لا يكتب في هذا الهامش

تمارين الفصل الأول : تكامل ريمان

1) اعتبر $f:[0,\frac{\pi}{2}]\to\mathbb{R}$ ، $f(x)=\cos^2(x)$ حيث $x\in[0,\frac{\pi}{2}]$
أثبت أن $f\in\mathcal{R}[0,\frac{\pi}{2}]$ و f واحسب تكاملها .

2) أثبت أن $f(x)=|x|$ تنتمي إلى $\mathcal{R}[-1,1]$ ، وأحسب تكاملها

3) لدينا الدالة: $f:[0,1]\to\mathbb{R}$ ، $f(x)=\begin{cases}1 ; & x=\frac{1}{n}, n\in\mathbb{N}\\ 0 ; & \text{خلاف ذلك}\end{cases}$
أثبت أن $f\in\mathcal{R}[0,1]$ ، وأحسب $\int_0^1 f(x)dx$.

4) أعط مثالاً لدالة $f:[a,b]\to\mathbb{R}$ تنتمي إلى $\mathcal{R}[c,b]$ لكل $c\in(a,b)$ لكن $f\notin\mathcal{R}[a,b]$.

5) أوجد النهاية : $\lim_{x\to 0^+}\frac{1}{x}\int_0^x\sqrt{4+t^3}\,dt$ (بدون استخدام قاعدة لوبيتال)

6) ابحث تقارب وتباعد التكامل $\int_0^\infty \sin(x^2)\,dx$

7) هل التكامل $\int_0^\infty\frac{\sin^2(x)}{x^2}dx$ متقارب ؟ برر إجابتك .

8) أدرس تقارب التكامل المعتل $\int_1^\infty\frac{\cos(t)}{t^p}dt$ (حيث $-\infty<p<1$)

9) أدرس تقارب التكامل المعتل $\int_1^\infty\frac{\cos(t)}{t^p}dt$ (حيث $p>0$) .

10) أحسب التكامل : $\int_0^3\frac{\sqrt{x}}{\sqrt{27-x^3}}dx$.

لا يكتب في هذا الهامش

تمارين الفصل الثالث: متتاليات ومتسلسلات الدوال

1) ناقش التقارب النقطي لكلٍ من:

(أ) $f_n(x) = xe^{-nx}$; $x \geq 0$ ، (ب) $f_n(x)=\frac{\sin(nx)}{1+nx}$; $x\geq 0$

(ج) $f_n(x) = \left(1-\frac{1}{n}\right)^n x^n$; $x\in(0,1]$

2) اختبر التقارب المنتظم لمتتالية الدوال f_n:

(أ) $f_n(x)=\sqrt{x^2+\frac{1}{n}}$; $x\in\mathbb{R}$

(ب) $f_n(x)=\sin\left(\frac{n}{nx+1}\right)$; $x\in[a,\infty)$ (حيث $a>0$).

(ج) $f_n(x)=\sqrt{n}\,(1-x^2)^n$; $\frac{1}{2}\leq|x|\leq 1$

3) أثبت أن $f_n(x)=\frac{nx+1}{n}$ متقاربة بانتظام على $\mathbb{R}$. هل f_n^2 متقاربة بانتظام على $\mathbb{R}$؟ ماذا تستنتج؟

4) لأي عدد ما $a>0$ أحسب النهاية معللاً خطواتك:

$$\lim_{n\to\infty}\int_a^{\pi} \frac{\sin(nx)}{nx}\,dx$$

5) اختبر تقارب متسلسلات الدوال التالية:

(أ) $\sum_{n=0}^{\infty} (-1)^n\frac{x^{2n+1}}{(2n+1)!}$ ، $x\in\mathbb{R}$ (ب) $\sum_{n=1}^{\infty}\left(\frac{x}{x+1}\right)^n$ ، $x\geq 0$

(ج) $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$ ، $x\in\mathbb{R}$ (د) $\sum_{n=1}^{\infty} (-1)^n\frac{x^2+n}{n^2}$ ، $x\in[a,b]$

(هـ) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(1+x^{2n})}$ ، $x\in\mathbb{R}$

6) أوجد النهاية التالية:

$$\lim_{x\to 1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^x}$$

7) برهن أن: $\forall x>0$:

$$\frac{e^{-x}}{(1-e^{-x})^2}=\sum_{n=1}^{\infty} n e^{-nx}$$

8) هات مثالاً لدالة f قابلة للاشتقاق بأي رتبة على $\mathbb{R}$، لكن لا يمكن تمثيلها بمتسلسلة قوى حول $x=0$ تتقارب إلى f.

9) اكتب الدالة $f(x)=\frac{x^2}{4-x^2}$ كمتسلسلة قوى حول $c=0$ وحدد فترة تقاربها.

