4.5 Dimension

Number of Vectors in a Basis

THEOREM:

All bases for a finite-dimensional vector space have the same number of vectors.

THEOREM:

Let V be an n-dimensional vector space, and let $\{v_1,v_2,...,v_n\}$ be any basis.

(a) If a set in V has more than n vectors, then it is linearly dependent.

(b) If a set in V has fewer than n vectors, then it does not span V.

Definition:

The **dimension** of a finite-dimensional vector space V is denoted by dim(V) and is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

EXAMPLE 1

dim(R^n) = n **[The standard basis has n vectors.]**

dim(P_n) = n + 1 **[The standard basis has n + 1 vectors.]**

dim(M_{mn}) = mn **[The standard basis has mn vectors.]**

EXAMPLE 2

If S = $\{v_1, v_2, \ldots, v_r\}$ then every vector in span(S) is expressible as a linear combination of the vectors in S. Thus, if the vectors in S are linearly independent, they automatically form a basis for span(S), from which we can conclude that

$$\dim[\text{span}\{v_1, v_2, \ldots, v_r\}] = r$$

EXAMPLE 3

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\
2x_1 + 6x_2 \qquad\quad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Some Fundamental Theorems

THEOREM: Plus/Minus Theorem

Let S be a nonempty set of vectors in a vector space V.

(a) If S is a linearly independent set, and if **v** is a vector in V that is outside of span(S), then the set S ∪ {**v**} that results by inserting **v** into S is still linearly independent.

(b) If **v** is a vector in S that is expressible as a linear combination of other vectors in S, and if S − {**v**} denotes the set obtained by removing **v** from S, then S and S − {**v**} span the same space; that is,

$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\})$$

EXAMPLE 5

Show that $\boldsymbol{p_1} = 1 - x^2, \boldsymbol{p_2} = 2 - x^2, and \boldsymbol{p_3} = x^3$ are linearly independent vectors.

THEOREM:

Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

THEOREM:

Let S be a finite set of vectors in a **finite-dimensional** vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

THEOREM:

If W is a subspace of a finite-dimensional vector space V, then:

(a) W is finite-dimensional.

(b) $\dim(W) \leq \dim(V)$.

(c) $W = V$ if and only if $\dim(W) = \dim(V)$.

Exercise Set 4.5

In Exercises 1–6, find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

1. $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ -2x_1 - x_2 + 2x_3 &= 0 \\ -x_1 \qquad + x_3 &= 0 \end{aligned}$$

2. $$\begin{aligned} 3x_1 + x_2 + x_3 + x_4 &= 0 \\ 5x_1 - x_2 + x_3 - x_4 &= 0 \end{aligned}$$

3. $$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ x_1 \qquad + 5x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

5. $$\begin{aligned} x_1 - 3x_2 + x_3 &= 0 \\ 2x_1 - 6x_2 + 2x_3 &= 0 \\ 3x_1 - 9x_2 + 3x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

7. In each part, find a basis for the given subspace of R^3, and state its dimension.
   (a) The plane $3x - 2y + 5z = 0$.
   (b) The plane $x - y = 0$.
   (c) The line $x = 2t,\ y = -t,\ z = 4t$.
   (d) All vectors of the form (a, b, c), where $b = a + c$.

8. In each part, find a basis for the given subspace of R^4, and state its dimension.
   (a) All vectors of the form $(a, b, c, 0)$.
   (b) All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.
   (c) All vectors of the form (a, b, c, d), where $a = b = c = d$.

9. Find the dimension of each of the following vector spaces.
   (a) The vector space of all diagonal $n \times n$ matrices.
   (b) The vector space of all symmetric $n \times n$ matrices.
   (c) The vector space of all upper triangular $n \times n$ matrices.

10. Find the dimension of the subspace of P_3 consisting of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

11. (a) Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.
    (b) Make a conjecture about the dimension of W.
    (c) Confirm your conjecture by finding a basis for W.

12. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.
    (a) $\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (1, -2, -2)$
    (b) $\mathbf{v}_1 = (1, -1, 0),\ \mathbf{v}_2 = (3, 1, -2)$

13. Find standard basis vectors for R^4 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^4.

$$\mathbf{v}_1 = (1, -4, 2, -3), \quad \mathbf{v}_2 = (-3, 8, -4, 6)$$

14. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

15. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^3.

16. The vectors $\mathbf{v}_1 = (1, 0, 0, 0)$ and $\mathbf{v}_2 = (1, 1, 0, 0)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^4.

17. Find a basis for the subspace of R^3 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 0, 0), \quad \mathbf{v}_2 = (1, 0, 1), \quad \mathbf{v}_3 = (2, 0, 1), \quad \mathbf{v}_4 = (0, 0, -1)$$

18. Find a basis for the subspace of R^4 that is spanned by the vectors

$$\mathbf{v}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = (2, 2, 2, 0), \quad \mathbf{v}_3 = (0, 0, 0, 3), \quad \mathbf{v}_4 = (3, 3, 3, 4)$$

19. In each part, let $T_A\colon R^3 \to R^3$ be multiplication by A and find the dimension of the subspace of R^3 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

20. In each part, let T_A be multiplication by A and find the dimension of the subspace R^4 consisting of all vectors $\mathbf{x}$ for which $T_A(\mathbf{x}) = \mathbf{0}$.

(a) $A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 4 & 0 & 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Working with Proofs

21. (a) Prove that for every positive integer n, one can find $n+1$ linearly independent vectors in $F(-\infty, \infty)$. [*Hint:* Look for polynomials.]

(b) Use the result in part (a) to prove that $F(-\infty, \infty)$ is infinite-dimensional.

(c) Prove that $C(-\infty, \infty)$, $C^m(-\infty, \infty)$, and $C^\infty(-\infty, \infty)$ are infinite-dimensional.

22. Let S be a basis for an n-dimensional vector space V. Prove that if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ form a linearly independent set of vectors in V, then the coordinate vectors $(\mathbf{v}_1)_S, (\mathbf{v}_2)_S, \ldots, (\mathbf{v}_r)_S$ form a linearly independent set in R^n, and conversely.