PHYSICS 507 - Spring 2021

## $4^{\text {th }}$ HOMEWORK-Solutions

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Hand in: Wednesday 10 ${ }^{\text {th }}$ of March 2021

1. Two point charges, $q_{1}=3 q$ and $q_{2}=-q$ are located at points $(0,0,2 a)$ and ( $0,0,-a$ ) respectively. Find (i) the monopole moment, (ii) the dipole moment, and (iii) the approximate potential (in spherical coordinates) at large $r$ (include both monopole and dipole contributions). (Hint: read carefully pages 149 and 150 of our textbook)

## Solution:

(i) the monopole term is the total charge so $Q=2 q$.
(ii) The dipole moment is given by:

$$
\mathbf{p}=\sum_{i=1}^{2} q_{i} \mathbf{r}_{i}^{\prime}=3 q(0,0,2 a)-q(0,0,-a)=6 q a \hat{\mathbf{k}}+q a \hat{\mathbf{k}}=7 q a \hat{\mathbf{k}}
$$

(iii) The approximate potential at large distances is given by the potential of the monopole term and the potential due to the dipole term:
$V \approx \frac{1}{4 \pi \varepsilon_{0}}\left(\frac{Q}{r}+\frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^{2}}\right)=\frac{1}{4 \pi \varepsilon_{0}}\left[\frac{2 q}{r}+\frac{(7 q a \hat{\mathbf{k}}) \cdot \hat{\mathbf{r}}}{r^{2}}\right]=\frac{1}{4 \pi \varepsilon_{0}}\left[\frac{2 q}{r}+\frac{7 q a \cos \theta}{r^{2}}\right]=$ $\frac{q}{4 \pi \varepsilon_{0} r}\left[2+\frac{7 a \cos \theta}{r}\right]$
2. Find the surface charge density on the conducting plane, which is at the plane $x y$.


## Solution:

If we follow the image method the problem is solved by placing a charge $-q$ at $z=-3 d$ and a charge $+2 q$ at $z=-d$. In this case the total electric potential is given by:

$$
\begin{aligned}
& V(x, y, z)=\frac{q}{4 \pi \varepsilon_{0}}\left[\frac{1}{\sqrt{x^{2}+y^{2}+(z-3 d)^{2}}}-\frac{2}{\sqrt{x^{2}+y^{2}+(z-d)^{2}}}\right. \\
& \left.-\frac{1}{\sqrt{x^{2}+y^{2}+(z+3 d)^{2}}}+\frac{2}{\sqrt{x^{2}+y^{2}+(z+d)^{2}}}\right]
\end{aligned}
$$

The surface charge density is given by:

$$
\begin{aligned}
& \sigma=-\left.\varepsilon_{0} \frac{\partial V}{\partial z}\right|_{z=0} \\
& \sigma=-\frac{q}{4 \pi}\left[\frac{3 d}{\left(x^{2}+y^{2}+9 d^{2}\right)^{3 / 2}}-\frac{2 d}{\left(x^{2}+y^{2}+d^{2}\right)^{3 / 2}}\right. \\
& \left.+\frac{3 d}{\left(x^{2}+y^{2}+9 d^{2}\right)^{3 / 2}}-\frac{2}{\left(x^{2}+y^{2}+d^{2}\right)^{3 / 2}}\right]
\end{aligned}
$$

$$
\sigma=-\frac{q d}{2 \pi}\left[\frac{3}{\left(x^{2}+y^{2}+9 d^{2}\right)^{3 / 2}}-\frac{2}{\left(x_{\mathrm{\circ}}^{2}+y^{2}+d^{2}\right)^{3 / 2}}\right]
$$

3. A colloid consists of a suspension in water of small charged particles which, though microscopic, from an atomic point of view are still very large. If the colloidal particles were not charged, they would tend to coagulate into large lumps; but because of their charge, they repel each other and remain in suspension. Now if there is also some salt dissolved in the water, it will be dissociated into positive and negative ions. (Such a solution of ions is called an electrolyte.) The negative ions are attracted to the colloid particles (assuming their charge is positive) and the positive ions are repelled. The potential $V$ in a colloid arises in part from the same charges. The resulting effects influence in an important way the behavior of colloids.