10) لتكن $f(x)=\frac{5}{1+9x^4}$. اكتب f كمتسلسلة قوى على الصورة $S(x)=\sum_{n=1}^{\infty} a_n x^n$ وحدد فترة تقاربها وفترة تباعدها. احسب $\lim_{x\to\frac{1}{2}} S(x)$.
هل $\lim_{x\to\frac{1}{\sqrt{3}}^-} S(x)$ موجودة؟

المسموحة ضوئيا بـ CamScanner

لا يكتب في هذا الهامش

تمارين الفصلين الرابع: قياس لبيق، والخامس: تكامل لبيق

١) إذا كانت $\mathcal{B}$ جبر سيجما على مجموعة X وكانت $Y \subset X$ فبرهن أن $\mathcal{B}_Y = \{E \cap Y \mid E \in \mathcal{B}\}$ جبر سيجما على Y.

٢) لأي مجموعة متراصة $E \subseteq \mathbb{R}$ بيّن أن $m(E) < \infty$.

٣) إذا كانت $f: \mathbb{R} \to \mathbb{R}$ متزايدة فأثبت أنها قابلة للقياس

٤) هات مثالاً لدالتين f و g قابلتين للتكامل على مجموعة X (بمفهوم لبيق) لكن الدالة $f.g$ ليست قابلة للتكامل.

٥) تحقق من أن الدالة:

$$f(x) = \begin{cases} 0 & ; \ x = 0 \\ \sqrt[3]{x} & ; \ 0 < x \leq 1 \end{cases}$$

قابلة لتكامل لبيق، واحسب تكاملها $\int_{[0,1]} f\, dm$.

٦) أثبت أن: $\int_{\mathbb{Q}} \sin(x)\, dm = 0$ ($\mathbb{Q}$ الأعداد النسبية)

٧) أوجد النهايات التالية:

أ) $\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, dm$ ب) $\lim_{n\to\infty} \int_{[0,1]} \frac{n\sin^2(nx)}{1+n^2\sqrt{x}}\, dm$

ج) $\lim_{n\to\infty} \int_{\mathbb{R}} \sin\left(\frac{x}{n}\right) f(x)\, dm$ (حيث $f: \mathbb{R} \to \mathbb{R}$ دالة قابلة لتكامل لبيق).

٨) أثبت أن تكامل لبيق $\int_{(0,1]} \frac{x^2 \ln(x)}{(1+x^2)^2}\, dm$ موجود.

٩) أحسب النهايات التالية إن وجدت:

أ) $\lim_{n\to\infty} \int_{[0,1]} \frac{n\sin^2(nx)}{1+n^2\sqrt{x}}\, dm$ ب) $\lim_{n\to\infty} \int_0^1 \frac{nx}{1+n^2x^2}\, dx$

ج) $\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx$ د) $\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\, dx$

هـ) $\lim_{n\to\infty} \int_{[a,b]} \sin^n(x)\, dm$ و) $\lim_{n\to\infty} \int_0^1 x^n f(x)\, dx$ (حيث $f \in R[0,1]$)

١٠) إذا كانت $f: [a,b] \to \mathbb{R}$ دالة متصلة فبرهن أنها قابلة لتكامل لبيق

خالد التبال
433100322
Math 481 (H-W)

1] we have $f(x) = \cos^2(x)$ is cont on $\mathbb{R}$ $\Rightarrow$ $f(x)$ is cont on $[0, \frac{\pi}{2}]$, so we have $f(x)$ is riemann Integrable on $[0, \frac{\pi}{2}]$ since $f(x)$ is cont, we use FTC to compute the Integral

i.e $\int_0^{\frac{\pi}{2}} \cos^2(x)\,dx = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2x)}{2}\,dx = \frac{1}{2}x + \frac{\sin(2x)}{4}\Big]_0^{\frac{\pi}{2}}$

$= \boxed{\frac{\pi}{4}}$ the answer.

2] $f(x) = |x|$ is continuous function on $\mathbb{R}$ $\Rightarrow$ it is cont on $[-1, 1]$ $\Rightarrow$ it is riemann Integrable, we have $f(x) \in R[-1,1]$

Since it is cont we use FTC to compute the Integral

$\int_{-1}^{1} |x|\,dx = 2\int_0^1 x\,dx = x^2\Big]_0^1 = \boxed{1}$ the answer.

