{\infty} \frac{1}{n}$, hence the series $\sum_{n=1}^{\infty} a_n$ diverges by CT .

(b) Denote by $a_n = \int_1^2 e^{-nx^2}$. On the interval $[1,2]$ we have $e^{-nx^2} = \frac{1}{e^{nx^2}} \le \frac{1}{e^n}$

Therefore $$a_n = \int_1^2 e^{-nx^2}dx \le \int_1^2 \frac{1}{e^n}dx = \frac{1}{e^n}$$

Because the series $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges (geometric with $|r| = \frac{1}{e^n} < 1$), the given series converges by CT.

Question4 (5). (a) Calculate the following limit or show that it does not exist:

$$\lim_{x \to 0} x^2 \sin\frac{1}{x^2}$$

(b) Let $f(x) = x(x+1)(x+2)(x+3)$. Prove that all solutions of the equation $f'(x) = 0$ are real.

Solution. (a) Since $\left| x^2 \sin\frac{1}{x^2} \right| \le x^2\ \ \forall x \ne 0$ and $\lim_{x \to 0} x^2 = 0$, we conclude that $\lim_{x \to 0} x^2 \sin\frac{1}{x^2} = 0$ by squeezing rule.

(b) The function $f(x)$, as a polynomial, is continuous and differentiable over any interval.

Further, $f(-3) = f(-2) = f(-1) = f(0) = 0$. Using Rolle's theorem we get the following:

$$\exists c_1, c_2, c_3 \in \mathbb{R}\ ,\ c_1 \in (-3,-2),\ c_2 \in (-2,-1), c_3 \in (-1,0)\ \ st\ \ f'(c_1) = f'(c_2) = f'(c_3) = 0$$

Obviously, $f'(x)$ is a third-degree polynomial and cannot have more than three roots. So all solutions of the equation $f'(x) = 0$ are real.

Question5 (5). Decide whether the following function is uniformly continuous:

$$f(x)=\frac{\tan 3x}{x\cos 3x} \quad on\ (0,1)$$

Solution. Define the function $g(x)=\begin{cases}\dfrac{\tan 3x}{x\cos 3x} & , \quad x\in(0,1]\\ 3 & , \quad x=0\end{cases}$

Because $\lim_{x\to 0} g(x)=3$, the function $g(x)$ is continuous on the interval $[0,1]$.

Furthermore $g(x)\equiv f(x)$ on the interval $(0,1)$.

Using Continuous Extension Theorem we conclude that the function $f(x)=\dfrac{\tan 3x}{x\cos 3x}$ is uniformly continuous on the interval $(0,1)$.

Question6 (5). Determine whether the integral $\int_3^{\infty}\frac{1}{3+\sin x+\ln x}dx$ converges or not.

Solution. First we note that $\sin x\le x \ \ \forall x\ge 0;\ \ln x<x\ \forall x\ge 1$. So we can write

$2\le 3+\sin x\le 3+x \ \ \forall x\ge 0;\ \ln x<x\ \forall x\ge 1$

Therefore $\quad \dfrac{1}{3+\sin x+\ln x}>\dfrac{1}{3+x+x}=\dfrac{1}{3+2x}\ge\dfrac{1}{x+2x}=\dfrac{1}{3x}\ \forall x\ge 3$.

The integral $\int_3^{\infty}\frac{1}{x}\,dx$ diverges as it is p - integral of type1 with $p=1$. Hence the given integral diverges by direct comparison test.

Question7 (5). Study the U-convergence of the function sequence $f_n(x)=\dfrac{nx}{1+nx}$ on the following intervals: (a) $[0,\infty)$ (b) $[1,\infty)$.

Solution. (a) The pointwise limit of the sequence is $f(x)=\begin{cases}0 & if \quad x=0\\ 1 & if \quad x>0\end{cases}$.

Each function $f_n(x)$ is continuous on $[0,\infty)$ and the limit function $f(x)$ is discontinuous at $x=0$ which implies that the convergence is not uniform on the interval $[0,\infty)$.

(b) Here the pointwise limit of the sequence is $f(x)=1$.

In addition to that we have $\left|f_n(x)-f(x)\right|=\left|\frac{nx}{1+nx}-1\right|=\frac{1}{1+nx}<\frac{1}{nx}\le\frac{1}{n} \quad \forall x\ge 1$, with $\lim_{n\to\infty}\frac{1}{n}=0$.
It follows from M-test that the convergence of $f_n(x)$ is uniformly on the interval $[1,\infty)$.

Question8 (5). (a) Find the sum of the power series $\sum_{n=0}^{\infty}(n+1)x^n$ over the interval of convergence. (b) Find the sum of the number series $\sum_{n=1}^{\infty}\frac{n}{3^n}$.

Solution. (a) Recall $\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}$, $x\in(-1,1)$.

Differentiating term by term we get $\sum_{n=1}^{\infty}nx^{n-1}=\frac{1}{(1-x)^2}$, $x\in(-1,1)$ (*)

But the last sum is just that $\sum_{n=1}^{\infty}nx^{n-1}=\sum_{n=0}^{\infty}(n+1)x^n$, therefore $\sum_{n=0}^{\infty}(n+1)x^n=\frac{1}{(1-x)^2}$.

(b) The equality (*) is true for any $x\in(-1,1)$. In a particular it is true for the number $x=\frac{1}{3}\in(-1,1)$. Substituting $\frac{1}{3}$ instead of x, we get $\sum_{n=1}^{\infty}\frac{n}{3^{n-1}}=\frac{1}{(1-\frac{1}{3})^2}=\frac{9}{4} \Rightarrow \sum_{n=1}^{\infty}\frac{n}{3^n}=\frac{3}{4}$.

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