PHYSICS 507 – SPRING 2021
3rd HOMEWORK- Solutions
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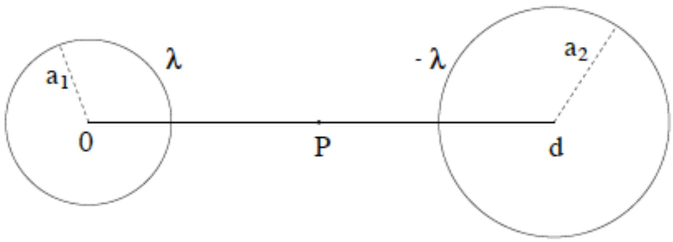
Hand in: Wednesday 17th of February at 23:59

1. Two very long cylindrical conductors, separated by a distance d, form a capacitor. Cylinder 1 has surface charge density λ and radius a_1, and cylinder number 2 has surface charge density $-\lambda$ and radius a_2 (see figure). The electric field for each of the cylinders is radially directed outward and is given by (for each of the conductors) $\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 |\mathbf{r}|}\hat{\rho}$:



   (a) Calculate the total electric field at a generic point P (at a distance ρ from 0).
   (b) Consider that d is far larger than the radii a_1 and a_2 ($d >> a_1, a_2$ so in this case you can consider them equal to a). In this case calculate the potential difference between the two cylindrical surfaces.
   (c) Calculate the capacitance per unit length of the system.

Solution:

At a generic point P which is at a distance ρ from 0 we have the superposition of the two fields of the two cylinders as:

$$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{1}{\rho} + \frac{1}{d-\rho}\right)\hat{\rho}$$

The potential difference between the two cylinders is:

$$V_{a2} - V_{a1} = -\frac{\lambda}{2\pi\varepsilon_0}\int_{d-a2}^{a1}\left(\frac{1}{\rho} + \frac{1}{d-\rho}\right)d\rho = -\frac{\lambda}{2\pi\varepsilon_0}\left[\ln a_1 + \ln\left(d-a_1\right) + \ln\left(d-a_2\right) - \ln a_2\right]$$

If the radii are almost equal then

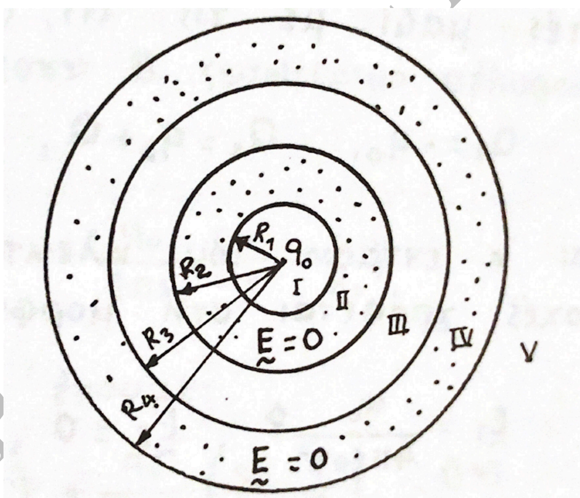
$$V_{a2}-V_{a1}=-\frac{\lambda}{2\pi\varepsilon_0}\left[\ln a+\ln\left(d-a\right)+\ln\left(d-a\right)-\ln a\right]=-\frac{\lambda}{\pi\varepsilon_0}\left[\ln\left(d-a\right)-\ln a\right]$$

$$=-\frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d-a}{a}\right)\approx-\frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)\approx\frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{a}{d}\right)$$

b) The capacitance per unit length is

$$C=\frac{\lambda}{V_{a2}-V_{a1}}=\frac{\lambda}{\frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{d}{a}\right)}=\frac{\pi\varepsilon_0}{\ln\left(\frac{d}{a}\right)}$$

3. The spherical shell with inner and outer radii R_1 and R_2 carries an electric charge Q. The spherical shell with inner and outer radii R_3 and R_4 carries a charge q, while at the centre there is a point-like charge q_0. The shells are made of conducting material. (i) Find the electric field in all five regions of space (ii) Find the electric potential in all five regions of space.