3] $f(x) = \begin{cases} 1 , & x = \frac{1}{n}, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$, $f(x) : [0, 1] \to \mathbb{R}$

for any Partition $P = \{x_0, \dots, x_n\}$ for $[0, 1]$, we have

$L(f, P) = 0$ because every Interval has an Irrational $\Rightarrow$

$m_K(f) = 0$ because $f(\alpha) = 0$ for every $\alpha \in [\mathbb{Q}^c \cap [0, 1]]$

So we need to show for $\varepsilon > 0$ $\exists$ P_ε s.t if $mesh(P_\varepsilon) < \delta$ $\Rightarrow$ $U(f, P_\varepsilon) < \varepsilon$, Let $\varepsilon > 0$ be given we can assume that $\varepsilon < 1$, Let x_0 be 0 and choose $x_1 = \frac{\varepsilon}{2}$, now assume we have $\frac{1}{K_1}, \frac{1}{K_2}, \dots, 1$ are the numbers of the form $\frac{1}{n}$, $n \in \mathbb{N}$ in $(\frac{\varepsilon}{2}, 1]$ Let ε_0 be $\min\{|\frac{1}{K_i} - \frac{1}{K_j}|, i \neq j\}$

note: there are finitely many $\frac{1}{K_i} \in (\frac{\varepsilon}{2}, 1]$ assume $\varepsilon_1 = |\frac{\varepsilon}{2} - \frac{1}{K_1}|$ where $\frac{1}{K_1}$ is the smallest number of the form $\frac{1}{n}$ in $(\frac{\varepsilon}{2}, 1]$

Let $\varepsilon' = \min\{\frac{\varepsilon}{2}, \varepsilon_0, \varepsilon_1\}$, assume we have L elements of the form $\frac{1}{K_1}, \frac{1}{K_2}, \dots \frac{1}{K_L} = 1$ in $(\frac{\varepsilon}{2}, 1]$ Let

$\varepsilon'' = \frac{\varepsilon'}{2L}$, and choose $x_2 = \frac{1}{K_1} - \varepsilon''$, $x_3 = \frac{1}{K_1} + \varepsilon''$ and so $x_4 = \frac{1}{K_2} - \varepsilon''$, $x_5 = \frac{1}{K_2} + \varepsilon''$, L Intervals around $\frac{1}{K_i}$

2L+1 points of partition on $\Rightarrow$ $U(f, P_\varepsilon) = M_1(x_1 - x_0) + M_1(x_2 - x_1) + \dots\dots$

where $M_K \neq 0$ Since there are ~~$2L-1$~~ $\cancel{(x_K - x_{K-1})}$ $K = 3, \dots, L+2$

$= \frac{\varepsilon}{2} + L\varepsilon'' = \frac{\varepsilon}{2} + \frac{L\varepsilon'}{2L} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

so we've concluded $U(F,P_\varepsilon) - L(F,P_\varepsilon) < \varepsilon$

$\Rightarrow$ f is Riemann Integrable and $\int_0^1 f\,dx = L(F) = 0$ $\square$

4] $f(x) = \begin{cases} \frac{1}{x} & , x \in (0,1] \\ 0 & , x = 0 \end{cases}$, $f(x) : [0,1] \to \mathbb{R}$

for every $c \in (0,1]$ f is Integrable on $[c,1]$ but $f(x) \notin R[0,1]$

5] Since $\sqrt{4+t^3}$ is continuous for $t \in \mathbb{R}^+ \cup \{0\} \Rightarrow$

$g(x) = \int_0^x \sqrt{4+t^3}\,dt$ is differentiable function $\Rightarrow$ the right hand

derivative for $x=0$ exist and is by definition $\lim_{x\to 0^+} \frac{g(x)-g(0)}{x-0}$

$= \lim_{x\to 0^+} \frac{1}{x}\int_0^x \sqrt{4+t^3}\,dt$, because $g(0)=0$, $g'(0) =$

$g'(x)\Big|_{x=0} = \sqrt{4+x^3}\Big|_{x=0} = \sqrt{4} = 2 \Rightarrow \lim_{x\to 0^+} \frac{1}{x}\int_0^x \sqrt{4+t^3}\,dt = \boxed{2}$

6] we have $\sin(x^2)$ is cont & bound on $[0,1]$, so we'll Investigate

$\int_1^\infty \sin(x^2)\,dx$, make substitution $t = x^2 \Rightarrow x = \sqrt{t} \Rightarrow dx = \frac{dt}{2\sqrt{t}}$

$\Rightarrow \int_1^\infty \sin(x^2)\,dx = \int_1^\infty \frac{\sin(x)}{2\sqrt{x}}\,dx = \lim_{\alpha\to\infty} \frac{-\cos(\alpha)}{2\sqrt{\alpha}} + \frac{\cos(1)}{2} + \frac{1}{2}\int_1^\infty \frac{-\cos(x)}{2x^{\frac{3}{2}}}\,dx$

$\lim_{\alpha\to\infty} \frac{-\cos(\alpha)}{2\sqrt{\alpha}} = 0$ we also have $\int_1^\infty \left|\frac{-\cos(x)}{2x^{\frac{3}{2}}}\right| dx \leq$

$\int_1^\infty \frac{1}{2x^{\frac{3}{2}}}\,dx$ which is convergent $\Rightarrow \int_1^\infty \sin(x^2)\,dx$ converges.

