L^p Spaces

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Introduction to L^p Spaces

Let $(X, \mathscr{A}, \mu)$ be a measure space and , $0 < p < +\infty$.

Definition

We define the space $\mathcal{L}^p(\mu)$ to be the set of all measurable functions $f : X \longrightarrow \bar{\mathbb{R}}$ such that $\int_X |f(x)|^p d\,\mu(x) < \infty$.

Remark

If μ is the counting measure on a countable set X, then $\int_X f(x)d\,\mu(x) = \sum_{x\in X} f(x)$. In this case, $\mathcal{L}^p$ is usually denoted ℓ^p, the set of sequences $(x_n)_n$ such that $\sum_{n=1}^{+\infty} |x_n|^p < +\infty$.

Definition

We define the relation $\sim$ on $\mathcal{L}^p(\mu)$ as follows $f \sim g$ if $f = g$ a.e. on X.

Proposition

The relation $\sim$ on $\mathcal{L}^p(\mu)$ is an equivalence relation.

Proof

It is evident that $f \sim f$ and that, if $f \sim g$, then $g \sim f$. Now, if $f \sim g$ and $g \sim h$, then there exist $A, B \in \mathscr{A}$ such that A^c and B^c are null sets and $f = g$ on A and $g = h$ on B. It results that $\mu((A \cap B)^c) = 0$ and $f = h$ on $A \cap B$ and, hence, $f \sim h$.
The relation $\sim$ defines the equivalence classes. The equivalence class $[f]$ of $f \in \mathcal{L}^p(\mu)$ is the set of all $g \in \mathcal{L}^p(\mu)$ which are equivalents to f $[f] = \{g \in \mathcal{L}^p(\mu);\ g \sim f\}$.

Definition

We define $L^p(\mu) = \mathcal{L}^p(\mu)/\sim = \{[f];\ f \in \mathcal{L}^p(\mu)\}$.

Proposition

For $p \geq 1$, the space $L^p(\mu)$ is a vector space.

Proof

We shall use the trivial inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, for $p \geq 1$ and $a, b \in \mathbb{R}$.
For $p = 1$ the statement is obvious. For $p > 1$ the function $y = x^p$; $x > 0$ is convex since $y'' \geq 0$. Therefore $\left(\frac{|a|+|b|}{2}\right)^p \leq \frac{|a|^p + |b|^p}{2}$.
Assume that f, g are in $L^p(\mu)$. Then both functions f and g are finite a.e. on X and, hence, $f+g$ is defined a.e. on X. If $f+g$ is any measurable definition of $f+g$, then, using the above elementary inequality, $|(f+g)(x)|^p \leq 2^{p-1}(|f(x)|^p + |g(x)|^p)$ for a.e. $x \in X$ and, hence,

$$\int_X |f(x)+g(x)|^p d\,\mu(x) \leq 2^p \int_X |f(x)|^p d\,\mu(x) + 2^p \int_X |g(x)|^p d\,\mu(x) < +\infty$$

Therefore $f+g \in L^p(\mu)$.

If $f \in L^p(\mu)$ and $\lambda \in \mathbb{R}$, then $\int_X |\lambda f(x)|^p d\,\mu(x) = |\lambda|^p \int_X |f(x)|^p d\,\mu(x) < +\infty$. Therefore, $\lambda f \in L^p(\mu)$.

L^∞ Space

Definition

Let $f: X \longrightarrow \bar{\mathbb{R}}$ be measurable. We say that f is essentially bounded over X with respect to the measure μ if there exists $M < +\infty$ such that $|f| \leq M$ a.e. on X.

Proposition

Let $f : X \longrightarrow \bar{\mathbb{R}}$ be measurable function. If f is essentially bounded over X with respect to the measure μ, there exists a smallest M with the property $|f| \leq M$ a.e. on X. This smallest M_0 is characterized by

i) $|f| \leq M_0$ a.e. on X,

ii) $\mu(\{x \in X;\ |f(x)| > m\}) > 0$ for every $m < M_0$.

