KING SAUD UNIVERSITY
DEPARTMENT OF MATHEMATICS
FINAL EXAMINATION, SEM. I, 1427-1428
MATH 570: TOPOLOGY and CALCULUS in $\Re^n$
TIME: $3H$ FULL MARKS: 40

Question #1

(a) Let $X = I \times I (= [0,1] \times [0,1])$ be the unit square in the plane. Define $\sim$ to be the equivalence relation that identifies two pints on the vertical edges of X haveing same height. Thus for $(x,y),(u,v) \in X$:

$$(x,y) \sim (u,v) \Leftrightarrow (x,y) = (u,v) or (x = 0, u = 1, y = v) or (x = 1, u = o, y = v)$$

verify that the quotient space $X/\sim$ is homeomorphic to $S_1 \times I$.

(b) Define $f : \Re^n \longrightarrow \Re$ by the equation $f(x,y) = x + y^2$. Show that f is a quotient map.

(c) Prove or disprove that a quotient of a Hausdorff space is Hausdorff.

(d) Show that $p : X(= \Re^{n+1} \setminus \{0\}) \longrightarrow X/\sim (= \Re P_n)$ is an oppen mapping, where $\Re P_n$ denotes the real projective space.

Question #2

(a) Show that a subspace Y of the real line $\Re$ with usual topology is connected if and only if Y is an interval.

(b) If X is a topological space then prove or disprove that the component C_x of X is connected.

(c) In a topological space X, define $x \sim y$ if there is no separation $X = A \cup B$ of X into disjoint open sets such that $x \in A$ and $y \in B$. Show that this is an equivalent relation. Show that each component of X lies in a quasicomponent of X.

(d) Give an example of locally connected space which is not path-connected.

Question #3

(a) If $f : \Re^n \longrightarrow \Re^m$, then prove that f is differentiable at $a \in \Re^n$ if and only if each f^i is.

(b) Let $f : \Re^n \longrightarrow \Re$ be a function defined by $f(x,y) = \sqrt{|xy|}$. Check whether f is deifferentiable at $(0,0)$.

(c) State the Inverse Function Theorem. If the function $g : \Re^2 \longrightarrow \Re^2$ is given by $g(x,y) = (2ye^{2x}, xe^y)$ and $f : \Re^2 \longrightarrow \Re^3$ is given by $f(x,y) = (2x-y^2, 2x+y, xy+y^3)$, then decide whether there is a neighborhood U of $(0,1)$ that g carries in a one-to-one fashion onto a neighborhood V of $(2,0)$. Calculate $D(f \circ g)$.

Question #4

(a) Define the notion of n-dimensional topological manifold and and smooth manifold and give at least two examples.

(b) Prove that S^n is an n-dimensional smooth manifold.

(c) If M is a smooth manifold of dimension n and $p \in M$. Show that the tangent space $\mathcal{T}_p(M)$ is also of dimension n.