---

7) Since we have $\frac{\sin^2(x)}{x^2}$ is bounded near Zero and $\lim_{x \to 0^+} \frac{\sin^2(x)}{x^2} = 1$ $\Rightarrow$ $\int_0^1 \frac{\sin^2(x)}{x^2} dx$ exist, now for $\int_1^\infty \frac{\sin^2(x)}{x^2} dx$, Since $0 \leq \sin^2(x) \leq 1$ and since $\int_1^\infty \frac{1}{x^2} dx$ converges to 1 $\Rightarrow$ $\int_1^\infty \frac{\sin^2(x)}{x^2} dx$ converges $\Rightarrow$ $\int_0^\infty \frac{\sin^2(x)}{x^2} dx$ converges □

8] for $0 < P < 1$ $I = \int_1^\infty \frac{\cos(t)}{t^P} dt$ if we Integrate by Parts taking $v = \frac{1}{t^P}$, $du = \cos(t)$ $\rightarrow$

$$I = \lim_{\alpha \to \infty} \frac{1}{t^P} \sin(t) \Big]_1^\alpha + P \int_1^\infty \frac{\sin(t)}{t^{P+1}} dt$$

, the first limit since $P > 0$ converges to $-\sin(1)$ aslo since $P > 0 \Rightarrow P+1 > 1$

$\Rightarrow P \int_1^\infty \frac{\sin(t)}{t^{P+1}} dt$ is absolutely convergent $\Rightarrow$ it is convegent

$\Rightarrow I$ converges □

for $P < 0 \Rightarrow \frac{1}{t^P} = t^{-P} = t^s$ for positive s

$\Rightarrow \int_1^\infty t^s \cos(t) dt$ diverges □

now for $P = 0$ $\int_1^\infty \cos(t) dt$ diverges □

9] $I = \int_1^\infty \frac{\cos(t)}{t^P} dt$ for $P > 0$, for $P > 1$

I is absolutely convergent because $0 \leq |\cos(t)| \leq 1$ $\forall t$ and $\int_1^\infty \frac{1}{t^P} dt$ convergs for $P > 1$ $\Rightarrow$ I is convergent

for $0 < P < 1$ it converges from "8"

for $p = 1$ we have $I = \int_1^\infty \frac{\cos(t)}{t}\,dt$, by parts

$I = \lim_{\alpha\to\infty} \frac{1}{t}\sin(t)\Big]_1^\alpha + \int_1^\infty \frac{\sin(t)}{t^2}\,dt$, the first limit exist

and $\int_1^\infty \frac{\sin(t)}{t^2}\,dt$ is absolutely convergent $\Rightarrow$ I converges

$\Rightarrow \int_1^\infty \frac{\cos(t)}{t^p}\,dt$ converges for all $p > 0$ $\square$

10] to compute $\int_0^3 \frac{\sqrt{x}}{\sqrt{27 - x^3}}\,dx$, we make

the substitution $u = x^{\frac{3}{2}} \Rightarrow x^3 = u^2$ also , $x = 0 \Rightarrow u = 0$
$x = 3 \Rightarrow u = \sqrt{27}$

$du = \frac{3}{2}\sqrt{x}\,dx \Rightarrow \sqrt{x}\,dx = \frac{2}{3}\,du \Rightarrow$ the Integral

becomes $L = \frac{2}{3}\int_0^{\sqrt{27}} \frac{du}{\sqrt{27 - u^2}}$ note this Improper Integral because

the denominator vanishes at $\sqrt{27}$

$L = \lim_{\alpha\to\sqrt{27}} \frac{2}{3}\sin^{-1}\left(\frac{u}{\sqrt{27}}\right)\Big]_0^\alpha = \frac{2}{3}\sin^{-1}(1) = \frac{2}{3}\frac{\pi}{2}$ the

answer $\Rightarrow$ the Integral converges $\square$.

