King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(5.2)

Graph Terminology and Special Types of Graphs

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Basic Terminology

First, we give some terminology that describes the vertices and edges of undirected graphs.

DEFINITION 1 Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.

DEFINITION 2 The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) =$ _$v \in A$ $N(v)$.

DEFINITION 3 The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by deg(v).

EXAMPLE 1 What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

Solution: In G, deg(a) = 2, deg(b) = deg(c) = deg(f) = 4, deg(d) = 1, deg(e) = 3, and deg(g) = 0.

The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$.

In H, deg(a) = 4, deg(b) = deg(e) = 6, deg(c) = 1, and deg(d) = 5. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

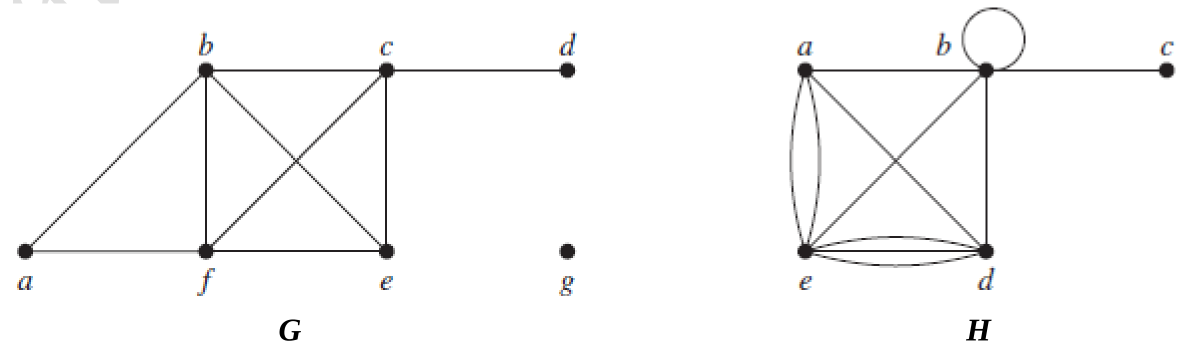


FIGURE 1 The Undirected Graphs *G* and *H*.

THEOREM 1

THE HANDSHAKING THEOREM Let $G = (V, E)$ be an undirected graph with m edges $m = |E|$. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

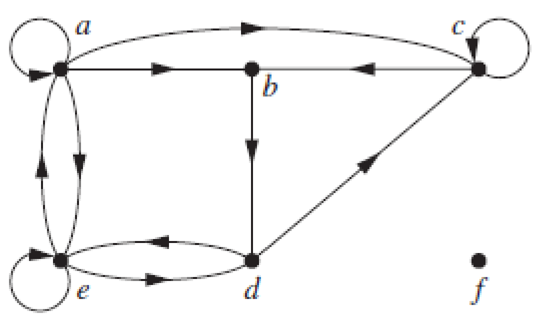
EXAMPLE 2 How many edges are there in a graph with 10 vertices each of degree six?
Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$.

THEOREM 2 An undirected graph has an even number of vertices of odd degree.

DEFINITION 4 When (u, v) is an edge of the graph G with directed edges, u is said to be *adjacent to* v and v is said to be *adjacent from* u. The vertex u is called the *initial vertex* of (u, v), and v is called the *terminal* or *end vertex* of (u, v). The initial vertex and terminal vertex of a loop are the same.

DEFINITION 5 In a graph with directed edges the *in-degree of a vertex v*, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of* v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

EXAMPLE 3 Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.



G

FIGURE 2 The Directed Graph *G*.

Solution: The in-degrees in G are $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$, and $\deg^-(f) = 0$. The out-degrees are $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, and $\deg^+(f) = 0$.

