PHYS 404
2nd Midterm Exam
Tuesday 3rd November 2015

Solutions

Please answer all questions

1. Use the relation $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$ to find the polynomials (a) $H_2(x)$ and (b) $H_3(x)$.

(5 marks)

Solution:

a) $H_2(x) = (-1)^2 e^{x^2} \frac{d^2}{dx^2}\left(e^{-x^2}\right) = e^{x^2} \frac{d}{dx}\left(-2xe^{-x^2}\right) = -e^{x^2} e^{-x^2}\left(4x^2 - 2\right) =$
$4x^2 - 2$

b) $H_3(x) = (-1)^3 e^{x^2} \frac{d^3}{dx^3}\left(e^{-x^2}\right) = -e^{x^2} \frac{d^2}{dx^2}\left(-2xe^{-x^2}\right) = -e^{x^2} \frac{d}{dx}\left[e^{-x^2}\left(4x^2 - 2\right)\right] =$
$-e^{x^2} e^{-x^2}\left(-8x^3 + 12x\right) = 8x^3 - 12x$

2. We know that $L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n \left(e^{-x}x^n\right)$ and also $L_n^k(x) \equiv (-1)^k \left(\frac{d}{dx}\right)^k L_{n+k}(x)$. Use these relations to find the associated Laguerre polynomials $L_2^1(x)$

(5 marks)

Solution:

$$L_n^k(x) = (-1)^k \left(\frac{d}{dx}\right)^k L_{n+k}(x) \underset{n=2,k=1}{\Rightarrow} L_2^1(x) = (-1)^1 \left(\frac{d}{dx}\right)^1 L_3(x) \Rightarrow$$

$$L_2^1(x) = -\left(\frac{d}{dx}\right) L_3(x)$$

But

$$L_n(x)=\frac{e^x}{n!}\left(\frac{d}{dx}\right)^n\left(e^{-x}x^n\right)\Rightarrow L_3(x)=\frac{e^x}{3!}\left(\frac{d}{dx}\right)^3\left(e^{-x}x^3\right)\Rightarrow$$

$$L_3(x)=\frac{e^x}{6}\left(\frac{d}{dx}\right)^2\left[e^{-x}\left(-x^3+3x^2\right)\right]\Rightarrow L_3(x)=\frac{e^x}{6}\left(\frac{d}{dx}\right)\left[e^{-x}\left(x^3-6x^2+6x\right)\right]$$

$$L_3(x)=\frac{e^x}{6}\left[-e^{-x}\left(x^3-6x^2+6x\right)+e^{-x}\left(3x^2-12x+6\right)\right]\Rightarrow$$

$$L_3(x)=\frac{1}{6}\left(-x^3+9x^2-18x+6\right)$$

So by substitution we get

$$L_2^1(x)=-\left(\frac{d}{dx}\right)L_3(x)\Rightarrow L_2^1(x)=-\frac{1}{6}\left(\frac{d}{dx}\right)\left(-x^3+9x^2-18x+6\right)$$

$$L_2^1(x)=-\frac{1}{6}\left(-3x^2+18x-18\right)=\frac{1}{2}x^2-3x+3$$

3. Show that the Hermite polynomial $H_2(x)$ satisfies the differential equation: $y''-2xy'+2ny=0$.

(5 marks)

Solution:

$$y''-2xy'+2ny\underset{y=L_2(x)}{\overset{n=2}{=}}H_2''(x)-2xH_2'(x)+4H_2(x)=$$

$$\left(4x^2-2\right)''-2x\left(4x^2-2\right)'+4\left(4x^2-2\right)=$$

$$8-16x^2+16x^2-8=0$$

4. Use the series form of the Laguerre polynomials to show that $L_n'(0)=-n$

$$L_n(x)=\sum_{k=0}^{n}\frac{(-1)^{n-k}\,n!}{k!(n-k)!(n-k)!}x^{n-k}$$

(5 marks)

Solution:

The problem has been solved in the class