4.3
Linear Independence

Linear Independence and Dependence

DEFINITION:

If $S = \{v_1, v_2, \ldots, v_r\}$ is a set of two or more vectors in a vector spaceV , then S is said to be a **linearly independent set** if no vector in S can be expressed as a linear combination of the others. A set that is not linearly independent is said to be **linearly dependent**.

THEOREM:

nonempty set S = $\{v_1, v_2, \ldots, v_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1 v_1 + k_2 v_2 + \cdots + k_r v_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \cdots, k_r = 0$.

EXAMPLE 1

Linear Independence of the Standard Unit Vectors in R^n

The most basic linearly independent set in $\boldsymbol{R^n}$ is the set of standard unit vectors

$$= (0, 0, 0, . . . , 1) = (0, 1, 0, . . . , 0), . . . , e_n = (1, 0, 0, . . . , 0), e_2\ e_1$$

, consider the standard unit vectors To illustrate this in $\boldsymbol{R^3}$

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

To prove linear independence we must show that the only coefficients satisfying the vector equation

$$k_1\,\mathbf{i} + k_2\,\mathbf{j} + k_3\,\mathbf{k} = \mathbf{0}$$

are $k_1 = 0$, $k_2 = 0$, $k_3 = 0$. But this becomes evident by writing this equation in its component form $(k_1, k_2, k_3) = (0, 0, 0)$

You should have no trouble adapting this argument to establish the linear independence . of the standard unit vectors in $\boldsymbol{R^n}$

EXAMPLE 2

Determine whether the vectors

$$\boldsymbol{v_1} = (1, -2, 3),\ \boldsymbol{v_2} = (5, 6, -1),\ \boldsymbol{v_3} = (3, 2, 1) \qquad (2)$$

are linearly independent or linearly dependent in $\boldsymbol{R^3}$.

Solution

The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = \mathbf{0} \qquad \textbf{(3)}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (3) in the component form

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 5\,k_2 + 3\,k_3 &= 0 \\ -2\,k_1 + 6\,k_2 + 2\,k_3 &= 0 \\ 3\,k_1 - k_2 + k_3 &= 0 \end{aligned} \qquad (4)$$

Thus, our problem reduces to determining whether this system has nontrivial solutions. There are various ways to do this; one possibility is to simply solve the system, which yields

$$k_1 = \frac{-1}{2}t \text{ , } k_2 = \frac{-1}{2}t \text{ , } k_3 = t$$

(we omit the details). This shows that the system has nontrivial solutions and hence that the vectors are linearly dependent. A second method for establishing the linear dependence is to take advantage of the fact that the coefficient matrix

$$A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

is square and compute its determinant. We leave it for you to show that det(A) = 0 from which it follows that (4) has nontrivial solutions by parts (b) and (g) of Theorem 2.3.8. Because we have established that the vectors v_1,v_2 and v_3 in (2) are linearly dependent, we know that at least one of them is a linear combination of the others. We leave it for you to confirm, for example, that

$$v_3 = \frac{1}{2}v_1 + \frac{1}{2}v_2$$

EXAMPLE 3

Determine whether the vectors

$$\boldsymbol{v_1} = (1, 2, 2, -1),\ \boldsymbol{v_2} = (4, 9, 9, -4),\ \boldsymbol{v_3} = (5, 8, 9, -5)$$

in $\boldsymbol{R^4}$ are linearly dependent or linearly independent.

EXAMPLE 4

Show that the polynomials $1, x, x^2, \ldots, x^n$ form a linearly independent set in $\boldsymbol{P_n}$.

EXAMPLE 5

Determine whether the polynomials

$p_1 = 1 - x, \quad p_2 = 5 + 3x - 2x^2, \quad p_3 = 1 + 3x - x^2$

are linearly dependent or linearly independent in P_2.

Sets with One or Two Vectors

THEOREM:

(a) A finite set that contains **0** is linearly dependent.

(b) A set with exactly one vector is linearly independent if and only if that vector is not **0**.

(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

A Geometric Interpretation of Linear Independence

THEOREM:

Let $S = \{v_1, v_2, \dots, v_r\}$ be a set of vectors in $\boldsymbol{R}^n$. If $r > n$, then S is linearly dependent.

Proof Suppose that

$$\begin{aligned} \mathbf{v}_1 &= (v_{11}, v_{12}, \dots, v_{1n}) \\ \mathbf{v}_2 &= (v_{21}, v_{22}, \dots, v_{2n}) \\ &\vdots \\ \mathbf{v}_r &= (v_{r1}, v_{r2}, \dots, v_{rn}) \end{aligned}$$

and consider the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

If we express both sides of this equation in terms of components and then equate the corresponding components, we obtain the system

$$\begin{aligned} v_{11}k_1 + v_{21}k_2 + \cdots + v_{r1}k_r &= 0 \\ v_{12}k_1 + v_{22}k_2 + \cdots + v_{r2}k_r &= 0 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\ \ \vdots \\ v_{1n}k_1 + v_{2n}k_2 + \cdots + v_{rn}k_r &= 0 \end{aligned}$$

This is a homogeneous system of n equations in the r unknowns $k_1, \dots, k_r$. Since $r > n$, it follows from Theorem 1.2.2 that the system has nontrivial solutions. Therefore, $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r\}$ is a linearly dependent set. ◀

Linear Independence of Functions

DEFINITION 2 If $\mathbf{f}_1 = f_1(x), \mathbf{f}_2 = f_2(x), \ldots, \mathbf{f}_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the ***Wronskian*** of $f_1, f_2, \ldots, f_n$.