Solution:
In the figure we see five regions. In regions II and IV we have **E** = 0 because we are inside a conductor.

All the charge Q of the shell II rests on its inner and outer surface. **Since we do not know these charges** we write:

$Q=Q_1+Q_2$ (1)

All the charge q of the shell IV rests on its inner and outer surface. Since we do not know these charge we write:

$$q = Q_3 + Q_4 \quad (2)$$

Thus the surfaces of radii R_1, R_2, R_3, R_4 carry charges Q_1, Q_2, Q_3 and Q_4 respectively that are distributed uniformly on them so the create electric fields with radial direction. If we apply Gauss law for the five regions (drawing proper Gaussian surfaces) we get:

$$E_{\mathrm{I}} 4\pi r^2 = \frac{q_0}{\varepsilon_0} \Rightarrow \mathbf{E}_{\mathrm{I}} = \frac{q_0}{4\pi\varepsilon_0 r^2}\hat{r} \quad (3)$$

$$E_{\mathrm{II}} 4\pi r^2 = \frac{(q_0 + Q_1)}{\varepsilon_0} \Rightarrow \mathbf{E}_{\mathrm{I}} = \frac{(q_0 + Q_1)}{4\pi\varepsilon_0 r^2}\hat{r} \quad (4)$$

$$E_{\mathrm{III}} 4\pi r^2 = \frac{(q_0 + Q_1 + Q_2)}{\varepsilon_0} \Rightarrow \mathbf{E}_{\mathrm{III}} = \frac{(q_0 + Q_1 + Q_2)}{4\pi\varepsilon_0 r^2}\hat{r} \quad (5)$$

$$E_{\mathrm{IV}} 4\pi r^2 = \frac{(q_0 + Q_1 + Q_2 + Q_3)}{\varepsilon_0} \Rightarrow \mathbf{E}_{\mathrm{IV}} = \frac{(q_0 + Q_1 + Q_2 + Q_3)}{4\pi\varepsilon_0 r^2}\hat{r} \quad (6)$$

$$E_{\mathrm{V}} 4\pi r^2 = \frac{(q_0 + Q_1 + Q_2 + Q_3 + Q_4)}{\varepsilon_0} \Rightarrow \mathbf{E}_{\mathrm{V}} = \frac{(q_0 + Q_1 + Q_2 + Q_3 + Q_4)}{4\pi\varepsilon_0 r^2}\hat{r} \quad (7)$$

Since in the conducting regions II and IV we have zero electric field then form (4) and (6) we have:

$$q_0 + Q_1 = 0 \quad (8) \quad \text{and} \quad q_0 + Q_1 + Q_2 + Q_3 = 0 \quad (9)$$

Combining (1), (2) with (8) and (9) we get:

$$Q_1 = -q_0, \quad Q_2 = q_0 + Q, \quad Q_3 = -q_0 - Q, \quad Q_4 = q_0 + Q + q$$

Commented [VL1]: This part is the most important part of the solution

Thus the electric fields now are given by:

$$\mathbf{E}_{\mathrm{I}} = \frac{q_0}{4\pi\varepsilon_0 r^2}\hat{r}, \quad \mathbf{E}_{\mathrm{II}} = 0, \quad \mathbf{E}_{\mathrm{III}} = \frac{(q_0 + Q)}{4\pi\varepsilon_0 r^2}\hat{r}, \quad \mathbf{E}_{\mathrm{IV}} = 0, \quad \mathbf{E}_{\mathrm{V}} = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 r^2}\hat{r}$$

(ii) If we take as our reference point the infinity then

a) For $r \geq R_4$

$$V_V(r) - V(\infty) = \int_r^{+\infty} \mathbf{E}_{\mathrm{V}} \cdot d\mathbf{r} \Rightarrow V_V(r) = \int_r^{+\infty} \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 r^2} \hat{r} \cdot (dr)\hat{r}$$

$$\Rightarrow V_V(r) = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0} \int_r^{+\infty} \frac{dr}{r^2} \Rightarrow V_V(r) = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 r}$$