Proof

We set $A = \{M;\ |f| \leq M \text{ a.e. on } X\}$ and $M_0 = \inf A$. The set A is non-empty by assumption and is included in $[0, +\infty[$ and, hence, M_0 exists. We take a decreasing sequence $(M_n)_n$ in A with $\lim\limits_{n\to+\infty} M_n = M_0$. From $M_n \in A$, the set $A_n = \{x \in X;\ |f(x)| > M_n\}$ is a null set for every n and, since $\{x \in X;\ |f(x)| > M_0\} = \bigcup_{n=1}^{+\infty} A_n$, we conclude that $\{x \in X;\ |f(x)| > M_0\}$ is a null set. Therefore, $|f| \leq M_0$ a.e. on X.

If $m < M_0$, then $m \notin A$ and, hence, $\mu(\{x \in X;\ |f(x)| > m\}) > 0$.

□

Definition

Let $f\colon X \longrightarrow \bar{\mathbb{R}}$ be a measurable function. If f is essentially bounded, then the smallest M with the property that $|f| \leq M$ a.e. on X is called the essential supremum of f over X with respect to the measure μ and it is denoted by $\mathrm{ess.sup}_X(f)$ or $||f||_\infty$.
$||f||_\infty$ is characterized by the properties

1. $|f| \leq ||f||_\infty$ a.e. on X,
2. for every $m < ||f||_\infty$, $\mu(\{x \in X;\ |f(x)| > m\}) > 0$.

Definition

We define $L^\infty(\mu)$ to be the set of all equivalence class of measurable functions $f\colon X \longrightarrow \bar{\mathbb{R}}$ which are essentially bounded over X with respect to the measure μ.

Proposition

The space $L^\infty(\mu)$ is a space over $\mathbb{R}$.

Proof

If f, g in $L^\infty(\mu)$, then there exist two subsets $A_1, A_2 \in \mathscr{A}$ such that $\mu(A_1^c) = \mu(A_2^c) = 0$ and $|f| \leq ||f||_\infty$ on A_1 and $|g| \leq ||g||_\infty$ on A_2. If we set $A = A_1 \cap A_2$, then we have $\mu(A^c) = 0$ and $|f+g| \leq |f|+|g| \leq ||f||_\infty + ||g||_\infty$ on A. Hence $f+g$ is essentially bounded over X with respect to the measure μ and

$$||f+g||_\infty \leq ||f||_\infty + ||g||_\infty.$$

If $f \in L^\infty(\mu)$ and $\lambda \in \mathbb{R}$, then there exists $A \in \mathscr{A}$ with $\mu(A^c) = 0$ such that $|f| \leq ||f||_\infty$ on A. Then $|\lambda f| \leq |\lambda| ||f||_\infty$ on A. Hence λf is essentially bounded over X with respect to the measure μ and $||\lambda f||_\infty = |\lambda| ||f||_\infty$.

Hölder Inequality

Definition

For all $1 < p < +\infty$, we define the real number $q = \dfrac{p}{p-1}$, if $p = 1$ and if $p = \infty$, $q = 1$. q is called the conjugate of p or the dual of p.

p, q are related by the symmetric equality

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma

Let p and q be two conjugate real numbers such that $p > 1$. Then for all $a > 0$; $b > 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof

Note that the function $\varphi(t) = \frac{t^p}{p} + \frac{1}{q} - t$ with $t \geq 0$ has the only minimum at $t = 1$. It follows that $t \leq \frac{t^p}{p} + \frac{1}{q}$.

For $t = ab^{-\frac{1}{p-1}}$, we have $\dfrac{a^p b^{-q}}{p} + \dfrac{1}{q} \geq ab^{-\frac{1}{p-1}}$; and the result follows. $\square$

Theorem

(Hölder's inequalities)
Let $1 \leq p, q \leq +\infty$ and p, q be conjugate to each other. If $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$ and

$$\int_X |fg(x)| d\,\mu(x) \leq \left(\int_X |f(x)|^p d\,\mu(x)\right)^{\frac{1}{p}} \left(\int_X |g(x)|^q d\,\mu(x)\right)^{\frac{1}{q}}$$

$$\int_X |fg(x)| d\,\mu(x) \leq ||g||_\infty \int_X |f(x)| d\,\mu(x), \quad p = 1, q = +\infty.$$