THEOREM 3 Let $G = (V, E)$ be a graph with directed edges. Then

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Some Special Simple Graphs

EXAMPLE 4 Complete Graphs A **complete graph on *n* vertices**, denoted by K_n, is a simple graph that contains exactly one edge between each pair of distinct vertices. The graphs K_n for $n = 1, 2, 3, 4, 5, 6$, are displayed in Figure 3. A simple graph for which there is at least one pair of distinct vertex not connected by an edge is called **noncomplete .**

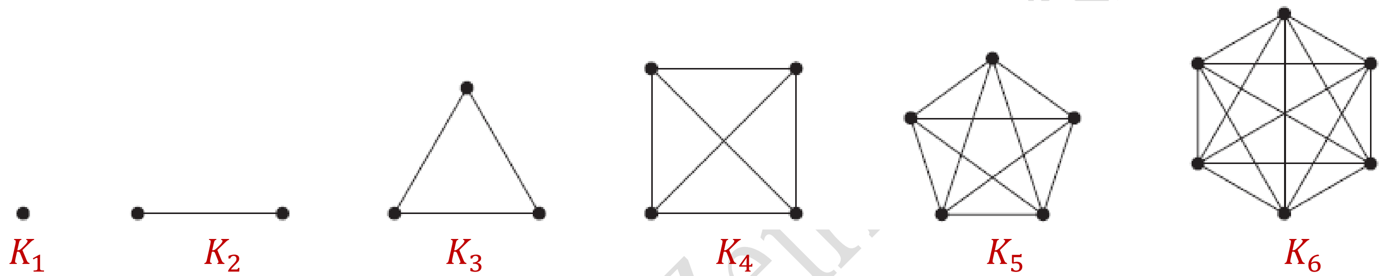


FIGURE 3 The Graphs K_n for $1 \leq n \leq 6$.

EXAMPLE 5 Cycles A **cycle** C_n , $n \geq 3$, consists of *n* vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4.

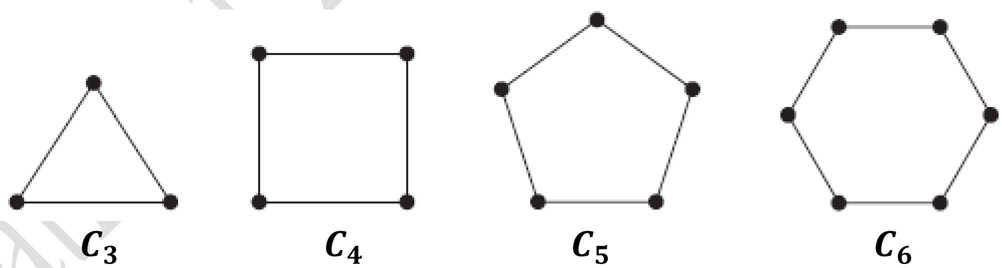


FIGURE 4 The Cycles C_3, C_4, C_5, and C_6 .

EXAMPLE 6 Wheels We obtain a **wheel** W_n when we add an additional vertex to a cycle C_n, for $n \geq 3$, and connect this new vertex to each of the n vertices in C_n, by new edges. The wheels W_3, W_4, W_5 and W_6 are displayed in Figure 5.