Assume that the colloid extends in one dimension (say along $x$ ) and it has a temperature $T$. The charge density in the colloidal is given by $\rho=|e| n_{0}\left(e^{-\mid e V V / k T}-e^{|e| V / k T}\right)$, where $|e|$ is the charge of electron, $k$ is the Boltzmann constant and $n_{0}$ is the initial of the colloidal in the absence of the electric field. (i) Find the potential as a function of the position $x$ in the colloidal in the limit where it is small compared to $k T$. (ii) plot a qualitative graph of the potential as a function of $x$.

## Solution:

Since the problem is one dimensional we know that:
$\frac{d^{2} V}{d x^{2}}=-\frac{\rho}{\varepsilon_{0}} \Rightarrow \frac{d^{2} V}{d x^{2}}=-\frac{|e| n_{0}}{\varepsilon_{0}}\left(e^{-e \mid V / k T}-e^{|e| V T}\right)$
but if the potential is small compared to $k T$ then
$e^{ \pm e \mid V / k T}=1 \pm \frac{|e| V}{k T}$
thus we get
$\frac{d^{2} V}{d x^{2}}=-\frac{|e| n_{0}}{\varepsilon_{0}}\left(1-\frac{|e| V}{k T}-1-\frac{|e| V}{k T}\right) \Rightarrow \frac{d^{2} V}{d x^{2}}=\frac{2|e|^{2} n_{0}}{\varepsilon_{0} k T} V$
The solution of this differential equation is:
$V(x)=A e^{-x / D}+B e^{+x / D}, \quad D^{2}=\frac{2|e|^{2} n_{0}}{\varepsilon_{0} k T}$
But since at infinity the potential must be zero then $B$ should be zero. Thus the solution is:
$V(x)=A e^{-x / D}$, with $D^{2}=\frac{2|e|^{2} n_{0}}{\varepsilon_{0} k T}$.
The plot for potential is the following:

4. Find the general solution of the Laplace equation in spherical coordinates when the potential has the form $V_{0}(\theta)$ on the sphere. Study the special case where $V_{0}(\theta)=k \sin ^{2} \theta / 2$. We are interested in the potential outside the sphere.

## Solution:

The solution of such a problem has the following form:

$$
\begin{aligned}
& V_{\text {in }}(r, \theta)=\sum_{\ell=0}^{\infty} A_{\ell} r^{\ell} P_{\ell}(\cos \theta), \quad(r \leq R) \\
& V_{\text {out }}(r, \theta)=\sum_{\ell=0}^{\infty} \frac{B_{\ell}}{{ }^{\ell+1}} P_{\ell}(\cos \theta), \quad(r>R)
\end{aligned}
$$

The two solutions must agree at $r=R$. Thus

$$
\begin{aligned}
& \sum_{\ell=0}^{\infty} A_{\ell} R^{\ell} P_{\ell}(\cos \theta)=\sum_{\ell=0}^{\infty} \frac{B_{\ell}}{R^{\ell+1}} P_{\ell}(\cos \theta) \Rightarrow \sum_{\ell=0}^{\infty} A_{\ell} R^{\ell} P_{\ell}(\cos \theta)-\sum_{\ell=0}^{\infty} \frac{B_{\ell}}{R^{\ell+1}} P_{\ell}(\cos \theta)=0 \\
& \sum_{\ell=0}^{\infty}\left(A_{\ell} R^{\ell}-\frac{B_{\ell}}{R^{\ell+1}}\right) P_{\ell}(\cos \theta)=0 \Rightarrow A_{\ell} R^{\ell}-\frac{B_{\ell}}{R^{\ell+1}}=0 \Rightarrow \\
& B_{\ell}=A_{\ell} R^{2 \ell+1}
\end{aligned}
$$

