King Saud University, College of Science, Department of Mathematics
Math-280 (Introduction to Real Analysis)
Final Exam [Time: 3 Hours]/ 1^{st} Semester, 1436–1437 H.

Exercise 1 [3+3+3=9 Marks]:

1. Determine the following infimum: $\inf\left\{z = 2^x + 2^{\frac{1}{x}},\ \ x > 0\right\}$.
2. Find the limit of the sequence: $\lim_{n\to\infty} \sqrt[n]{a^n + b^n}$, where $a,\ b \geq 0$.
3. Decide whether the following series is convergent or divergent: $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

Exercise 2 [2+1+3+3=9 Marks]:

1. Using the $\varepsilon-\delta$–definition of the limit, show that $\lim_{x\to a} f(x-a) = \lim_{x\to 0} f(x)$.
2. Calculate the limit: $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1}$.
3. Let f be continuous on $[0,1]$, and suppose that $f(0) = f(1)$. Show that there is a point $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.
4. Find the extrema of $g(x) = 3x^4 - 8x^3 + 6x^2$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$

Exercise 3 [4+2+4=10 Marks]:

1. Show that if $f \in \mathcal{R}(0,1)$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx$.
2. Use (1) to calculate the limit: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$.
3. Test the convergence of the improper integral: $I = \int_0^{\infty} \frac{4x}{1+x^6}dx$.

Exercise 4 [(4)+(3+3+2)=12 Marks]:

1. Study the uniform convergence of the sequence of functions: $f_n(x) = nx^n(1-x),\ n \in \mathbb{N}$, on $\mathcal{D} = [0,1]$.
2. Let $f_n : [1,2] \longrightarrow \mathbb{R}$, be such that $f_n(x) = \frac{x}{(1+x)^n},\ n \in \mathbb{N}$.

   (a) Show that $\sum_{n=1}^{\infty} f_n(x)$ is convergent $\forall x \in [1,2]$.

   (b) Show that this convergence is uniform.

   (c) Verify the identity: $\int_1^2 \left(\sum_{n=1}^{\infty} f_n(x)\right) dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x)dx$.

.... Good Luck ....

Typical answers to the final exam problems, Math 280
First semester 1436/1437H
2015/2016 G.

Exercise 1:

1) For any $a, b > 0$, we have $\frac{a+b}{2} \geqslant \sqrt{ab}$. So put $a = 2^x$ and $b = 2^{\frac{1}{x}} \Rightarrow$

$$2^x + 2^{\frac{1}{x}} \geqslant 2\sqrt{2^x 2^{\frac{1}{x}}} = 2\sqrt{2^{x+\frac{1}{x}}} \geqslant 2\sqrt{2^2} = 4$$

Since the minimum value of $x + \frac{1}{x}$ on $(0, \infty)$ is attained at $x = 1$ and equals 2:

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$$\left(x + \frac{1}{x}\right)' = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$$

$\left(x + \frac{1}{x}\right)'' = \frac{2}{x^3}$ whis is > 0 for $x = 1$ so it is a local min.

Thus $\inf\{z = 2^x + 2^{\frac{1}{x}}, x > 0\} = 4$ (with equality if and only if $x = 1$).

2) Suppose, without loss of generality, that $a \geqslant b$.

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So $\sqrt[n]{a^n + b^n} = a \sqrt[n]{1 + \underbrace{\left(\frac{b}{a}\right)^n}_{\to 0}} \longmapsto a$ as $n \to \infty$

So $\lim\limits_{n \to \infty} \sqrt[n]{a^n + b^n} = \max\{a, b\}$.

3) We use the limit comparison test:

So, consider the series $\sum b_n$, $b_n = \frac{1}{n}$ which is divergent. We have

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$$\lim_{n \to \infty} \frac{\frac{1}{n^{1+1/n}}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{\cancel{\frac{1}{n}} \cdot \frac{1}{n^{1/n}}}{\cancel{\frac{1}{n}}} = \lim_{n \to \infty} \frac{1}{n^{1/n}} = 1.$$

So the series $\sum \frac{1}{n^{1+1/n}}$ is divergent.

Exercise 2 :-

1) $\lim_{x \to a} f(x-a) = l \iff$

$\forall \varepsilon > 0, \exists \delta > 0, |x-a| < \delta \Rightarrow |f(x-a) - l| < \varepsilon$

put $y = x - a$, then the latter becomes:

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$\forall \varepsilon > 0, \exists \delta > 0, |y| < \delta \Rightarrow |f(y) - l| < \varepsilon$

which is equivalent to $\lim_{y \to 0} f(y) = l$.

Thus $\lim_{x \to a} f(x-a) = \lim_{x \to 0} f(x)$.

2) According to (1) put $y = x^2 - 1$, so we obtain

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$$\lim_{x \to 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x \to 1} (x+1) \frac{\sin(x^2-1)}{x^2-1}$$

$$= \lim_{x \to 1} \underbrace{(x+1)}_{\to 2} \lim_{y \to 0} \underbrace{\frac{\sin y}{y}}_{\to 1} = 2.$$

3) Set $g(x) = f(x) - f(x + \frac{1}{2})$ on $[0, \frac{1}{2}]$.

g is clearly continuous on $[0, \frac{1}{2}]$, and

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$g(0) = f(0) - f(\frac{1}{2})$, $g(\frac{1}{2}) = f(\frac{1}{2}) - f(1)$

$= f(\frac{1}{2}) - f(0)$

$= -g(0)$.

Thus 0 lies between $g(0)$ and $g(\frac{1}{2}) = -g(0)$.
By the intermediate value theorem there is a $c \in [0, \frac{1}{2}]$ such that $g(c) = 0$.
i.e. $f(c) - f(c+\frac{1}{2}) = 0$ or $f(c) = f(c+\frac{1}{2})$.