The sequence is Pointwis convergent to $f(x)=0$, if $x=0$ $\Rightarrow f_n(0)=0$. If $x>0$ $f_n(x)=x\frac{1}{e^{nx}}$ where $x>0 \Rightarrow e^{nx}\to\infty$ $\Rightarrow f_n(x)\to 0$ □

(ب) $f_n(x)=\frac{\sin(nx)}{1+nx}$ converges Pointwise to $f(x)=0$, if $x=0$ $\Rightarrow f_n(0)=0$. If $x>0 \Rightarrow \sin(x)$ is bounded $\wedge$ $1+nx\to\infty \Rightarrow$ $f_n(x)=0$ □

ج.) $f_n(x)=\cancel{\sqrt{n}(1-x^2)^n}$ $(1-\frac{1}{n})^n x^n$, we $0\leq(1-\frac{1}{n})^n<1$ $\forall n\in\mathbb{N}$ for $x\in(0,1)$, $x^n\to 0$. Hence, $f_n(x)\to 0 \Rightarrow f(x)=0$. Also for $x=1$ $f_n(1)=(1-\frac{1}{n})^n\to\frac{1}{e} \Rightarrow f(x)=\frac{1}{e}$ $\Rightarrow f_n(x)$ Converges Pointwise to $f(x)=\begin{cases}0, & x\in(0,1)\\ \frac{1}{e} & x=1\end{cases}$ □

Q 2] (أ) The sequence is Pointwise convergent to $f(x)=|x|$

now $\sup|f_n(x)-f(x)| = \sup\left|\sqrt{x^2+\frac{1}{n}}-\sqrt{x^2}\right| =$

$\sup\left|\frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+\sqrt{x^2}}\right| \leq \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}}$ $\forall x\in\mathbb{R}$

$=\frac{1}{\sqrt{n}}\to 0 \Rightarrow \sup|f_n(x)-f(x)|\to 0 \Rightarrow$ The convergence is uniform □

(ب) $f_n(x)=\sin\left(\frac{n}{nx+1}\right)$, $\forall x\in\mathbb{R}^+$ $f_n(x)\to\sin(\frac{1}{x})$

now $\sup|f_n(x)-f(x)| = \sup\left|\sin(\frac{n}{nx+1})-\sin(\frac{1}{x})\right| =$

$\sup\left|\frac{n}{nx+1}-\frac{1}{x}\right||\cos(\alpha)| \leq \sup\left|\frac{n}{nx+1}-\frac{1}{x}\right|$

$=\sup\left|\frac{-1}{nx+1}\right| \leq \sup\left|\frac{1}{na+1}\right|\to 0$ as $n\to\infty$

hence the convergence is uniform □

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2] $f_n(x) = \sqrt{n}\,(1-x^2)^n, \ \forall x \in [\frac{1}{2},1]$ $\quad f_n(x) \to 0$

$\sup |f_n(x)| \leq \sqrt{n}\,(1-\frac{1}{2}^2)^n = \sqrt{n}\,(\frac{3}{4})^n \to 0 \quad \Rightarrow$

$f_n(x)$ is uniformely convergent.

Q3] $|f_n(x) - f(x)| = \left|\frac{nx+1}{n} - x\right|, \quad f_n(x) \to x \quad \forall x \in \mathbb{R}$

$= |\frac{1}{n}|$. Hence $\sup |f_n(x) - f(x)| = |\frac{1}{n}| \to 0$

$\Rightarrow f_n(x)$ is uniformely convergent.

Now $f_n^2(x) = \dfrac{n^2x^2 + 2nx + 1}{n^2} \longrightarrow x^2 \quad \forall x \in \mathbb{R}$

$|f_n^2(x) - f^2(x)| = \frac{2x}{n} + \frac{1}{n^2} \not\to 0$ for arbitraty large x

$\Rightarrow \sup |f_n^2(x) - f^2(x)| \not\to 0 \Rightarrow f_n^2$ is not uniformely converges

we conclude if f_n, g_n converges uniformely for $x \in A$

$\not\Rightarrow f_n g_n$ converges uniformely to fg $\quad \square$

Q4] we can easily see that $g_n(x) = \dfrac{\sin(nx)}{nx}$ converges uniformely to $g(x) = 0 \quad \forall x \in [a,\infty) \quad a > 0 \Rightarrow [a,\pi]$

$\Rightarrow$ we can Interchange the limits and Integral $\Rightarrow$

$$\lim_{n\to\infty} \int_a^{\pi} \frac{\sin(nx)}{nx}\,dx = \int_a^{\pi} \lim_{n\to\infty} \frac{\sin(nx)}{nx}\,dx = \int_a^{\pi} 0\,dx = 0 \quad \square$$

Q5] (i) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ we have $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ converges

$\forall x \in \mathbb{R}$ by ratio test $\Rightarrow \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ converges pointwise $\forall x \in \mathbb{R}$

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ii) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{x}{x+1}\right)^n, \ \forall x > 0 \ \sum_{n=1}^{\infty} \left(\frac{x}{x+1}\right)^n$ is a geometric

Series with $0 < q = \frac{x}{x+1} < 1$ hence the series converges pointwise, it is not uniformely convergent

for $\displaystyle\sup_{x \in \mathbb{R}^+ \cup \{0\}} \sum_{k=n}^{n+r} f_k(x) = r \not\to 0$ for

arbitrary large n, r

2.] $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}$, for any $x \in \mathbb{R}$ $x \sum_{n=1}^{\infty} \frac{1}{(n+x^2)^2}$ converges