This relation holds also for $r = R_4$ and since a conductor has a constant potential it is true for al region IV thus

$$V_V(R_4) = V_{\mathrm{IV}} = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 R_4}$$

b) For $R_2 \le r \le R_3$

$$V_{\mathrm{III}}(r) - V(R_3) = \int_r^{R_3} \mathbf{E}_{\mathrm{III}} \cdot d\mathbf{r} \Rightarrow V_{\mathrm{III}}(r) - V(R_3) = \int_r^{R_3} \frac{(q_0 + Q)}{4\pi\varepsilon_0 r^2} \hat{r} \cdot (dr)\hat{r}$$

$$\Rightarrow V_{\mathrm{III}}(r) - V(R_3) = \frac{(q_0 + Q)}{4\pi\varepsilon_0} \int_r^{R_3} \frac{1}{r^2} \hat{r} \cdot (dr)\hat{r}$$

$$\Rightarrow V_{\mathrm{III}}(r) - V(R_3) = \frac{(q_0 + Q)}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{R_3} \right)$$

But

$$V(R_3) = V(R_4) = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 R_4} \text{ so}$$

$$V_{\mathrm{III}}(r) - \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 R_4} = \frac{(q_0 + Q)}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{R_3} \right) \Rightarrow$$

$$V_{\mathrm{III}}(r) = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 R_4} + \frac{(q_0 + Q)}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{R_3} \right)$$

The relation is true for $r = R_2$

$$V(R_2) = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 R_4} + \frac{(q_0 + Q)}{4\pi\varepsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_3} \right)$$

and this is the potential for the region II which is a conductor

$$V_{\mathrm{II}} = \frac{(q_0 + Q + q)}{4\pi\varepsilon_0 R_4} + \frac{(q_0 + Q)}{4\pi\varepsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_3} \right)$$

c) For $r \le R_1$

$$V_{\mathrm{I}}(r)-V\left(R_1\right)=\int_r^{R_1}\mathbf{E}_{\mathrm{I}}\cdot d\mathbf{r}\Rightarrow V_{\mathrm{I}}(r)-V\left(R_1\right)=\int_r^{R_1}\frac{q_0}{4\pi\varepsilon_0 r^2}\hat{r}\cdot(dr)\hat{r}$$

$$\Rightarrow V_{\mathrm{I}}(r)-V\left(R_1\right)=\frac{q_0}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{R_1}\right)$$

But

$$V\left(R_1\right)=V\left(R_2\right)=V_{\mathrm{II}}=\frac{\left(q_0+Q+q\right)}{4\pi\varepsilon_0 R_4}+\frac{\left(q_0+Q\right)}{4\pi\varepsilon_0}\left(\frac{1}{R_2}-\frac{1}{R_3}\right) \text{ so}$$

$$V_{\mathrm{I}}(r)=\frac{\left(q_0+Q+q\right)}{4\pi\varepsilon_0 R_4}+\frac{\left(q_0+Q\right)}{4\pi\varepsilon_0}\left(\frac{1}{R_2}-\frac{1}{R_3}\right)+\frac{q_0}{4\pi\varepsilon_0}\left(\frac{1}{r}-\frac{1}{R_1}\right)$$

Notes:

1. We see that in such problems there are **unknown charges** on the surface of the conductors which we calculate from the condition that inside a conductor the field is zero and from the given charge of each conductor. **This is very important**.

2. If the charge of a shell is zero then we follow the same process. We have unknown charges on the conductor surfaces but the total charge is zero.

3. If a shell is grounded then its potential is zero. Its charge is unknown. The charges on its surfaces are also unknown. Then in order to find the charges we use the fact the electric field is zero inside the conductor and the equation of the potential with value equal to zero. The same we do if the potential has a constant value V_0 on the conductor.