Proof

We start with the case $1 < p, q < +\infty$. If $\int_X |f(x)|^p d\,\mu(x) = 0$ or if $\int_X |g(x)|^q d\,\mu(x) = 0$, then either $f = 0$ a.e. on X or $g = 0$ a.e. on X and the inequality is trivially true.
So we assume that $A = \int_X |f(x)|^p d\,\mu(x) > 0$ and $B = \int_X |g(x)|^q d\,\mu(x) > 0$. From the lemma (16) with $a = \frac{|f|}{A^{\frac{1}{p}}}$, $b = \frac{|g|}{B^{\frac{1}{q}}}$, we have that

$$\frac{|fg|}{A^{\frac{1}{p}} B^{\frac{1}{q}}} \leq \frac{1}{p}\frac{|f|^p}{A} + \frac{1}{q}\frac{|g|^q}{B}$$

a.e. on X. After integration we find

$$\frac{1}{A^{\frac{1}{p}}B^{\frac{1}{q}}}\int_X |fg(x)|d\,\mu(x) \le \frac{1}{p}+\frac{1}{q} = 1$$

Then

$$\int_X |fg(x)|d\,\mu(x) \le \Big(\int_X |f(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}}\Big(\int_X |g(x)|^q d\,\mu(x)\Big)^{\frac{1}{q}}$$

Let now $p = 1$ and $q = +\infty$. Since $|g| \le ||g||_\infty$ a.e. on X, $|fg| \le |f|||g||_\infty$ a.e. on X. Integrating, we find the desired inequality

The Minkowski's Inequality

Theorem

(Minkowski's inequality)
Let $1 \leq p < +\infty$. If f, g in $L^p(\mu)$, then

$$\Big(\int_X |f(x)+g(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}} \leq \Big(\int_X |f(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}} + \Big(\int_X |g(x)|^p d\,\mu(x)\Big)$$

Proof

The case $p = 1$ is trivial. Hence, we assume that $1 < p < +\infty$. We write $|f + g|^p \leq (|f| + |g|)|f + g|^{p-1} = |f||f + g|^{p-1} + |g||f + g|^{p-1}$ a.e. on X and, applying Hölder's inequality, we find

$$\begin{aligned}
\int_X |f(x) + g(x)|^p d\,\mu(x) &\leq \Big(\int_X |f(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}} \Big(\int_X |f(x) + g(x)|^{(p-1} \\
&\quad + \Big(\int_X |g(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}} \Big(\int_X |f(x) + g(x)|^{(p} \\
&= \Big(\int_X |f(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}} \Big(\int_X |f(x) + g(x)|^p d\,\mu
\end{aligned}$$

Simplifying, we get the inequality we want to prove.

Corollary

The mapping $f \longmapsto ||f||_p = \left(\int_X |f(x)|^p\right)^{\frac{1}{p}}$ is a norm on $L^p(\mu)$ and $(L^p(\mu),\ ||\ ||_p)$ is a normed vector space.

Properties of the L^p Spaces

Pointwise Convergence

Definition

Let A be an arbitrary non empty set and $(f_n\colon A \longrightarrow \bar{\mathbb{R}})_n$ be a sequence of functions defined on A.

1. We say that the sequence $(f_n)_n$ converges pointwise on A to a function $f\colon A \longrightarrow \bar{\mathbb{R}}$ if $\lim\limits_{n\to+\infty} f_n(x) = f(x)$ for all $x \in A$.
   In case $f(x)$ is finite, this means that $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}$ such that $|f_n(x) - f(x)| \leq \varepsilon;\ \forall n \geq N$.
2. Let $(X, \mathscr{A}, \mu)$ be a measure space. We say that the sequence $(f_n)_n$ converges to f (pointwise) a.e. on $A \in \mathscr{A}$ if there exists a set $B \in \mathscr{A}$, $B \subset A$, such that $\mu(A \setminus B) = 0$ and $(f_n)_n$ converges to f pointwise on B.

Remark

If $(f_n)_n$ converges to both f and g a.e. on A, then $f = g$ a.e. on A.

Convergence in L^p

Definition

Let $(f_n)_n$ be a sequence in $L^p(\mu)$ and $f \in L^p(\mu)$. We say that $(f_n)_n$ converges to f in $L^p(\mu)$ if $\lim_{n\to+\infty} ||f_n - f||_p = 0$.
We say that $(f_n)_n$ is Cauchy in $L^p(\mu)$ if $\lim\limits_{n,m\to+\infty} ||f_n - f_m||_p = 0$.

