The Fubini Theorem

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Generalities on Product Spaces

Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two measure spaces. We intend to construct the product measure on a suitable σ-algebra contained in the power set of the Cartesian product $X = X_1 \times X_2$.
By a rectangular set R in X we mean any set of the form $R = A \times B$ where $A \in \mathscr{A}_1$ and $B \in \mathscr{A}_2$. We denote by $\mathcal{R}$ the set of all rectangles in X. The product $\sigma-$algebra of $\mathscr{A}_1$ and $\mathscr{A}_2$ on X is the $\sigma-$algebra generated by $\mathcal{R}$ and will be denoted by $\mathscr{A}_1 \otimes \mathscr{A}_2$.
$\mathscr{A}_1 \otimes \mathscr{A}_2$ is the smallest $\sigma-$algebra such that the projections $\pi_1 : X_1 \times X_2 \longrightarrow X_1$ and $\pi_2 : X_1 \times X_2 \longrightarrow X_2$ are measurable. π_1 and π_2 are defined by: $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$
In the same way if $(X_j, \mathscr{A}_j)$, $j = 1, \ldots, n$ are n measurable spaces, we define the $\sigma-$algebra $\otimes_{j=1}^n \mathscr{A}_j$ on the space $X = \prod_{j=1}^n X_j$, and for the remainder of this course, we provide the product space X with this $\sigma-$algebra.

Proposition

Let X, Y be two separable[a] metric spaces. Then

$$\mathscr{B}_{X\times Y} = \mathscr{B}_X \otimes \mathscr{B}_Y,$$

where $\mathscr{B}_X$ is the Borel $\sigma-$algebra on X.

[a] separable means that there exists a countable dense subset

Proof

The inclusion $\mathscr{B}_X \otimes \mathscr{B}_Y \subset \mathscr{B}_{X\times Y}$ holds whenever X, Y to be separable since π_1 and π_2 are continuous and then measurable with respect to the $\sigma-$algebra $\mathscr{B}_{X\times Y}$.
Let $(x_n)_n$ and $(y_n)_n$ two dense sequence respectively in X and Y. We consider the set of balls of center $(x_n)_n$ and radius rational. This family is countable. We denote this family $(U_k)_k$. Then any open subset of X is a finite or countable union of $(U_k)_k$. We consider in the same way we consider a sequence $(V_k)_k$ of open subsets in Y. If $\mathcal{O}$ is an open subset of $X \times Y$ and all $(x, y) \in \mathcal{O}$ there exists an open subset of X which contains x and an open subset V of Y which contains y and $U \times V \subset \mathcal{O}$. Then any open subset of $X \times Y$ is a finite or countable union of the open subsets in $\{U_n \times V_m;\ n, m \in \mathbb{N}\}$. Then any open subset of $X \times Y$ is in $\mathscr{B}_X \otimes \mathscr{B}_Y$ and then $\mathscr{B}_{X\times Y} \subset \mathscr{B}_X \otimes \mathscr{B}_Y$.

□

Definition

If $E \subset X_1 \times X_2$; we define the $x-$section of E by

$$E_x = \{y \in X_2;\ (x, y) \in E\},\ \ y \in X_2$$

and the $y-$section by

$$E^y = \{x \in X_1;\ (x, y) \in E\},\ \ y \in X_2.$$

Similarly, if $f : X \longrightarrow \bar{\mathbb{R}}$, then the x and y-sections of f are the mappings $f_x : X_2 \longrightarrow \bar{\mathbb{R}}$ and $f^y : X_1 \longrightarrow \bar{\mathbb{R}}$ defined by $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.

Proposition

If $E \in \mathscr{A}_1 \otimes \mathscr{A}_2$, then the sections E_x and E^y respectively belong to $\mathscr{A}_1$ and $\mathscr{A}_2$ for each $x \in X_1$, and to $\mathscr{A}_1$ for each $y \in X_2$. If f is measurable with respect to the product algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$, then its sections f_x and f^y are measurable with respect to the factors $\mathscr{A}_2$ and $\mathscr{A}_1$ respectively.

Proof

Let $\mathscr{B}$ be the collection of all subsets $E \subset X$ such that $E_x \in \mathscr{A}_2$ for all $x \in X_1$ and $E^y \in \mathscr{A}_1$ for all $y \in X_2$. Then $(A \times B)_x = B$ if $x \in A$ and $(A \times B)_x = \emptyset$ if $x \in A^c$. Similarly for the section $(A \times B)^y$. Hence $\mathscr{B}$ contains all rectangles. Moreover, $\mathscr{B}$ is a $\sigma-$ algebra, since $\left(\bigcup_{j=1}^{+\infty} E_j\right)_x = \bigcup_{j=1}^{+\infty}(E_j)_x$ and $(E_x)^c = (E^c)_x$, and similarly for $y-$sections. Therefore $\mathscr{B} = \mathscr{A}_1 \otimes \mathscr{A}_2$.

The measurability of f_x and f^y follows from the first statement and the relationships

$$(f_x)^{-1}(B) = (f^{-1}(B))_x; (f^y)^{-1}(B) = (f^{-1}(B))^y.$$

□

Lemma

Let $\mathscr{C}$ be the family of elementary sets for the product measure space

$$\mathscr{C} = \{E = \bigcup_{j=1}^{n} R_j;\ R_j = A_j \times B_j, A_j \in \mathscr{A}_1,\ B_j \in \mathscr{A}_2\}, \qquad (1)$$

where R_j are disjoint rectangles and n is an arbitrary natural number. Then
i) $\mathscr{C}$ is an algebra,
ii) $\sigma(\mathscr{C}) = \mathscr{A}_1 \otimes \mathscr{A}_2$.