Pointwise, how $\sup |f_n(x)|$, to compute it

$$f_n'(x) = \frac{(n+x^2)^2 - 2(x)(2x)[n+x^2]}{(n+x^2)^2} = \frac{(n+x^2)[n+x^2-4x^2]}{(n+x^2)^4}$$

$$= 0 \Rightarrow n - 3x^2 = 0 \Rightarrow x = \pm\sqrt{\frac{n}{3}}$$

it is easy to see that at $x = \sqrt{\frac{n}{3}}$ $f_n(x)$ attains

The maximum $f_n(\sqrt{\frac{n}{3}}) = \frac{\sqrt{\frac{n}{3}}}{(n+\frac{n}{3})^2} \to 0$

$\Rightarrow$ also we have by p series $\sum_{n=1}^{\infty} \frac{\sqrt{\frac{n}{3}}}{(n+\frac{n}{3})^2}$

converges $\Rightarrow$ the series converges uniformly

$\checkmark$

3) $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$, $\forall x \in [a,b]$ the series

$= \sum_{n=1}^{\infty} \frac{(-1)^n x}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ the formers is absolutely convergent, the

Latter is convergent by alternating serie test

also $\forall$ bounded Interval the serie $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$

converges uniformly

c

3) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n(1+x^{2n})}$, the series converges $\forall x \in \mathbb{R}$

$0 < \frac{1}{(1+x^{2n})} \leq 1$. Since $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges $\Rightarrow$

$\forall \varepsilon > 0$ $\exists N$, s.t, $m, n \geq N$ $\left| \sum_{k=n}^{m} \frac{(-1)^k}{k} \right| < \varepsilon$

$$\Rightarrow \frac{1}{1+x^{2n}} \left| \sum_{k=n}^{m} \frac{(-1)^k}{k} \right| < \varepsilon \quad \forall x \in \mathbb{R}$$

$\Rightarrow$ the series is uniformly convergent

Q6] the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x}$ converges uniformly In

neighborhood of 1 $\Rightarrow \lim_{x \to 1} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x} = \sum_{n=1}^{\infty} \lim_{x \to 1} \frac{(-1)^{n-1}}{n^x}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$, by noticing $\ln(1+x) =$

$$\int \frac{1}{1+x} dx = \int \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}, x=1$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$$

convergent

Q7] noticing that $ne^{-nx} = -(e^{-nx})'$, now

$-\sum_{n=1}^{\infty} (e^{-nx})$ is geometric series converges $\forall x > 0$

and its summation is $-\left(\frac{1}{1-e^{-x}}\right) + 1$

now differentiating the series we get $\sum_{n=1}^{\infty} n e^{-nx}$

$$= \left(-\left(\frac{1}{1-e^{-x}}\right) + 1\right)' = \frac{e^{-x}}{(1-e^{-x})^2}$$ $\square$

Q8] $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Q9] $f(x) = \frac{x^2}{4-x^2} = \frac{x^2}{4}\left[\frac{1}{1-(\frac{x}{2})^2}\right] =$

$\frac{x^2}{4} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^{2n}$ converges for $\left(\frac{x}{2}\right)^2 < 1$

$\Rightarrow x^2 < 4 \Rightarrow |x| < 2$

$\Rightarrow$ the series converges for $x \in (-2, 2)$

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Q 10] $f(x) = 5\left(\frac{1}{1+(3x^2)^2}\right) = 5\left(\frac{1}{1-[(-1)(3x^2)^2]}\right)$

$= 5\sum_{n=0}^{\infty}(-1)^n(3x^2)^{2n}$

conrges for $|3x^2|^2 < 1$

$\Rightarrow 9x^4 < 1 \Rightarrow x^4 < \frac{1}{9} \Rightarrow$

$x \in \left(-\frac{1}{\sqrt[4]{9}}, \frac{1}{\sqrt[4]{9}}\right) = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$S(x)$ is uniformely convergent for neighborhood of $\frac{1}{2}$ $N_g(\frac{1}{2}) \subset \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\Rightarrow \lim_{x\to\frac{1}{2}} S(v) = 5\sum_{n=0}^{\infty}\lim_{x\to\frac{1}{2}}(-1)^n(3x^2)^{2n}$

$= 5\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{4}\right)^{2n} = 5\sum_{n=0}^{\infty}\left(-\frac{9}{16}\right)^n$

$= \frac{1}{1+\frac{9}{16}} = 5\,\frac{16}{25} = \frac{16}{5}$

~~no $\lim_{x\to\frac{1}{\sqrt{3}}-} S(x)$ doesn't exist~~ yes $\lim_{x\to\frac{1}{\sqrt{3}}-}$ $S(x)$ Exists

