King Saud University, College of Science, Department of Mathematics
Math-280 (Introduction to Real Analysis)
First Midterm Exam [Time: 90 Minutes]/ 1^{st} Semester, 1436–1437 H.

Exercise 1 [3+3+2=8 Marks]:

1. Determine the "sup" and "inf" of the following set $\mathbb{A} = \left\{\frac{m}{n},\ m, n \in \mathbb{N},\ m < 3n\right\}$, and justify your answer.

2. Find $\sup\left\{y = x + \frac{1}{x},\ x < 0\right\}$.

3. Find sup, inf, max, min, (if exist), of the set $\mathbb{B} = \left\{x,\ 0 \leq x \leq \sqrt{2},\ x \text{ is rational}\right\}$.

Exercise 2 [2+3+3=8 Marks]:

1. Find the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n = \dfrac{2^n + (-1)^n}{2^{n+1} + (-1)^{n+1}}$.

2. Show that the sequence $(y_n)_{n\in\mathbb{N}}$ defined by $y_1 = 1,\ y_{n+1} = \sqrt{2 + y_n}$, $\forall n \in \mathbb{N}$, is convergent and find its limit.

3. Let $0 < a_1 < b_1$ and define $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = \dfrac{a_n + b_n}{2}$.

   (a) Prove that each one of the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converges.

   (b) Prove that they have the same limit.

Exercise 3 [2+4+3=9 Marks]:

1. Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$.

2. Show that the series $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ is **conditionally** convergent.

3. Prove that if $0 \leq z_n < 1$, then $\displaystyle\sum_{n=1}^{\infty} z_n$ is convergent if and only if $\displaystyle\sum_{n=1}^{\infty} \frac{z_n}{1 + z_n}$ is convergent.

.... Good Luck ....

Answers for 1^{st} midterm exam
M-280, 1^{st} semester 1436/1437 H.

Exercise 1:

1- $m < 3n \Longrightarrow \frac{m}{n} < 3$, whence 3 is an upper bound.
Let $\varepsilon > 0$, then for any $N > [\frac{3}{\varepsilon}]$ we have
$\frac{3(N-1)}{N} > 3 - \varepsilon$, and $\frac{3(N-1)}{N} \in A$.

Thus $\sup A = 3$.

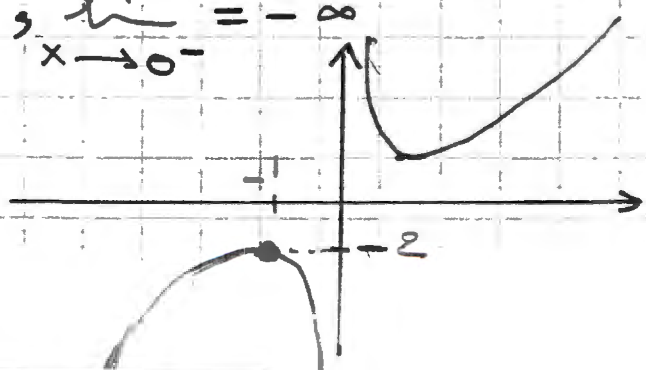
$\inf A = 0$, because: given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$, (by Archedes) and $\frac{1}{N} \in A$.

2- This set consists of the values of the function $f(x) = x + \frac{1}{x}$, so let us determine its extrema. $f'(x) = 1 - \frac{1}{x^2} = 0 \Longrightarrow \frac{x^2-1}{x^2} = 0$
$\Longrightarrow x = \pm 1$

Sice we are interested in $x < 0$, consider $x = -1$
$f''(x) = \frac{2}{x^3}$, $f''(-1) = -2 < 0$, so that
$f(-1) = -1 + \frac{1}{-1} = -2$ is a local maximum.

$\lim_{x \to -\infty} x + \frac{1}{x} = -\infty$, $\lim_{x \to 0^-} = -\infty$

So $\sup A = -2$



3) B consists of rationals between 0 and $\sqrt{2}$.

So $\inf B = \operatorname{Min} B = 0 \in B$

$\operatorname{Sup} B = \sqrt{2} \notin B$.

$\operatorname{Max} B$ does not exist because for any $x \in B$, $\exists\ y \in B$ such that $x < y < \sqrt{2}$.

Exercise 2 :

1- $$\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{\cancel{2^n}\left(1 + \frac{(-1)^n}{2^n}\right)}{\cancel{2^n}\left(2 + \frac{(-1)^{n+1}}{2^n}\right)} = \frac{1}{2}.$$

2- Let us show that (y_n) is bounded above by 2.

we have $y_1 = 1 \leqslant 2$

Suppose $y_n \leqslant 2 \implies 2 + y_n \leqslant 4$

$$\implies y_{n+1} = \sqrt{2 + y_n} \leqslant \sqrt{4} = 2.$$

So $y_n \leqslant 2$, $\forall n \in \mathbb{N}$ (bounded above).