Proof

i) $\mathscr{C}$ is closed under intersection and $\mathscr{C}$ is closed under complementarity.

$X_1 \times X_2 \in \mathscr{C}$ is trivial.

Let $E, F \in \mathscr{C}$, write $E = \bigcup_{j=1}^{n} A_j \times B_j$ and $F = \bigcup_{j=1}^{m} C_j \times D_j$, where $A_1, \ldots, A_n$, $C_1, \ldots, C_m$ in $\mathscr{A}_1$, $B_1, \ldots, B_n$, $D_1, \ldots, D_m$ in $\mathscr{A}_2$ and both unions are disjoint. Then

$$
\begin{aligned}
E \cap F &= \bigcup_{j=1}^{n} A_j \times B_j \cap \bigcup_{k=1}^{m} C_k \times D_k = \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} A_j \times B_j \cap C_k \times D_k \\
&= \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} (A_j \cap C_k) \times (B_j \cap D_k)
\end{aligned}
$$

The set $E \cap F$ is clearly a finite union of $\mathscr{A}_1 \times \mathscr{A}_2-$sets. To see that the union is disjoint, pick distinct $(j, k), (j', k') \in \{1, \ldots, n\} \times$

Construction of the Product Measure

Theorem

Let $(X_1, \mathscr{A}_1, \mu_1)$ and $(X_2, \mathscr{A}_2, \mu_2)$ be two $\sigma-$finite measure spaces.
a) There exists a unique measure μ on $(X_1 \times X_2, \mathscr{A}_1 \otimes \mathscr{A}_2)$ such that

$$\mu(A \times B) = \mu_1(A)\mu_2(B). \tag{2}$$

This measure is $\sigma-$finite and denoted $\mu_1 \otimes \mu_2$.

$$(E) = \sum_{j=1}^{n} \mu_1(A_j)\mu_2(B_j),$$

for each elementary set $E \in \mathscr{C}$ as defined by the equation (1).
b) For all $E \in \mathscr{A}_1 \otimes \mathscr{A}_2$

$$\mu_1 \otimes \mu_2(E) = \int \mu_2(E_x)d\mu(x) = \int \mu_2(E^y)d\nu(y). \tag{3}$$

Proof

Uniqueness

There exists an increasing sequence $(A_n)_n$ of X_1 and an increasing sequence $(B_n)_n$ of X_2 such that $X_1 = \cup_{n=1}^{+\infty} A_n$, $X_2 = \cup_{n=1}^{+\infty} B_n$, $\mu_1(A_n) < +\infty$ and $\mu_2(B_n) < +\infty$. Then $X_1 \times X_2 = \cup_{n=1}^{+\infty} A_n \times B_n$. If μ and ν are two measure which fulfills the equation (3), then

$$\mu(A_n \times B_n) = \nu(A_n \times B_n) < +\infty, \quad \forall n \in \mathbb{N}.$$

Since the class of measurable rectangles is closed under finite intersection and by Theorem (**??**) Chapter IV, $\mu = \nu$.

Existence For all $C \in \mathscr{A}_1 \otimes \mathscr{A}_2$, we set

$$\mu(C) = \int_{X_1} \mu_2(C_x) d\mu_1(x). \tag{4}$$

To prove the formula (5), we must prove firstly that $x \longmapsto \mu_2(C_x)$ is measurable.

Suppose that μ_2 is finite and define

$$\mathscr{A} = \{C \in \mathscr{A}_1 \otimes \mathscr{A}_2;\ x \longmapsto \mu_2(C_x) \text{ is measurable } \}.$$

$\mathscr{A}$ contains the measurable rectangles $C = A \times B$ since $\mu_2(C_x) = \chi_A(x)\mu_2(B)$. Moreover $\mathscr{A}$ is a monotone class: if $C \subset C'$, $\mu_2(C' \setminus C)_x = \mu_2(C'_x) - \mu_2(C_x)$ since μ_2 is finite, and if $(C_n)_n$ is an increasing sequence

$$\mu_2(\cup_{k=1}^{+\infty} C_n)_x = \lim_{n\to+\infty} \mu_2(C_n)_x.$$

By Theorem (**??**) Chapter IV, $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$.
In the general case where μ_2 is $\sigma-$finite, we take as above the sequence $(B_n)_n$ and define $\mu_{2,n}(B) = \mu_2(B \cap B_n)$. Then $\mu_2(C_x) = \lim_{n\to+\infty} \mu_{2,n}(C_x)$ which is measurable.

To prove that μ is a measure on $\mathscr{A}_1 \otimes \mathscr{A}_2$, let $(C_n)_n$ a sequence of disjoint measurable subsets in $\mathscr{A}_1 \otimes \mathscr{A}_2$, then $((C_n)_x)_n$ are disjoint for all $x \in X_1$ and

$$\begin{aligned}\mu(\cup_{n=1}^{+\infty} C_n) &= \int_{X_1} \mu_2(\cup_{n=1}^{+\infty} (C_n)_x) d\mu_1(x) \\ &= \int_{X_1} \sum_{n=1}^{+\infty} \mu_2((C_n)_x) d\mu_1(x) \\ &= \sum_{n=1}^{+\infty} \int_{X_1} \mu_2((C_n)_x) d\mu_1(x) \\ &= \sum_{n=1}^{+\infty} \mu(C_n).\end{aligned}$$

Moreover $\mu(A \times B) = \mu_1(A)\mu_2(B)$.
In the same way, if we define