Theorem

If $(f_n)_n$ is Cauchy sequence in $L^p(\mu)$, then there exists $f \in L^p(\mu)$ such that $(f_n)_n$ converges to f in $L^p(\mu)$. (In other words $L^p(\mu)$ is a Banach space.)
Moreover, there exists a subsequence $(f_{n_k})_k$ which converges to f a.e. on X.

Corollary

If $(f_n)_n$ converges to f in $L^p(\mu)$, there is a subsequence $(f_{n_k})_k$ which converges to f a.e. on X.

Proof

a) We consider the first case $1 \leq p < +\infty$.
Since each f_n is finite a.e. on X, there exists $A \in \mathscr{A}$ such that $\mu(A^c) = 0$ and all f_n are finite on A. Then for every k, there exists n_k such that $\displaystyle\int_X |f_n(x) - f_m(x)|^p d\,\mu(x) < \frac{1}{2^{kp}}$ for every $n, m \geq n_k$.
Since we may assume that each n_k is large enough, then we can take $n_k < n_{k+1}$ for every k. Therefore, $(f_{n_k})_k$ is a subsequence of $(f_n)_n$.

From the construction of n_k and from the fact that $n_k < n_{k+1}$, we get $\int_X |f_{n_{k+1}}(x) - f_{n_k}(x)|^p d\,\mu(x) < \frac{1}{2^{kp}}$ for every k. We define the measurable function G by

$$G = \sum_{k=1}^{+\infty} |f_{n_{k+1}} - f_{n_k}|, \text{ on } A \text{ and } G = 0, \text{ on } A^c.$$

Let $G_N = \sum_{k=1}^{N} |f_{n_{k+1}} - f_{n_k}|$ on A and $G_N = 0$ on A^c, then $\left(\int_X G_N^p(x) d\,\mu(x)\right)$

$\sum_{k=1}^{N} \left(\int_X |f_{n_{k+1}}(x) - f_{n_k}(x)|^p d\,\mu(x)\right)^{\frac{1}{p}} < 1$, by Minkowski's inequality.

Since $(G_N)_N$ increases to G on X, $\int_X G^p(x) d\,\mu(x) \leq 1$ and, thus, $G < +\infty$ a.e. on X. It follows that the series $\sum_{k=1}^{+\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$

On B we have $f = f_{n_1} + \lim_{N\to+\infty} \sum_{k=1}^{N}(f_{n_{k+1}} - f_{n_k}) = \lim_{N\to+\infty} f_{n_N}$ and, hence, $(f_{n_k})_k$ converges to f a.e. on X. We, also, have on B

$$\begin{aligned} |f_{n_N} - f| &= |f_{n_N} - f_{n_1} - \sum_{k=1}^{+\infty}(f_{n_{k+1}} - f_{n_k})| \\ &= |\sum_{k=1}^{N-1}(f_{n_{k+1}} - f_{n_k}) - \sum_{k=1}^{+\infty}(f_{n_{k+1}} - f_{n_k})| \\ &\leq \sum_{k=N+1}^{+\infty} |f_{n_{k+1}} - f_{n_k}| \leq G \end{aligned}$$

for every N and, hence, $|f_{n_N} - f|^p \leq G^p$ a.e. on X for every N. Since $\int_X G^p(x)d\,\mu(x) < +\infty$ and $\lim_{N\to+\infty} |f_{n_N} - f| = 0$ a.e. on X, we use the Dominated Convergence Theorem we find that

$$\lim_{N\to+\infty} \int_X |f_{n_N}(x) - f(x)|^p d\,\mu(x) = 0$$

If $n_k \longrightarrow +\infty$, we get

$$\lim_{k\to+\infty} \Big(\int_X |f_k(x) - f(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}} \leq \lim_{k\to+\infty} \Big[\Big(\int_X |f_k(x) - f_{n_k}(x)|^p d\,\mu$$
$$+\Big(\int_X |f_{n_k}(x) - f(x)|^p d\,\mu(x)\Big)^{\frac{1}{p}}$$

and we conclude that $(f_n)_n$ converges to f in $L^p(\mu)$.

