Change of Variables in $\mathbb{R}^n$

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Transfer of Measure

Theorem

Let $(X, \mathscr{A}, \mu)$ be measure space and $(Y, \mathscr{B})$ be measurable space. Let $g\colon X \to Y$ be a measurable function. We define the measure ν on Y by

$$\nu(B) = \mu(g^{-1}(B)) \text{ for all measurable subset } B \in \mathscr{B}.$$

ν is called the transport measure of μ or the pullback measure of μ. For all function $f\colon Y \to \overline{\mathbb{R}}$

$$\int_Y f(y)d\nu(y) = \int_X (f \circ g)(x)d\mu(x). \tag{1}$$

Proof

First suppose $f = \chi_B$. Let $A = g^{-1}(B) \subseteq X$. Then $f \circ g = \chi_A$, and we have

$$\int_Y f(y)d\nu(y) = \int_Y \chi_B(y)d\nu(y) = \nu(B) = \mu(g^{-1}(B)) = \mu(A) = \int_X (f \circ g$$

Since both sides of this equation are linear in f, the equation holds whenever f is simple. Applying the standard procedure, the equation is then proved for all measurable function f.

Remark

If $(X, \mathscr{A})$ be measurable space and $(Y, \mathscr{B}, \nu)$ is a measure space. Let $g\colon X \to Y$ be a bijective function and its inverse is measurable. We define the measure μ on X by $\mu(A) = \nu(g(A))$, and it follows that

$$\int_Y f(y)d\nu(y) = \int_X (f \circ g)(x)d\mu(x). \tag{2}$$

The Factorization Theorem

This section will be devoted to prove the Change of Variables in $\mathbb{R}^n$ Theorem. For this we prove two fundamental lemmas. The first called the Factorization of Diffeomorphism Lemma and the second is called the Volume Differential Lemma.

Lemma

Factorization of Diffeomorphism Lemma
Let $\varphi\colon U \longrightarrow V$ be a diffeomorphism of open sets in $\mathbb{R}^n$, $n \geq 2$. For any $a \in U$, there is a neighborhood Ω of a where φ can be expressed as the composition:

$$\varphi_{\restriction_\Omega} = u \circ v \tag{3}$$

of a diffeomorphism u that fixes some $1 \leq m \leq n-1$ coordinates of $\mathbb{R}^n$ and another diffeomorphism v that fixes the other $n-m$ coordinates.

Proof

We have to solve the above equation for the appropriate diffeomorphism $v\colon \Omega \to v(\Omega)$ and $u\colon v(\Omega) \to \varphi(\Omega)$. Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^{n-m}$ be the coordinate values. We must have $u(x, y) = (x, u_2(x, y))$ and $v(x, y) = (v_1(x, y), y)$, where $u_1(x, y) \in \mathbb{R}^m$ and $v_1(x, y) \in \mathbb{R}^{n-m}$. We have then

$$\begin{aligned}
\varphi(x,y) &= u\big(v(x,y)\big) \\
&= \Big(u_1\big(v_1(x,y), v_2(x,y)\big), u_2\big(v(x,y)\big)\Big) \\
&= \Big(u_1\big(v_1(x,y), y\big), u_2\big(v(x,y)\big)\Big) \\
&= \Big(v_1(x,y), u_2\big(v(x,y)\big)\Big)
\end{aligned}$$

Then

$$g_1 = v_1, \quad g_2 = u_2 \circ v.$$

The first equation determines the solution function v trivially. The second equation can be inverted by the inverse function theorem:

$$u_2 = g_2 \circ v^{-1}.$$

So given $(x_0, y_0) \in U$, we can define v^{-1} on some open set $\tilde{U}$ containing $v(x_0, y_0)$. Then we can take $\Omega = v^{-1}(\tilde{U})$. $\square$