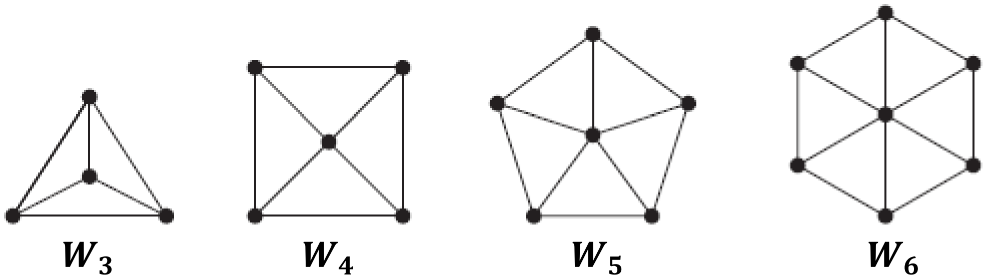


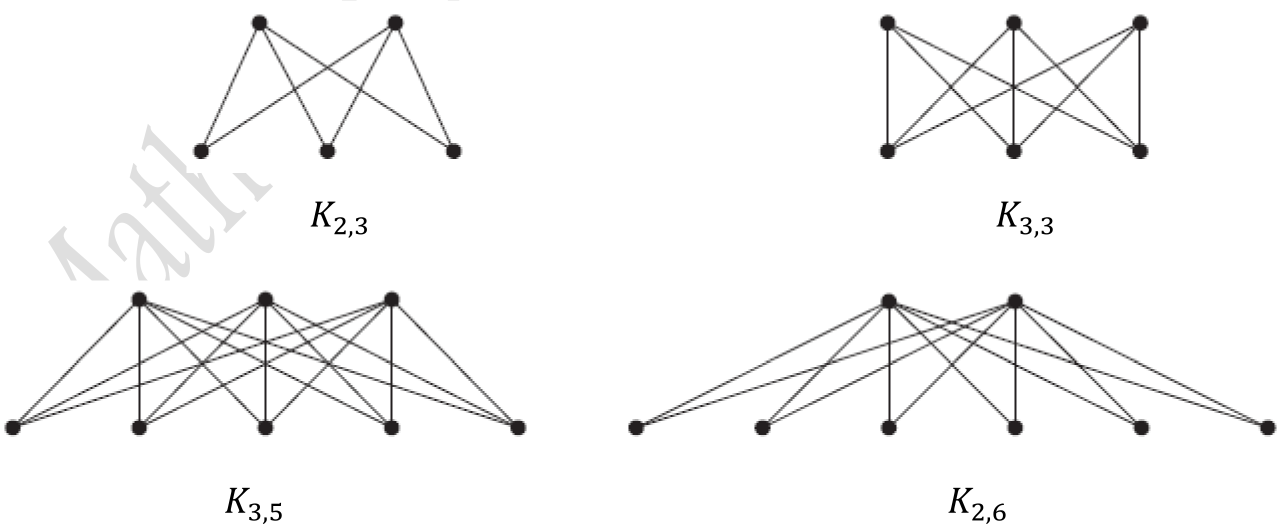
FIGURE 5 The Wheels W_3, W_4, W_5 and W_6

Bipartite Graphs

DEFINITION 6 A simple graph *G* is called ***bipartite*** if its vertex set *V* can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in *G* connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set *V* of *G*.

THEOREM 4 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

EXAMPLE 7 Complete Bipartite Graphs A **complete bipartite graph** $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of *m* and *n* vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.
The complete bipartite graphs $K_{2,3}$, $K_{3,3}$, $K_{3,5}$, and $K_{2,6}$ are displayed in Figure 9.



DEFINITION 7 A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.

DEFINITION 8 Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.

EXAMPLE 8 The graph G shown in Figure 15 is a subgraph of K_5. If we add the edge connecting c and e to G, we obtain the subgraph induced by $W = \{a, b, c, e\}$.

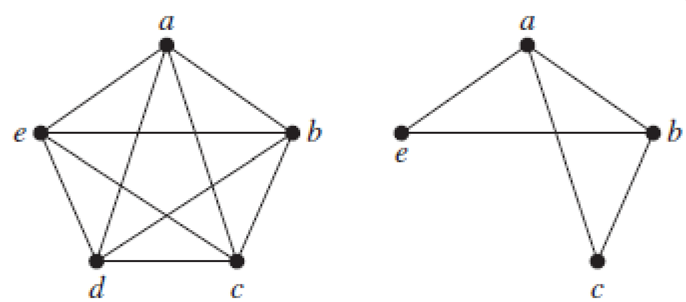


FIGURE 6 A Subgraph of K_5.

DEFINITION 9 A simple graph is called **regular** if every vertex of this graph has the same degree. A regular graph is called ***r*-regular** if every vertex in this graph has degree ***r***.

THEOREM 5 If $K_n(V, E)$ then $|E| = \frac{n(n-1)}{2}$

THEOREM 6 If $K_{m,n}(V_1 \cup V_2, E)$ where $|V_1| = m$, $|V_2| = n \Rightarrow |E| = mn$

Adjacency Matrices

Suppose that G = (V ,E) is a simple graph where |V| = n. Suppose that the vertices of G are listed arbitrarily as v1, v2, . . . , vn. The adjacency matrix A (or AG) of G, with respect to this listing of the vertices, is the n x n zero–one matrix with 1 as its (i, j)th entry when vi and vj are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is A = [aij], then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 9 Use an adjacency matrix to represent the graph shown in Figure 7.

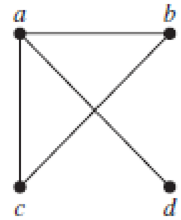


FIGURE 7
Simple Graph.

Solution: We order the vertices as *a, b, c, d*. The matrix representing this graph is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

EXAMPLE 10 Draw a graph with the adjacency matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Solution: A graph with this adjacency matrix is shown in Figure 8.