b) Now, let $p = +\infty$. For each n, m we have a set $A_{n,m} \in \mathscr{A}$ such that $\mu(A_{n,m}^c) = 0$ and $|f_n - f_m| \leq ||f_n - f_m||_\infty$ on $A_{n,m}$.
Let $A = \bigcap_{n,m\geq 1} A_{n,m}$, then $\mu(A^c) = 0$ and $|f_n - f_m| \leq ||f_n - f_m||_\infty$ on A for every n, m. This gives that $(f_n)_n$ is Cauchy sequence for the norm $||\ ||_\infty$ on A and, hence, there exists a mapping f such that $(f_n)_n$ converges to f uniformly on A. Now,

$$\lim_{n\to+\infty} \|f_n - f\|_\infty \leq \lim_{n\to+\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$$

Convergence in Measure

Let $(X, \mathscr{A}, \mu)$ is a measure space.

Definition

1. Let $f, f_n \colon X \longrightarrow \bar{\mathbb{R}}$ be measurable functions. We say that $(f_n)_n$ converges to f in measure on $A \in \mathscr{A}$ if all f, f_n are finite a.e. on A and for every $\varepsilon > 0$;

$$\lim_{n \to +\infty} \mu(\{x \in A;\ |f_n(x) - f(x)| \geq \varepsilon) = 0.$$

2. We say that $(f_n)_n$ is a Cauchy sequence in measure on $A \in \mathscr{A}$ if all f_n are finite a.e. on A and for every $\varepsilon > 0$

$$\lim_{n,m \to +\infty} \mu(\{x \in A;\ |f_n(x) - f_m(x)| \geq \varepsilon\}) = 0.$$

Remarks

1. The uniform convergence yields the convergence in measure
2. If we want to be able to write the values $\mu(\{x \in A;\ |f_n(x) - f(x)| \geq \varepsilon\})$ and $\mu(\{x \in A;\ |f_n(x) - f_m(x)| \geq \varepsilon\})$, we first extend the functions $|f_n - f|$ and $|f_n - f_m|$ outside the set $B \subset A$, where all f, f_n are finite, as functions defined on X and measurable. Then, since $\mu(A \setminus B) = 0$, we get that the above values are equal to the values $\mu(\{x \in B;\ |f_n(x) - f(x)| \geq \varepsilon\})$ and, respectively, $\mu(\{x \in B;\ |f_n(x) - f_m(x)| \geq \varepsilon\})$. Therefore, the actual extensions play no role and, hence, we may for simplicity extend all f, f_n as 0 on $X \setminus B$. Thus the replacement of all f, f_n by 0 on $X \setminus B$ makes all functions finite everywhere on A and does not affect the fact that $(f_n)_n$ converges to f in measure on A or that $(f_n)_n$ is Cauchy in measure on A.
3. Let $a, b > 0$ and $A = \{x \in A;\ |f(x)| > a\}$

Proposition

If $(f_n)_n$ converges to both f and g in measure on A, then $f = g$ a.e. on A.

Proof

We may assume that all f, g, f_n are finite on A. Applying the above remark we find that

$$\begin{aligned}\mu(\{x \in A;\ |f(x) - g(x)| \geq \varepsilon\}) &\leq \mu(\{x \in A;\ |f_n(x) - f(x)| \geq \frac{\varepsilon}{2}\}) \\ &+\mu(\{x \in A;\ |f_n(x) - g(x)| \geq \frac{\varepsilon}{2}\})\ _{n-}\end{aligned}$$

This implies that $\mu(\{x \in A;\ |f(x) - g(x)| \geq \varepsilon\}) = 0$ for every $\varepsilon > 0$. We now write

$$\{x \in A;\ f(x) \neq g(x)\} = \bigcup_{k=1}^{+\infty} \{x \in A;\ |f(x) - g(x)| \geq \frac{1}{k}\}.$$

Since each term in the union is a null set, we get $\mu(\{x \in A;\ f(x) \neq g(x)\}) = 0$ and we conclude that $f = g$ a.e. on A.

Proposition

If $(f_n)_n$ converges to f and $(g_n)_n$ converges to g in measure on A and if $\alpha \in \mathbb{R}$. Then
a) $(f_n + g_n)_n$ converges to $f + g$ in measure on A.
b) $(\alpha f_n)_n$ converges to αf in measure on A.
c) If there exists $M < +\infty$ such that $|f_n| \leq M$ a.e. on A, then $|f| \leq M$ a.e. on A.
d) If there exists $M < +\infty$ such that $|f_n|, |g_n| \leq M$ a.e. on A, then $(f_n g_n)_n$ converges to fg in measure on A.

