King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2016–2017
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Bachelor AFM: M. Eddahbi

Final exam Stochastic Processes: MATH. 380 (40%) (three pages)

Monday, May 15, 2017 / Sha'ban 19, 1438 (three hours 9–12 AM)

Use ballpoint or ink–jet pens

Problem 1 (8 marks)

Two biased coins are being flipped repeatedly. The probability that coin 1 comes up heads is $\frac{1}{4}$, while that of coin 2 is $\frac{3}{4}$. Each coin is being flipped until a head comes up. Let X and Y be the number of flips of coins 1 and 2 to come up heads for the first time.

1. **(2 mark)** Find the p.m.f. of X and Y.
2. **(2 mark)** What is the joint p.m.f. of (X, Y): (find $S_{(X,Y)}$ and $P(X = m, Y = n)$)
3. **(1 mark)** Calculate the covariance of X and Y.
4. **(Bonus 1 mark)** Calculate $\sum_{k=1}^{n-1} P(X = k, Y = n - k)$ for all $n \geq 1$.
5. **(Bonus 1 mark)** Deduce the $P(X + Y = n)$, for all $n \geq 1$.
6. **(Bonus 1 mark)** What is the p.m.f. the **total number** of flips until both coins come up heads.
7. **(2 mark)** Let X be random variable such that $E[X] = 1$ and $\mathrm{Var}(X) = 5$. Find $E[(2 + X)^2]$
8. **(1 mark)** Find $\mathrm{Var}(4 + 3X)$

Problem 2 (10 marks)

The $f_{(X,Y)}(x, y)$ be the joint density function of the random variable X and Y

$$f_{(X,Y)}(x, y) = \begin{cases} 2e^{-(x+2y)} & \text{for } 0 < x,\ 0 < y. \\ 0 & \text{otherwise} \end{cases}$$

1. **(1 mark)** Find the marginal p.d.f. of X and Y, respectively.
2. **(1 mark)** Are X and Y independent? Explain your answer.
3. **(1 mark)** Find the correlation coefficient ρ of X and Y.
4. **(1 mark)** Calculate $E[Y]$ and $\mathrm{Var}(Y)$.

5. **(1 mark)** Find the conditional density of Y given $X = x$.
6. **(1 mark)** Compute the conditional expectation of Y given $X = x$
7. **(1 mark)** Identify $E[Y \mid X]$ and $E[Y^2 \mid X]$
8. **(1 mark)** Use the properties of the conditional expectation to find without calculations, $\text{Var}(Y \mid X)$ and $\text{Var}(X \mid Y)$.
9. **(1 mark)** Calculate the moment generating function $M_Z(t)$ of an exponential random variable Z with parameter $\lambda > 0$. Precise the condition on t.
10. **(1 mark)** Deduce the MGFs of X and Y.
11. **(Bonus 2 mark)** Deduce from Q10 the moment generating function of $X + Y$.

Problem 3 (12 marks)

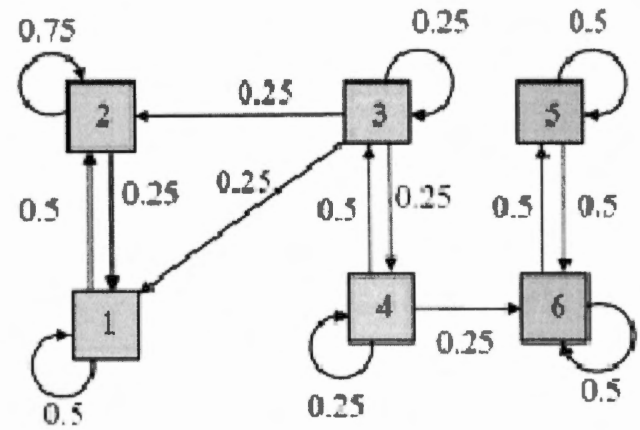
Consider an homogenous Markov chain $\{X_n, n \geq 0\}$ on states 1, 2, 3, 4 and transition matrix

$$P = \begin{pmatrix} a & \frac{1}{3} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & b & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & c \\ 0 & 0 & 0 & d \end{pmatrix}$$

1. **(1 mark)** Find a, b, c and d such that P is a transition probability matrix.
2. **(1 mark)** Assume that the X_0 has a uniform distribution. Calculate $P(X_0 = i)$, for $i \in \{1, 2, 3, 4\}$
3. **(1 mark)** Find the row vector α_1.
4. **(1 mark)** Find the distribution of X_2.
5. **(1 mark)** Calculate the expectation of X_2^3.
6. **(1 mark)** Calculate $P(X_0 = 3,\ X_1 = 2, X_2 = 1)$.
7. **(1 mark)** Calculate $P(X_0 = 3 \mid X_1 = 2, X_2 = 1)$.
8. **(1 mark)** Draw a **2–step** state transition diagram of the Markov chain.
9. **(1 mark)** Specify communicating classes of the Markov chain $\{X_n, n \geq 0\}$.
10. **(1 mark)** Find recurrent and transient classes if any
11. **(1 mark)** Find absorbing states if any
12. **(1 mark)** Is this Markov chain irreducible? Explain your answer

Problem 4 (10 marks)

Consider the following state transition diagram of a Markov chain



1. **(1 mark)** Give the state space E of $(X_n)_{n\geq 0}$.
2. **(1 mark)** Write down the corresponding transition matrix.
3. **(1 mark)** List the communicating classes.
4. **(1 mark)** Specify recurrent and transient classes
5. **(1 mark)** Is this Markov chain irreducible ? Explain your answer.
6. **(1 mark)** Change the arrows as minimal as possible in the previous diagram to make the Markov chain irreducible ?
7. **(1 mark)** Give the new transition matrix.
8. **(1 mark)** Give an example a transition diagram or transition matrix containing an absorbing state.
9. **(1 mark)** Is your new Markov chain irreducible ? Explain your answer.
10. **(1 mark)** If it is irreducible how can you make it reducible ? If it is reducible how can you make it irreducible ?