The discontinuity of the electric field implies that:

$$
\begin{aligned}
& \left.\left(\frac{\partial V_{e x}}{\partial r}-\frac{\partial V_{i n}}{\partial r}\right)\right|_{r=R}=-\frac{1}{\varepsilon_{0}} \sigma_{0}(\theta) \Rightarrow \\
& -\sum_{\ell=0}^{\infty} \ell A_{\ell} r^{\ell-1} P_{\ell}(\cos \theta)-\left.\sum_{\ell=0}^{\infty}(\ell+1) \frac{B_{\ell}}{r^{\ell+2}} P_{\ell}(\cos \theta)\right|_{r=R}=-\frac{1}{\varepsilon_{0}} \sigma_{0}(\theta) \Rightarrow \\
& -\sum_{\ell=0}^{\infty} \ell A_{\ell} R^{\ell-1} P_{\ell}(\cos \theta)-\sum_{\ell=0}^{\infty}(\ell+1) \frac{B_{\ell}}{R^{\ell+2}} P_{\ell}(\cos \theta)=-\frac{1}{\varepsilon_{0}} \sigma_{0}(\theta) \Rightarrow \\
& \sum_{\ell=0}^{\infty}\left(\ell A_{\ell} R^{\ell-1}+(\ell+1) \frac{B_{\ell}}{R^{\ell+2}}\right) P_{\ell}(\cos \theta)=\frac{1}{\varepsilon_{0}} \sigma_{0}(\theta) \Rightarrow A_{\ell} R^{2 \ell+1} \\
& \sum_{\ell=0}^{\infty}\left(\ell A_{\ell} R^{\ell-1}+(\ell+1) \frac{A_{\ell} R^{2 \ell+1}}{R^{\ell+2}}\right) P_{\ell}(\cos \theta)=\frac{1}{\varepsilon_{0}} \sigma_{0}(\theta) \Rightarrow \\
& \sum_{\ell=0}^{\infty} A_{\ell} R^{\ell-1}(2 \ell+1) P_{\ell}(\cos \theta)=\frac{1}{\varepsilon_{0}} \sigma_{0}(\theta)
\end{aligned}
$$

From this expression we get:

$$
\begin{aligned}
& A_{\ell} R^{(-1}(2 \ell+1)=\frac{(2 \ell+1)}{2 \varepsilon_{0}} \int_{\theta=0}^{\pi} \sigma_{0}(\theta) P_{\ell}(\cos \theta) \sin \theta d \theta \Rightarrow \\
& A_{\ell}=\frac{1}{2 \varepsilon_{0} R^{\ell-1}} \int_{\theta=0}^{\pi} \sigma_{0}(\theta) P_{\ell}(\cos \theta) \sin \theta d \theta
\end{aligned}
$$

Now for $V_{0}(\theta)=k \sin ^{2} \theta / 2$ we have that

$$
\begin{aligned}
& V_{0}(\theta)=k \sin ^{2} \theta / 2 \Rightarrow V_{0}(\theta)=\frac{k}{2}(1-\cos \theta) \Rightarrow \\
& V_{0}(\theta)=\frac{k}{2}-\frac{k}{2} \cos \theta \Rightarrow V_{0}(\theta)=\frac{k}{2} P_{0}(\cos \theta)-\frac{k}{2} P_{1}(\cos \theta)
\end{aligned}
$$

Thus

$$
\begin{aligned}
& A_{0} R^{0}=k / 2 \Rightarrow A_{0}=k / 2 \\
& A_{1} R^{1}=-k / 2 \Rightarrow A_{1}=-k / 2 R \\
& A_{2}=A_{3}=\ldots A_{n}=\ldots=0
\end{aligned}
$$

Thus for the solution outside the sphere we have:

$$
B_{\ell}=A_{\ell} R^{2 \ell+1} \Rightarrow\left\{\begin{array} { c } 
{ B _ { 0 } = A _ { 0 } R ^ { 2 \cdot 0 + 1 } } \\
{ B _ { 1 } = A _ { 1 } R ^ { 2 \cdot 1 + 1 } } \\
{ B _ { 2 } = B _ { 3 } = \ldots = 0 }
\end{array} \Rightarrow \left\{\begin{array}{c}
B_{0}=A_{0} R \\
B_{1}=A_{1} R^{3} \\
B_{2}=B_{3}=\ldots=0
\end{array} \Rightarrow\right.\right.
$$

$$
\left\{\begin{array}{c}
B_{0}=k R / 2 \\
B_{1}=-k R^{3} / 2 R=-k R^{2} / 2 \\
B_{2}=B_{3}=\ldots=0
\end{array}\right.
$$