Proof

We may assume that all f, f_n are finite on A.

a) We apply the remark 3

$$\mu(\{x \in A;\ |(f_n + g_n)(x) - (f+g)(x)| \geq \varepsilon\}) \leq \mu(\{x \in A;\ |f_n(x) - f($$

$$+\mu(\{x \in A;\ |g_n(x) - g($$

b) Also for $\alpha \neq 0$,

$$(\{x \in A;\ |\alpha f_n(x) - \alpha f(x)| \geq \varepsilon\}) = \mu(\{x \in A;\ |f_n(x) - f(x)| \geq \frac{\varepsilon}{|\alpha|}\}) \underset{n \to +\infty}{\longrightarrow}$$

c) For n large enough

$$\begin{aligned}\mu(\{x\in A;\ |f(x)|\geq M+\varepsilon\}) &\leq \mu(\{x\in A;\ |f_n(x)|\geq M+\frac{\varepsilon}{2}\})\\ &\quad +\mu(\{x\in A;\ |f_n(x)-f(x)|\geq\frac{\varepsilon}{2}\})\\ &= \mu(\{x\in A;\ |f_n(x)-f(x)|\geq\frac{\varepsilon}{2}\}) \underset{n\to+\infty}{\longrightarrow}\end{aligned}$$

Hence, $\mu(\{x\in A;\ |f(x)|\geq M+\varepsilon\})=0$ for every $\varepsilon>0$.
We have $\{x\in A;\ |f(x)|>M\}=\bigcup_{k=1}^{+\infty}\{x\in A;\ |f(x)|\geq M+\frac{1}{k}\}$ and, since all sets of the union are null sets, then $\mu(\{x\in A;\ |f(x)|> M\})=0$. Hence, $|f|\leq M$ a.e. on A.

d) Applying the result of c),

$$\begin{aligned}\mu(\{x \in A;\ |f_n(x)g_n(x) - f(x)g(x)| \geq \varepsilon\}) &\leq \mu(\{x \in A;\ |f_n(x)g_n(x) - \\ &+\mu(\{x \in A;\ |f_n(x)g(x) - \\ &\leq \mu(\{x \in A;\ |g_n(x) - g(x)| \\ &+\mu(\{x \in A;\ |f_n(x) - f(x\end{aligned}$$

□

Proposition

Let $(f_n)_n$ be a sequence of measurable functions on a measure space $(X, \mathcal{B}, \mu)$. If μ is finite and the sequence $(f_n)_n$ converges almost everywhere to f, then the sequence $(f_n)_n$ converges in measure to f.

Proof

Let $\varepsilon > 0$, we set

$$A_n(\varepsilon) = \{x; |f_n(x) - f(x)| \geq \varepsilon\}, \quad B_n(\varepsilon) = \bigcup_{k\geq n} A_k(\varepsilon)$$

and

$$B(\varepsilon) = \bigcap_{n\geq 1} B_n(\varepsilon) = \overline{\lim}_{n\to+\infty} A_n(\varepsilon)$$

If $x \in B(\varepsilon)$, then x belongs to an infinite of $A_n(\varepsilon)$. The sequence $(f_n(x))_n$ can not converges to $f(x)$ and then $\mu(B(\varepsilon)) = 0$. Moreover since μ is finite $\lim\limits_{n\longrightarrow+\infty} \mu(B_n(\varepsilon)) = 0$, and since $A_n(\varepsilon) \subset B_n(\varepsilon)$, then $\lim\limits_{n\longrightarrow+\infty} \mu(A_n(\varepsilon)) = 0$.

Theorem

If $(f_n)_n$ is a Cauchy sequence in measure on A, there exists $f : X \longrightarrow \bar{\mathbb{R}}$ such that $(f_n)_n$ converges to f in measure on A. Moreover, there is a subsequence $(f_{n_k})_k$ which converges to f a.e. on A.

Corollary

If $(f_n)_n$ converges to f in measure on A, there is a subsequence $(f_{n_k})_k$ which converges to f a.e. on A.