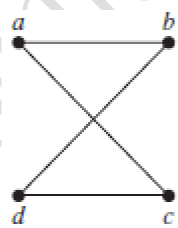


FIGURE 8
A Graph with the Given Adjacency Matrix.

EXAMPLE 11 Use an adjacency matrix to represent the pseudograph shown in Figure 9.

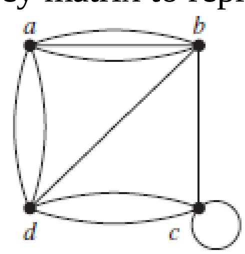


FIGURE 9
A Pseudograph.

Solution: The adjacency matrix using the ordering of vertices *a*, *b*, *c*, *d* is

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

Another common way to represent graphs is to use **incidence matrices**. Let $G = (V, E)$ be an undirected graph. Suppose that $v1, v2, \ldots, vn$ are the vertices and $e1, e2, \ldots, em$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix **M** = $[mij]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 12 Represent the graph shown in Figure 10 with an incidence matrix.

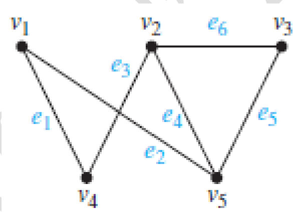


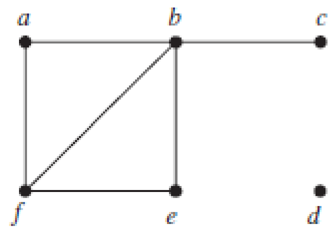
FIGURE 10 An Undirected Graph.

Solution: The incidence matrix is

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

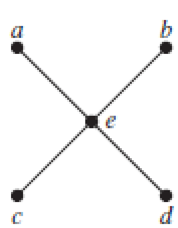
Exercises

1. Find the number of vertices, the number of edges, and the degree of each vertex in the given undirected graph. Identify all isolated and pendant vertices. Find the sum of the degrees of the vertices of the graph and verify that it equals twice the number of edges in the graph.



2. In Exercises (A) – (M) determine whether the graph is bipartite. You may find it useful to apply Theorem 4 and answer the question by determining whether it is possible to assign either red or blue to each vertex so that no two adjacent vertices are assigned the same color.

(A)



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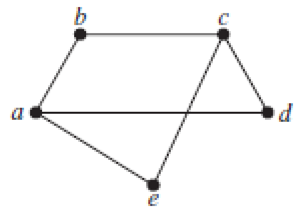
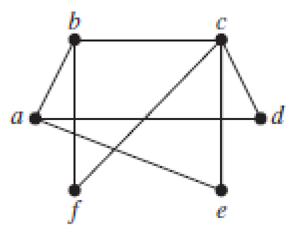
(B)

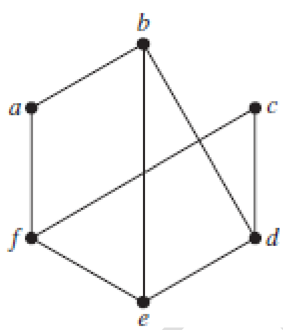


(C)

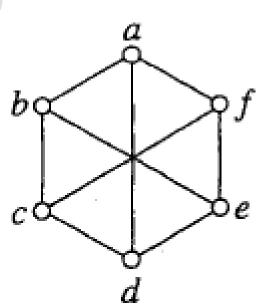


b and *f* two adjacent vertices are assigned the same sign. ∴ G is not a bipartite graph .

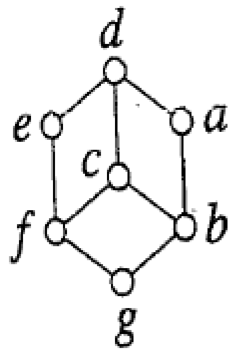
(D)



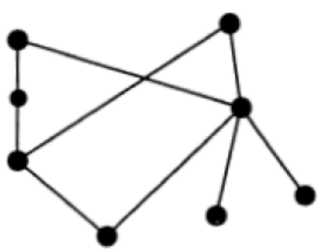
(E)



(F)

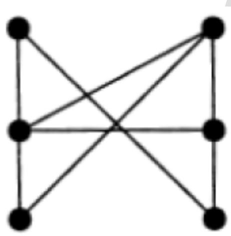


(G)

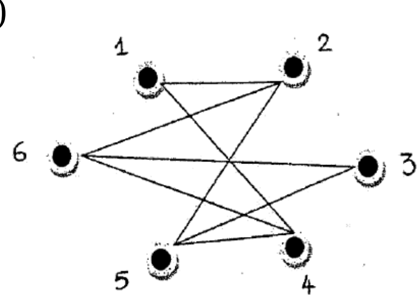


two adjacent vertices are assigned the same sign. ∴ G is not a bipartite graph .