Lemma

Volume Differential

Let $\varphi \colon U \to V$ be a diffeomorphism between open sets in $\mathbb{R}^n$. Then for all measurable subset $A \subseteq U$,

$$\lambda(\varphi(A)) = \int_{\varphi(A)} d\lambda(x) = \int_A |\det \mathcal{D}\varphi(x)|\, d\lambda(x). \tag{4}$$

Proof

First step It suffices to prove the lemma locally. That is, suppose there exists an open cover of U, $(U_k)_k$, so that the equation (4) holds for any measurable subset A contained inside one of the U_k. Then (4) holds for all measurable $A \subseteq U$.
Indeed, define the disjoint measurable sets $V_k = U_k \setminus \bigcup_{j=1}^{k-1} U_j$, which also cover U. And define the two measures:

$$\mu(A) = \lambda(\varphi(A)), \quad \nu(A) = \int_A |\det \mathcal{D}\varphi(x)| \, d\lambda(x).$$

Now let $A \subseteq U$ be any measurable set. We have $A \cap V_j \subseteq U_j$, so $\mu(A \cap V_j) = \nu(A \cap V_j)$ by hypothesis. Therefore,

$$\mu(A) = \mu\Big(\bigcup_{j=1}^{+\infty} A \cap V_j\Big) = \sum_{j=1}^{+\infty} \mu(A \cap V_j) = \sum_{j=1}^{+\infty} \nu(A \cap V_j) = \nu(A).$$

Second step Suppose the equation (4) holds for two diffeomorphism φ and ψ, and all measurable sets. Then it holds for the composition diffeomorphism $\varphi \circ \psi$, and all measurable sets.
Indeed, for any measurable subset A,

$$\begin{aligned}\int_{\varphi(\psi(A))} d\lambda(x) &= \int_{\psi(A)} |\det \mathcal{D}\varphi(x)|\, d\lambda(x) \\ &= \int_A |(\det \mathcal{D}\varphi)\circ\psi(x)| \cdot |\det \mathcal{D}\psi(x)|\, d\lambda(x) \\ &= \int_A |\det \mathcal{D}(\varphi\circ\psi)(x)|\, d\lambda(x).\end{aligned}$$

The second equality follows from the equation (2) applied to the diffeomorphism ψ, which is valid once we know $\lambda(\psi(B)) = \int_B |\det \mathcal{D}\psi(x)|\, d\lambda$ for all measurable subset B.

Proof of the lemma

We proceed to prove the lemma by induction, on the dimension n.

1. **Case $n = 1$.**
   Cover U by a countable set of bounded intervals I_k in $\mathbb{R}$. By the first reduction, it suffices to prove the lemma for measurable sets contained in each of the interval I_k individually. By the uniqueness of measures, it also suffices to show $\mu = \nu$ only for the closed intervals $[a, b]$.
   This is just the Fundamental Theorem of Calculus:

$$\int_{\varphi([a,b])} d\lambda(x) = |\varphi(b) - \varphi(a)| = \left| \int_a^b \varphi'(x) d\lambda(x) \right| = \int_a^b |\varphi'(x)| \, d\lambda($$

   For the last equality, remember that φ, being a diffeomorphism, must have a derivative that is positive on all of $[a, b]$ or negative on all of $[a, b]$.

2. **General case.**
   According to Lemma (2), the diffeomorphism φ can always be

Theorem

[Differential change of variables in $\mathbb{R}^n$]
Let $\varphi\colon U \to V$ be a diffeomorphism of open sets in $\mathbb{R}^n$. If $A \subseteq U$ is measurable subset, and $f\colon V \to \mathbb{R}$ is measurable, then

$$\int_{\varphi(A)} f(y)d\lambda(y) = \int_A f(\varphi(x)) \cdot |\det \mathcal{D}\varphi(x)| d\lambda(x).$$

(Substitute $y = g(x)$ and $d\lambda(y) = |\det \mathcal{D}g(x)| d\lambda(x)$.)

The theorem results from the theorem (1).