Let us show that (y_n) is increasing.

we have $y_1 = 1$, $y_2 = \sqrt{2+1} = \sqrt{3}$

whence $y_1 < y_2$.

suppose $y_{n-1} < y_n$ and show $y_n < y_{n+1}$.

$$y_{n-1} < y_n \implies 2 + y_{n-1} < 2 + y_n$$

$$\implies y_n = \sqrt{2 + y_{n-1}} < y_{n+1} = \sqrt{2 + y_n}$$

So (y_n) is increasing; whence it is convergent.

Let us calculate its limit, suppose that $\lim_{n \to \infty} y_n = y$.

then, $y_{n+1} = \sqrt{2+y_n} \Rightarrow \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} \sqrt{2+y_n}$

$$\Rightarrow y = \sqrt{2+y} \Rightarrow y^2 - y - 2 = 0$$

$$\Delta = (-1)^2 - 4(1)(-2) = 9 \Rightarrow y = \frac{1 \pm \sqrt{9}}{2}$$

$\Rightarrow y = 2$ or $y = -1$, since $y_n > 0$

we see that $\lim_{n \to \infty} y_n = y = 2$.

3. a) We have the inequality $\sqrt{ab} < \frac{a+b}{2}$

So, for any n, we have (since $0 < a_1 < b_1$):

$$a_{n+1} < \sqrt{a_n b_n} < \frac{a_n + b_n}{2} = b_{n+1}$$

So $a_n < b_n$, $\forall n$

Thus $a_n . a_n < a_n b_n \Rightarrow \sqrt{a_n^2} < \sqrt{a_n b_n}$

$$\Rightarrow a_n < a_{n+1}, \forall n.$$

Also $a_n < b_n \Rightarrow a_n + b_n < b_n + b_n$

$$\Rightarrow \frac{a_n + b_n}{2} < \frac{2b_n}{2} \Rightarrow b_{n+1} < b_n$$

Hence $a_1 < a_n < a_{n+1} < b_{n+1} < b_n < b_1$.

Thus (a_n) is increasing and bounded by b_1 and (b_n) is decreasing and bounded below by a_1. Hence they are both convergent sequences.

b) $\lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{a_n} \sqrt{b_n} \Rightarrow a = \sqrt{a} \sqrt{b}$

$\Rightarrow a^2 = ab \Rightarrow a = b$, they have same limit.

Exercise 3 :

1- we can use, for example, the root (or ratio) test)

$$\lim_{n\to\infty} \sqrt[n]{\frac{2^n n!}{n^n}} = \lim_{n\to\infty} \frac{2\,\sqrt[n]{n!}}{n} = \frac{2}{e} < 1$$

$\frac{\sqrt[n]{n!}}{n} \to \frac{1}{e}$

Thus the series converges.

2- Let us show that $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent.

$a_n = \frac{\ln n}{n} > 0 \ , \ n \geq 2$ and $\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

In addition $f(x) = \frac{\ln x}{x}$ is decreasing, since

$$f'(x) = \frac{\frac{1}{x}x - 1 \cdot \ln x}{x^2} = \frac{1 - \ln x}{x^2} < 0 \text{ for } x > e.$$

Thus (a_n) is decreasing as well.

By the alternating series test, $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent.

Let us show that $\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln n}{n} \right| = \sum_{n=2}^{\infty} \frac{\ln n}{n}$ is divergent. For, we use the integral test.

The function $\frac{\ln x}{x}$ is decreasing (by above) continuous and positive.

Also $\int_1^{\infty} \frac{\ln x}{x}\, dx = \left[\frac{1}{2}\ln^2 x\right]_1^{\infty} = \lim_{x\to\infty} \frac{1}{2}\ln^2 x = \infty$

divergent; so $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ also diverges.

The series $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent while $\sum_{n=2}^{\infty} \left| (-1)^n \frac{\ln x}{n} \right|$ is divergent, so it is conditionally convergent (not absolutely convergent but only convergent).

3. We have $0 \leq z_n < 1$. Then

$z_n < 1 \Rightarrow 1 + z_n < 2 \Rightarrow \frac{1}{2} < \frac{1}{1+z_n}$

$\Rightarrow \frac{z_n}{2} < \frac{z_n}{1+z_n}, \forall n.$

Also $z_n \geq 0 \Rightarrow 1 + z_n \geq 1 \Rightarrow 1 \geq \frac{1}{1+z_n}$

$\Rightarrow \frac{z_n}{1+z_n} \leq z_n, \forall n.$

Therefore $\frac{z_n}{2} < \frac{z_n}{1+z_n} < z_n, \forall n.$

By the Comparison test, if $\sum_{n=1}^{\infty} z_n$ is convergent then $\sum_{n=1}^{\infty} \frac{z_n}{1+z_n}$ is also convergent, and if $\sum_{n=1}^{\infty} \frac{z_n}{1+z_n}$ is convergent $\frac{1}{2}\sum_{n=1}^{\infty} z_n$, and thus $\sum_{n=1}^{\infty} z_n$, is convergent.

In conclusion $\sum_{n=1}^{\infty} z_n$ is convergent if and only if $\sum_{n=1}^{\infty} \frac{z_n}{1+z_n}$ is convergent.