Name
ID
Group

373
Final Exam
June, 2013

I) a) Let $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{a, c\}, \{a\}\}$. Let $A = \{a, c\}$. Find each of the following sets: $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$, $\mathrm{Bd}(A)$, $\mathrm{Ext}(A)$.

b) Show that $\mathcal{B} = \{\{x\} : x \in \mathbb{R}\}$ is a base for a topology on $\mathbb{R}$. Describe the topology.

c) Let $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \emptyset, \{a, b\}, \{a\}\}$. Let $A = \{a, c\}$. Find each of the following sets: $\mathrm{Cl}_A(\{a\})$, $\mathrm{Cl}_X(\{a\})$, $\mathrm{Ext}_A(\{a\})$, $\mathrm{Ext}_X(\{a\})$.

d) Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} x + 2, & x \leq 0 \\ 3, & x > 0 \end{cases}.$$

Is f $\mathcal{U} - \mathcal{U}$ continuous? Is f $\mathcal{U} - \mathcal{C}$ continuous? Justify your answers.

II) a) Let (X, d) be a metric space. Show that the function $e : X \times X \to \mathbb{R}$, given by $e(x, y) = \min\{1, d(x, y)\}$ is a metric on X. Show that $\mathcal{T}(d) = \mathcal{T}(e)$.

b) Prove that every metric space is a Hausdorff space.

c) Define a topological *metrizable* space. Is every topological space metrizable? Justify your answer.

III) a) Let X be a topological space and $A \subseteq X$. Prove that if there exists a sequence of points of A convergent to x, then $x \in \overline{A}$. Prove that the converse holds if X is metrizable.

b) Prove that if X is a compact metrizable space, then X is complete.

c) Prove that $(\mathbb{R}, \mathcal{U})$ is complete.

d) Is the completness a topological property? Justify your answer.

IV) a) Let $f : X \to \mathbb{R}$ be a continuous function, defined on a compact space X. Prove that f attains its maximum and minimum on X.

b) Let X be a compact Hausdorff space. Prove that X is normal.

c) Let $f : X \to Y$ be a one-to-one, open function, from the space X onto the space Y. Prove that if B is a compact subset of Y, then $f^{-1}(B)$ is a compact subset of X.

V) a) Prove that $f : X \to Y$ is continuous if and only if $f^{-1}(\mathrm{Int}(B)) \subseteq \mathrm{Int}(f^{-1}(B))$.

b) Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies for a space X. Prove that the identity function from $(X, \mathcal{T}_1)$ onto $(X, \mathcal{T}_2)$ is open if and only if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.