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| Department of Mathematics | math316 | Semester I |
| King Saud University | Final Exam | 2010-2011 |

1. Determine the constants $a$ and $b$ so that the function $f(x) = ae^x + be^{-x}$ is orthogonal to $g(x) = x$ in $L^2(-1,1)$, and $\|f\| = 1$.

2. (i) Expand the function $f(x) = x + 1$ in a Fourier series over $[-1, 1]$.

   (ii) Is the series uniformly convergent, and why?

3. Prove that $e^{-x/2} = 2\sum_{n=0}^{\infty} 3^{-n-1} L_n(x)$, where $L_n$ is the Laguerre polynomial of degree $n$ which is defined by

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right), \qquad x > 0.$$

   Hint: Show first that

$$\left\langle e^{-x/2}, L_n \right\rangle = \int_0^\infty e^{-x/2} L_n(x) e^{-x} dx = \frac{1}{n!2^n} \int_0^\infty x^n e^{-3x/2} dx,$$

   then show that

$$\int_0^\infty x^n e^{-3x/2} dx = n! (2/3)^{n+1}.$$

4. Use the substitution $y(x) = x^{-1/2} u(x)$ to transform Bessel's equation $x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$ to $u'' + u = 0$, and hence determine the general solution of Bessel's equation.

5. Prove that $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ for all $x > 0$. Sketch the graph of $J_{-1/2}(x)$, indicating its zeros.

6. (i) Express the function

$$f(x) = \begin{cases} \sin x, & |x| < \pi, \\ 0, & |x| > \pi, \end{cases}$$

   as a Fourier integral.

   (ii) Use the result of (i) to obtain the formula an integral representation of $\pi$.