THEOREM:

If the functions $\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n$ have $n-1$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is not identically zero on $(-\infty, \infty)$, then these functions form a linearly independent set of vectors in $C^{(n-1)}(-\infty, \infty)$.

Exercise Set 4.3

1. Explain why the following form linearly dependent sets of vectors. (Solve this problem by inspection.)

   (a) $\mathbf{u}_1 = (-1, 2, 4)$ and $\mathbf{u}_2 = (5, -10, -20)$ in R^3

   (b) $\mathbf{u}_1 = (3, -1)$, $\mathbf{u}_2 = (4, 5)$, $\mathbf{u}_3 = (-4, 7)$ in R^2

   (c) $\mathbf{p}_1 = 3 - 2x + x^2$ and $\mathbf{p}_2 = 6 - 4x + 2x^2$ in P_2

   (d) $A = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -2 & 0 \end{bmatrix}$ in M_{22}

2. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^3.

   (a) $(-3, 0, 4)$, $(5, -1, 2)$, $(1, 1, 3)$

   (b) $(-2, 0, 1)$, $(3, 2, 5)$, $(6, -1, 1)$, $(7, 0, -2)$

3. In each part, determine whether the vectors are linearly independent or are linearly dependent in R^4.

   (a) $(3, 8, 7, -3)$, $(1, 5, 3, -1)$, $(2, -1, 2, 6)$, $(4, 2, 6, 4)$

   (b) $(3, 0, -3, 6)$, $(0, 2, 3, 1)$, $(0, -2, -2, 0)$, $(-2, 1, 2, 1)$

4. In each part, determine whether the vectors are linearly independent or are linearly dependent in P_2.

(a) $2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$

(b) $1 + 3x + 3x^2,\ x + 4x^2,\ 5 + 6x + 3x^2,\ 7 + 2x - x^2$

5. In each part, determine whether the matrices are linearly independent or dependent.

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ in M_{22}

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ in M_{23}

6. Determine all values of k for which the following matrices are linearly independent in M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & k \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ k & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

7. In each part, determine whether the three vectors lie in a plane in R^3.

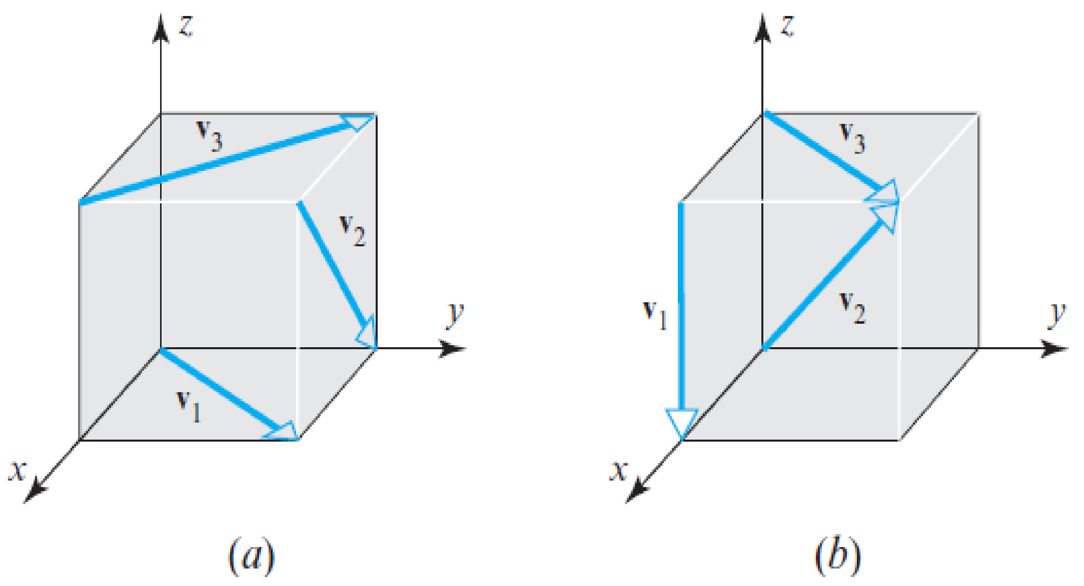
(a) $\mathbf{v}_1 = (2, -2, 0),\ \mathbf{v}_2 = (6, 1, 4),\ \mathbf{v}_3 = (2, 0, -4)$

(b) $\mathbf{v}_1 = (-6, 7, 2),\ \mathbf{v}_2 = (3, 2, 4),\ \mathbf{v}_3 = (4, -1, 2)$

whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent in R^3.

(a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & 0 \end{bmatrix}$

15. Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in part (*a*) of the accompanying figure linearly independent? What about those in part (*b*)? Explain.



▲ Figure Ex-15