4) $g'(x) = 12x^3 - 24x^2 + 12x = 0$

$$\Rightarrow 12x(x^2 - 2x + 1) = 0$$

$$\Rightarrow 12x(x-1)^2 = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

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Thus the critical points are $\{-\frac{1}{2}, 0, \frac{1}{2}\}$ as $1 \notin [-\frac{1}{2}, \frac{1}{2}]$.

$$g(-\tfrac{1}{2}) = \frac{43}{16}, \quad g(0) = 0, \quad g(\tfrac{1}{2}) = \frac{11}{16}$$

$\Rightarrow g(0) = 0$ is the minimum, and $g(-\frac{1}{2}) = \frac{43}{16}$ is the maximum of g on $[-\frac{1}{2}, \frac{1}{2}]$.

Exercise 3 :-

1) If $f \in R(0,1)$, let us choose a uniform partition P_n such that

then

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$$x_k = \frac{k}{n}, \quad x_{k+1} - x_k = \frac{1}{n}, \quad \omega_{k+1} = x_{k+1} = \frac{k+1}{n}$$

$$\Rightarrow S(f, P_n) = \sum_{k=1}^{n} f(\omega_k)\,\Delta x_k = \sum_{k=0}^{n-1} f\left(\frac{k+1}{n}\right)\cdot\frac{1}{n}$$

$$= \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$, (put $i = k+1$ and then return back to k).

passing to the limit in the latter as $n \to \infty$

we obtain

$$\lim_{n\to\infty} \frac{1}{n} \sum^{\infty} f\left(\frac{k}{n}\right) = \lim_{n\to\infty} S(f, P_n) = \int_0^1 f(x)\,dx .$$

2) write the given limit as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1+\left(\frac{k}{n}\right)^2} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

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where $f(x) = \dfrac{x}{1+x^2}$, and apply (1) above to get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\left[\ln(1+x^2)\right]_0^1$$

$$= \frac{\ln 2}{2} .$$

3) put $I = \underbrace{\int_0^1 \frac{4x}{1+x^6}\,dx}_{I_1} + \underbrace{\int_1^\infty \frac{4x}{1+x^6}\,dx}_{I_2}$

I_1 is a definite integral whose value is finite

Let us test the convergence of I_2:

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$\dfrac{4x}{1+x^6} \leqslant \dfrac{4x}{x^6} = \dfrac{4}{x^5}$, and we know that

$$\int_1^\infty \frac{4}{x^5}\,dx = \lim_{M\to\infty} \left[-\frac{1}{x^4}\right]_1^M = \lim_{M\to\infty}\left(1-\frac{1}{M^4}\right) = 1$$

So $\int_1^\infty \frac{4}{x^5}\,dx$ is convergent. By the Comparison test $I_2 = \int_1^\infty \frac{4x}{1+x^6}\,dx < \infty$ is convergent too.

Now both of I_1 and I_2 are finite; whence $I = \int_0^\infty \frac{4x}{1+x^6}\,dx$ is convergent.

Exercise 4 :

1) If $x = 0$ or $x = 1$, we have $f_n(x) = 0$, $\forall n \in \mathbb{N}$.

If $0 < x < 1$, then $n x^n (1-x) \longrightarrow 0$ as $n \longrightarrow \infty$.

So the pointwise limit is 0, i.e.

$$\lim_{n \to \infty} f_n(x) = 0.$$

on the other hand, for any $n \geq 2$, we have

$f_n'(x) = n\, x^{n-1} (n - (n+1)x)$.

4 So $f'(x) = 0 \implies x = 0$ or $x = \frac{n}{n+1}$.

Thus $\operatorname{Sup}\{|f_n(x) - 0|, x \in [0,1]\} = f_n\left(\frac{n}{n+1}\right)$.

as $f_n(0) = f_n(1) = 0$, while $f_n\left(\frac{n}{n+1}\right) > 0$, $n \in \mathbb{N}$.

But $\lim\limits_{n \to \infty} f_n\left(\frac{n}{n+1}\right) = \lim\limits_{n \to \infty} \left(\frac{n}{n+1}\right)^{n+1} = \frac{1}{e} \neq 0$

So the convergence of (f_n) to 0 is not uniform.

2)

a) If $x \in [1,2]$, then $|1+x| \geq 2 > 1$, and thus $\left|\frac{1}{1+x}\right| < 1$, so we have

3 $$\sum_{n=1}^{\infty} \frac{x}{(1+x)^n} = x \sum_{n=1}^{\infty} \frac{1}{(1+x)^n} = \frac{x}{1+x} \cdot \frac{1}{1 - \frac{1}{1+x}} = 1.$$

In particular $\sum\limits_{n=1}^{\infty} f_n(x)$ is convergent $\forall x \in [1,2]$.

b) Since $1 \leq x \leq 2$, we have $1 + x \geq 2$

$$\implies \frac{1}{(1+x)^n} \leq \frac{1}{2^n} \implies \frac{x}{(1+x)^n} \leq \frac{x}{2^n} \leq \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

So take $M_n = \frac{1}{2^{n-1}}$, whence $|f_n(x)| = \left|\frac{x}{(1+x)^n}\right| \leq \frac{1}{2^{n-1}}$

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Since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is convergent, then

$\sum_{n=1}^{\infty} \frac{x}{(1+x)^n}$ is uniformly convergent by Weirstrass.

c) The uniform convergence allows us to interchange the integral sign and the infinite sum, so

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$$\int_1^2 \underbrace{\Big(\sum_{n=1}^{\infty} f_n(x) \Big)}_{=1} dx = \sum_{n=1}^{\infty} \int_1^2 f_n(x) dx = 1 .$$