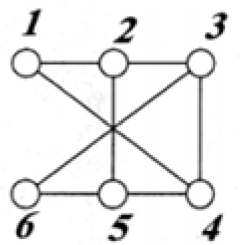
(H)



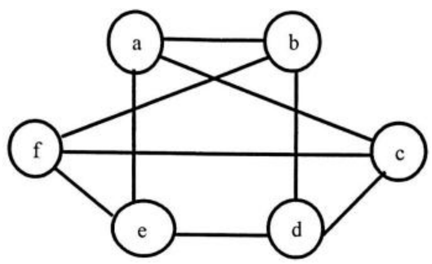
(K)



(L)



(M)



3. Let G be a graph have 6 edges and the given degree sequence $1, 3, x, x$, find the value of x ?

4. How many vertices does a regular graph of degree four with 10 edges have?

5. Can a simple graph exist with 15 vertices each of degree five?

6. How many vertices does a K_n graph with 10 edges have?

7. Can a bipartite graph exist with 6 vertices and 10 edges?

8. Show that $K_{m,n}$ is regular if and only if $m = n$?

9. If G is a simple regular graph of degree k with n vertices, show that k is even or n is even.

10. Let $K_{m,n}$ be a bipartite graph with 6 vertices and 8 edges, find the value of $,n$?

11. If $K_{3,n}$ have the same number of edges of K_n , find the value of n ?

12. Let G be a graph have 5 edges and the given degree sequence $2,2,2,2,x$. Decide whether G is a regular ?

13. How many vertices does a regular graph of degree 3 with 10 edges have?

14. How many vertices does a regular graph of degree four with 20 total vertices degrees , have?

15. Let G be a graph have 10 edges with two vertices of 4 degrees each, and the degree of each vertex is equal to 3. Find the number of vertices

16. Let G be a graph have 11 edges and the given degree sequence $n,n,n,n,2n,2n,3n$, find the value of n ?

17. If K_{m,m^2} have 42 vertices, find the number of edges ?

18. Can a simple regular graph of degree 3 exist with 9 edges? Draw the graph if it is exists ?

19. Let G be a graph with $3n$ vertices , where the degree of n vertices of each of them is 2 and the others $2n$ vertices is 1. Find n if G have 20 edges

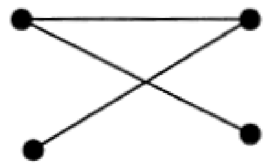
20. Let $K_{m,n}$ be a bipartite graph with 16 vertices and 64 edges, find the values of m , n ?

21. What is the degree sequence of K_n , where n is a positive integer? Explain your answer.

22. How many edges does a graph have if its degree sequence is 4 , 3 , 3 , 2 , 2 ? Draw such a graph.

23. How many edges does a graph have if its degree sequence is 5 , 2 , 2 , 2 , 2 , 1 ? Draw such a graph.

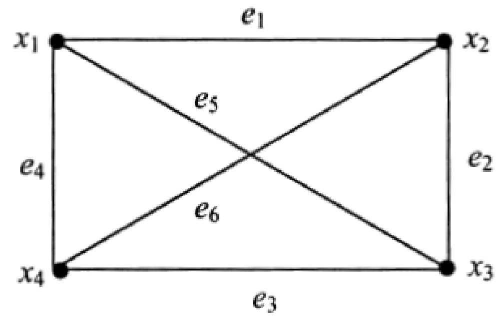
24. Use an adjacency matrix to represent the graph shown in Figure



25. Draw a graph with the adjacency matrix, decide whether it is a complete graph?

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

26. Represent the graph shown in the Figure with an incidence matrix.



27. Draw a graph with the incidence matrix, decide whether it is a regular graph?

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

28. Draw a graph with the adjacency matrix, decide whether it is a regular graph?

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$