Proof

As usual, we assume that all f_n are finite on A. We have, for all k, $\mu(\{x \in A;\ |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\}) \underset{n,m\to+\infty}{\longrightarrow} 0$. Therefore, there exists n_k such that $\mu(\{x \in A;\ |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}$ for every $n, m \geq n_k$. Since we may assume that each n_k is as large as we like, we may inductively take n_k such that $n_k < n_{k+1}$ for every k. Hence, $(f_{n_k})_k$ is a subsequence of $(f_n)_n$ and, from the construction of n_k and since $n_k < n_{k+1}$, we have that for every k;

$$\mu(\{x \in A;\ |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq \frac{1}{2^k}\}) < \frac{1}{2^k}$$

Let $E_k = \{x \in A;\ |f_{n_{k+1}}(x) - f_{n_k}(x)| \geq \frac{1}{2^k}\}$ and, hence, $\mu(E_k) < \frac{1}{2^k}$ for all k. Let $F_m = \bigcup_{k=m}^{+\infty} E_k$, $F = \bigcap_{m=1}^{+\infty} F_m = \overline{\lim}_{k\to+\infty} E_k$.

$\mu(F_m) \leq \sum_{k=m}^{+\infty} \mu(E_k) < \sum_{k=m}^{+\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}$ and, hence, $\mu(F) \leq \mu(F_m) < \frac{1}{2^{m-1}}$ for every m. This implies that $\mu(F) = 0$.

If $x \in A \setminus F$, there exists m such that $x \in A \setminus F_m$, which implies that $x \in A \setminus E_k$ for all $k \geq m$. Therefore, $|f_{n_{k+1}}(x) - f_{n_k}(x)| < \frac{1}{2^k}$ for all $k \geq m$, such that $\displaystyle\sum_{k=m}^{+\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < \frac{1}{2^{m-1}}$. Thus, the series $\displaystyle\sum_{k=m}^{+\infty} (f_{n_{k+1}}(x) - f_{n_k}(x))$ converges and we may define $f : X \longrightarrow \bar{\mathbb{R}}$ by

$$f = f_{n_1}(x) + \sum_{k=1}^{+\infty} (f_{n_{k+1}} - f_{n_k}),$$

on $A \setminus F$ and 0, on $A^c \cup F$.

$$f(x) = f_{n_1}(x) + \lim_{m \to +\infty} \sum_{k=1}^{m} (f_{n_{k+1}}(x) - f_{n_k}(x)) = \lim_{m \to +\infty} f_{n_m}(x)$$

for every $x \in A \setminus F$ and since $\mu(F) = 0$, we get $(f_{n_k})_k$ converges to f a.e.
Now, on $A \setminus F_m$; we have

$$\begin{aligned} |f_{n_m} - f| &= |f_{n_m} - f_{n_1} - \sum_{k=1}^{+\infty}(f_{n_{k+1}} - f_{n_k})| \\ &= |\sum_{k=1}^{m-1}(f_{n_{k+1}} - f_{n_k}) - \sum_{k=1}^{+\infty}(f_{n_{k+1}} - f_{n_k})| \\ &= \sum_{k=m}^{+\infty}|f_{n_{k+1}} - f_{n_k}| < \frac{1}{2^{m-1}}. \end{aligned}$$

Therefore, $\{x \in A;\ |f_{n_m}(x) - f(x)| \geq \frac{1}{2^{m-1}}\} \subset F_m$ and, hence,

$$\mu(\{x \in A;\ |f_{n_m}(x) - f(x)| \geq \frac{1}{2^{m-1}}\}) \leq \mu(F_m) < \frac{1}{2^{m-1}}.$$

Take an arbitrary $\varepsilon > 0$ and m_0 large enough such that $\frac{1}{2^{m-1}} \leq \varepsilon$. If $m \geq m_0$, $\{x \in A;\ |f_{n_m}(x) - f(x)| \geq \varepsilon\} \subset \{x \in A;\ |f_{n_m}(x) - f(x)| \geq \frac{1}{2^{m-1}}\}$ and, hence,

$$\mu(\{x \in A;\ |f_{n_m}(x) - f(x)| \geq \varepsilon\}) < \frac{1}{2^{m-1}} \underset{m \to +\infty}{\longrightarrow} 0.$$

This means that $(f_{n_k})_k$ converges to f in measure on A. Since $n_k \underset{k\to+\infty}{\longrightarrow} +\infty$, we have

$$\begin{aligned}\mu(\{x\in A;\ |f_k(x)-f(x)|\geq\varepsilon\}) &= \mu(\{x\in A;\ |f_k(x)-f_{n_k}(x)|\geq\frac{\varepsilon}{2}\}) \\ &\quad +\mu(\{x\in A;\ |f_{n_k}(x)-f(x)|\geq\frac{\varepsilon}{2}\})\end{aligned}$$

and we conclude that $(f_n)_n$ converges to f in measure on A.