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$\in B_Y$, now Let $\{E_i\}_{i\in I}$ be countable collection

of sets in $B_Y \Rightarrow E_i = Y \cap A_i$, where $A_i \in B$,

now $\bigcup_{i\in I} E_i = \bigcup_{i\in I} Y \cap A_i = Y \cap (\bigcup_{i\in I} A_i)$, but

since B is σ-Algebra $\Rightarrow \bigcup_{i\in I} A_i \in B \Rightarrow Y \cap (\bigcup_{i\in I} A_i)$
$\in B_Y$. $\forall$ $Y \cap E \in B_Y$;

$(Y \cap E)^c$ (note we want the complement in Y)

$\hookrightarrow = Y \cap E^c$, E^c In X $\Rightarrow$ since $E^c \in B$
$\Rightarrow B_Y$ is σ- Algebra $\square$

Q2] if E is compact $\Rightarrow$ it is closed and bounded
$\Rightarrow$ since E is closed it is measurable and

since E is bounded $E \subseteq [a,b]$ $\Rightarrow$
by monotonicity $\Rightarrow M(E) \leq M([a,b]) = b-a$

for some $a, b \in \mathbb{R}$ $\square$

Q3] the limits $A = \lim_{x\to-\infty} f(x)$ $\wedge$ $B = \lim_{x\to+\infty} f(x)$ Exist in $\overline{\mathbb{R}}$

since f is Increasing the set $f^{-1}((-\infty, a])$ is

Interval for every $a \in \mathbb{R}$ (because if $x \leq y$

then $f(x) \leq a \wedge f(y) \leq a) \Rightarrow \forall$ $x < t < y$

~~f(x)≤~~ $f(t) \leq f(y) \leq a$, so f is measurable $\square$

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$f(x) \leq f(t) \leq f(y) \leq a$, so f is measurable □

Q4] take $f(x) = \frac{1}{\sqrt{x}} = g(x)$ over $(0,1]$

f, g are Lebesgue Integrable, but $fg = \frac{1}{x}$ is not

Q5] $f(x)$ is monotone bounded therefore measurable Lebesgue Integrable, and since $f(x)$ is Riemann Integrab over $[0,1]$ $\int_0^1 f(x)\,dx = \frac{3}{4}$ => its Lebesgue Integration coincides with its Reimann => $\int_{[0,1]} f\,dm = \frac{3}{4}$

Q6] we have $\sin(x)$ is measurable bounded

$$\Rightarrow \quad -1\, m(Q) \leq \int_Q \sin(x)\,dm \leq m(Q)$$

but since Q is countable $m(Q) = 0$. Hence

$$\int_Q \sin(x)\,dm = 0 \qquad \square$$

Q7 i) each function is $\frac{nx}{1+n^2x^2}$ is Lebesgue Integrable

They're Riemann Integrable $\Rightarrow \int_{[0,1]} \frac{nx}{1+n^2x^2}\,dm$

$$= \int_{[0,1]} \frac{nx}{1+n^2x^2}\,dx = \frac{1}{2n} \ln(1+n^2x^2) \Big]_0^1 = \frac{1}{2n} \ln(1+n^2)$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2n} \ln(1+n^2) = 0 \qquad \square$$

Q7 u] $\lim_{n\to\infty} \int_{[0,1]} \frac{n \sin^2(nx)}{1+n^2\sqrt{x}} dm$, $f_n = \frac{n \sin^2(nx)}{1+n^2\sqrt{x}}$

$|f_n| \leq \frac{n}{1+n^2\sqrt{x}} \leq \frac{1}{1+\sqrt{x}}$ whic is Integrable on $[0,1]$

note that $f_n \to 0$ => by the dominate convergence thm

$\lim_{n\to\infty} \int_{[0,1]} \frac{n \sin^2(nx)}{1+n^2\sqrt{x}} dm = \int_{[0,1]} \lim_{n\to\infty} \frac{n \sin^2(nx)}{1+n^2\sqrt{x}} dm = 0$ □

Q7 2.] $\lim_{n\to\infty} \int_{\mathbb{R}} \sin\left(\frac{x}{n}\right) f(x)\, dm$

since f is Lebesgue int => $|f(x)|$ is => $|\sin(\frac{x}{n}) f(x)|$

$\leq |f|$ => by dominated convergne theorem

$\int_{\mathbb{R}} \lim_{n\to\infty} \sin\left(\frac{x}{n}\right) f(x)\, dm = 0$ □

Q8] $f(x) = \frac{x^2 \ln(x)}{(1+x^2)^2}$, note $\lim_{x\to 0} f(x) = 0$, by lohpital

the function is bonded cont on $(0,1]$ => it is riemann Integrable => it is Lebesgue Integrable