Remark

Consider the sequence $(f_n)_n$ defined by: $f_1 = \chi_{]0,1[}$, $f_2 = 2\chi_{]0,\frac{1}{2}[}$, $f_3 = 2\chi_{]\frac{1}{2},1[}$, and for all $n \in \mathbb{N}$, $f_{\frac{n(n+1)}{2}+k+1} = n\chi_{]\frac{k}{n+1},\frac{k+1}{n+1}[}$, for $k = 0, \dots, n$. If $0 < \varepsilon \leq 1$, $\mu(\{x \in]0,1[;\ |f_n(x)| \geq \varepsilon\}) \underset{n\to+\infty}{\longrightarrow} 0$. Therefore, $(f_n)_n$ converges to 0 in measure on $]0,1[$. But, as we have already seen, it is not true that $(f_n)_n$ converges to 0 a.e. on $]0,1[$.

Theorem

Let $1 \leq p < +\infty$.

1. The convergence in L^p implies convergence in measure.
2. If $\mu(X) < \infty$, then $L^q \subset L^p$ and the convergence in L^q implies convergence in L^p, for all $q \geq p$.

Proof

1. Suppose the sequence $(f_n)_n$ converges to f in L^p and let $\varepsilon > 0$, Then by the Markov inequality,

$$\mu\{|f_n - f| \geq \varepsilon\} \leq \frac{1}{\varepsilon^p}\int_X |f_n - f|^p d\mu = \frac{1}{\varepsilon^p}||f_n - f||_p^p$$

and 1) follows at once.

2. The Hölder inequality gives for any measurable function f,

$$\int_X |f(x)|^p d\mu(x) \leq \Big(\int_X |f(x)|^q d\mu(x)\Big)^{\frac{p}{q}} \Big(\int_X d\mu(x)\Big)^{\frac{q-p}{q}} = \|f\|_q^p(\mu($$

$$||f||_p \leq ||f||_q(\mu(X))^{\frac{q-p}{pq}}.$$

and 2) is proved.

□

Egoroff's Theorem

Theorem

(Egoroff)
Let $(X, \mathscr{B}, \mu)$ be a measure space. Assume that the measure μ is bounded and $(f_n)_{n\in\mathbb{N}}$ a sequence of real or complex measurable functions on X which converges point wise on X to a function f. For any $\varepsilon > 0$ there exists a set $A_\varepsilon \in \mathscr{B}$, such that $\mu(A_\varepsilon) \leq \varepsilon$ and the restriction of the sequence (f_n) on the complementary of A_ε is uniformly convergent.

Proof

The function f is measurable. For any integers (n, k), $k > 0$, let

$$E_n^{(k)} = \bigcap_{p=n}^{+\infty} \{x; |f_p(x) - f(x)| \leq \frac{1}{k}.\}$$

This set is measurable. For a given k, the sequence $(E_n^{(k)})_{n\in\mathbb{N}}$ is increasing and $\lim_{n\to+\infty} E_n^{(k)} = X$. (Because the sequence $(f_n)_{n\in\mathbb{N}}$ converges to f on X). As μ is bounded, $\lim_{n\to+\infty} \mu(E_n^{(k)})^c = 0$. Then there exists an integer $n(k)$ such that $\mu(E_{n(k)}^{(k)})^c \leq \varepsilon/2^k$. The set $A_\varepsilon = \bigcup_{k=1}^{+\infty}(E_{n(k)}^{(k)})^c$ is appropriate. In fact $\mu(A_\varepsilon) \leq \varepsilon$, and on the complementary of A_ε the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Remark

The requirement that μ is bounded is essential. For constructing a counterexample it suffices to take μ the Lebesgue measure on $\mathbb{R}$ and f_n the characteristic function of the interval $[n, +\infty[$.