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9(f) solved in 7) , 9(u) the same as 7(u)

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9 s) ~~Note that~~ $\cancel{(1+\frac{x}{n})^{-n} \to e^{-x}}$, ~~also~~

$\cancel{(1+\frac{x}{n})^{-n} \leq e^{-x}}$, ~~now~~ $\cancel{f_n = (1+\frac{x}{n})^{-n} \sin(\frac{x}{n})}$

$\cancel{|f_n| \leq e^{-x}}$ ~~which is Integrable on~~ $\cancel{[0,\infty)}$

=> ~~by dominate conrge theorem we get~~

$\cancel{\int_0^\infty \lim_{n\to\infty} (1+\frac{x}{n})^{-n} \sin(\frac{x}{n}) dx = \int_0^\infty e^{-x}\, 0\, dx = 0}$ □

Q9] 2.) for $n \geq 3$ $\int_0^\infty (1+\frac{x}{n})^{-n} \sin(\frac{x}{n})\, dx = n\int_0^\infty \frac{\sin(t)}{(1+t)^n}\, dt$

by parts $= \frac{n}{n-1} \int_0^\infty \frac{\cos(t)}{(1+t)^{n-1}}\, dt$

Since $\left|\frac{\cos(t)}{(1+t)^{n-1}}\right| \leq \frac{1}{(1+t)^2}$ which is Integrable, then

$\lim_{n\to\infty} \int_0^\infty (1+\frac{x}{n})^{-n} \sin(\frac{x}{n}) = 0$, by dominate convergence □

9 b] $\lim_{n\to\infty} \int_{[a,b]} \sin^n(x)\, dx$, now $\sin^n(x)$ is integrable

also $|\sin^n(x)| \leq 1$ which is Integrable on any bounded Interval

, also $\sin^n(x) \to 0$ almost everywhere

So by dominated convergence theorem

$\int_{[a,b]} \lim_{n\to\infty} \sin^n(x)\, dx = 0$ □

9 c] since $f(x) \in R[0,1] \Rightarrow |f(x)| \in R[0,1]$

$\Rightarrow$ for $x \in [0,1]$ $|f(x)\, x^n| \leq |f(x)|$ which is Integrable, note $x^n \to 0$ almost everywhere (actually everywhere except when $x=1$ $\Rightarrow$ by dominate convergence theorem $\int_0^1 \lim_{n\to\infty} x^n f(x)\, dx = 0$ □

Q 10] we'll proof that f is bounded on $I=[a,b]$
if $f \in R(a,b)$, then $f \in \mathcal{L}^1(I)$ and

$$\int_{[a,b]} f\,dm = \int_a^b f(x)$$

let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of I
define φ_p and ψ_p on I as follows

$$\varphi_p = \sum_{i=0}^{n-1} m_i \chi_{[x_i, x_{i+1})}$$

$$\psi_p = \sum_{i=0}^{n-1} M_i \chi_{[x_i, x_{i+1})}$$

$$m_i = \inf\{f(x) : x_i \le x \le x_{i+1}\}$$
$$M_i = \sup\{f(x) : x_i \le x \le x_{i+1}\}$$

observe $\varphi_p \le f \le \psi_p$, $\varphi_p, \psi_p \in S(I)$
from Riemann definition of Integral

$$\int_I \varphi_p\,dm = L(f, P)$$

$$\int_I \psi\,dm = U(f, P)$$

also $Q \supseteq P \Rightarrow \varphi_Q \ge \varphi_p$, $\psi_Q \le \psi_p$
choose sequence of partition (P_n) such that

$$P_{n+1} \supseteq P_n \qquad \|P_n\| \to 0$$

we conclude (ψ_n) is decreasing and φ_n is Increasing
their limits $\varphi(x) = \lim_{n\to} \varphi_n(x)$, $\psi(x) = \lim_{n\to\infty} \psi_n(x)$

are measurable and $\varphi \le f \le \psi$, f is bounded $\Rightarrow$
$\psi_n - \varphi_n$ is bounded and bounded convergence theorem $\Rightarrow$

$$\int_I (\psi - \varphi)\,dm = \lim_{n\to\infty} \int_I (\psi_n - \varphi_n)\,dm$$

$$= \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)]$$
$$= U(f) - L(f) = 0$$

hence $\varphi = \psi$ which means f is measurable

and $\int_I f\,dm = \int_I \varphi\,dm = \lim_{n\to\infty} \int \varphi_n\,dm = \lim_{n\to\infty} L(f, P_n)$

$$= L(f) = \int_a^b f(x)\,dx$$

by this f is Continuus => it is riemann Integrable

=> Lebesgue Integrable $\square$