بسـم الله الرحمن الرحـيم

College of Science
Dep. Statistics & Operations Research
OR 441 – Modeling and Simulation
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Final Exam

Name	KEY SOLUTION	Student Number	

والله ولي التوفيق

	Q. #1	Q. #2	Q. #3	Q. #4	Q.#5	Q.#6	Total
	15	10	10	15	20	20	90
Score							

Instructions

1. *Show your university ID*
2. *Exam period is 3 hours.*
3. *This Exam consists of 11 pages with 7 questions; each question should not take more than 25 min on average.*
4. *The answer of each question is on the same page, use the back of the pages if you need more space.*
5. *Answer all questions and Show all your work in the answer*
6. *Turn off your cell phones*
7. *Do not use your cell phone for calculations*

Question #1 : (15 points)

Answer the following with ***True*** or ***False***

False	1. Discrete event simulation is used to model systems that change over time in a continuous manner.
True	2. Inverse transform method requires the cumulative distribution function (CDF) to generate random numbers from a probability distribution.
False	3. The inverse transform method is only applicable for continuous probability distributions
True	4. Not always any distribution with CDF in functional form (closed form) has an inverse transform.
True	5. Discrete event simulation models the behavior of a system as a sequence of events sorted by the time of occurrence.
False	6. Inverse transform method doesn't require a random numbers generator from a uniform distribution to have random values from a distribution
False	7. The inverse transform method is used to generate random variables with non-negative values only.
False	8. Any shifted distribution must have a nonnegative shift value.
False	9. Discrete event simulation is used to simulate systems with a discrete state space.
False	10. The 95% Confidence Interval means that 95% of the data for the population will be between LL and UL
True	11. To apply the 95% confidence interval to determine the number of simulation runs to achieve a total width of the confidence interval equal 6 then we must take E = 3
True	12. The method of acceptance/rejection is a method for generating random numbers from any probability distribution.
False	13. The method of acceptance/rejection is a faster for generating random numbers than the inverse transform method
True	14. Discrete event simulation can be used to study the effects of changes in system parameters and inputs.
False	15. If the number X is accepted in acceptance/rejection, then this number follows the function g(x).
False	16. The rejected number in method of acceptance/rejection is always less than the accepted numbers.
False	17. To simulated from the truncated Weibull (α,β) distribution with range (δ,∞), then this means that the distribution is shifted by a value of δ
False	18. To determine the minimum number of runs for simulation to get a 95%-C.I. with half width less than E, then apply $\left(\frac{Z_\alpha\ S^2}{E}\right)^{\frac{1}{2}} \geq n$
TRUE	19. The 95% Confidence Interval means that if you construct 1000 samples from simulation then 950 of the intervals will have the exact parameter μ of the population.
False	20. An estimator is any statistic that is used to estimate an unknown quantity based on the all population
FALSE	21. The Convolution method is used if one random variable is defined by the sum of other independent random variables all from different distributions.

TRUE	22. In Acceptance/Rejection method, we must have at least two U(0,1) random number to give one random follows the function f(x).
FALSE	23. The Mixture of Distributions Method works the same as the convolution method but with using all weights (wi) equal 1.
FALSE	24. In *Acceptance/Rejection* method, we accept the new number w if the $\frac{g(w)}{f(w)} > u$ for $u\sim$ U(0,1).
FALSE	25. The function VLOOKUP in Excel always used to generate numbers from Normal distribution.
FALSE	26. The function RAND() in Excel is always gives numbers from Normal(0,1)
TRUE	27. It is always best to use simulation models to evaluate alternatives for complex or highly expensive systems.
FALSE	28. To generate one value of Erlang (k=2, λ=3) distribution we generate one value from Exponential (λ=3) and multiply it with 2.
FALSE	29. We can always use RANDBETWEEN(a,b) to generate integer numbers between (a) and (b) with any the probability distribution for random variable between (a,b).
FALSE	30. If X_1 , X_2, X_3 are random values generated from Shifted Exponential(λ=3) with shift value δ=5, then X_1 , X_2, X_3 all values must be less than 5.
TRUE	31. Any function in Excel end with ".INV" gives the inverse transform of random variable.
TRUE	32. We can determine number of simulation runs needed to make simulation results accurate by using confidence interval rule and a pilot sample from the simulation.
FALSE	33. The Function RANDBETWEEN(a,b) in Excel is used to generate continuous random number between a and b.

Question #2 : (10 points)

Customers arrive to a minimarket according to a Poisson process with arrival rate λ = 15 customers per hour. The arriving customers come to a single server checkout counter after they finish shopping. It is estimated that the checkout sever takes a random amount of time to finish the checkout for a customer. The service time follows an exponential distribution with mean 5 minutes. The simulation output for 10 customers is in the following table:

Col. 1	Col. 2	Col. 3	Col.4	Col. 5	Col. 6	Col. 7	Col. 8	Col. 9	Col. 10	Col. 11
Cust. #	U[0,1]	Time between arrivals (min)	Arrival time (min)	U[0,1]	Service time (min)	Service start (min)	Cust. Wait?	Wait Time (min)	EXIT time (min)	Cashier Idle Time (min)
1	0.059	0.24	0.24	0.105	0.33	0.24	0	0.00	0.58	0.24
2	0.159	0.69	0.93	0.503	2.10	0.93	0	0.00	3.03	0.36
3	0.186	0.82	1.76	0.958	9.51	3.03	1	1.27	12.54	0.00
4	0.852	7.63	9.39	0.759	4.27	12.54	1	3.15	16.81	0.00
5	0.550	3.19	12.58	0.755	4.22	16.81	1	4.23	21.03	0.00
6	0.342	1.67	14.26	0.377	1.42	21.03	1	6.77	22.45	0.00
7	0.716	5.03	19.29	0.152	0.49	22.45	1	3.16	22.94	0.00
8	0.554	3.23	22.52	0.399	1.53	22.94	1	0.42	24.47	0.00
9	0.742	5.42	27.94	0.527	2.25	27.94	0	0.00	30.19	3.48
10	0.918	10.01	37.96	0.415	1.61	37.96	0	0.00	39.56	7.77

Answer the following with **True** or **False**:

(b)	**1.** Using Excel the values of **col.2** is generated by: **(a)** using the function RANDBETWEEN(0,1) (b) using the function RAND() only **(c)** using the function VlOOKUP
(b)	**2.** The value of the arrival time for customer (3) is computed by: **(a)** arrival time of Cust.(3) + time between Cust.(3) and Cust.(4) (b) arrival time of Cust.(2) + time between Cust.(2) and Cust.(3) **(c)** Departure time Cust.(2) + time between Cust.(2) and Cust.(3)
(b)	**3.** Using Excel the values of **col.5** is generated by **(a)** using the function RANDBETWEEN(0,1) (b) using the function RAND() only **(c)** using the function VlOOKUP
(b)	**4.** The values of Col.3 is computed in EXCEL by **(a)** using the function RANDBETWEEN(a,b) **(b)** the function =(−60/15)* ln(1−RAND( )) with Rand() from clo.2 **(c)** the function = a + (b–a)*(RAND( )) with Rand() from clo.2
(c)	**5.** From the table, Col.# 8 the waiting time of customer (n) is zero if: **(a)** Cust.(n) arrival time is ≥ the departure (Exit) time of Cust.(n+1) **(b)** Cust.(n) arrival time is ≤ the departure (Exit) time of Cust.(n+1) **(c)** Cust.(n) arrival time is ≥ the departure (Exit) time of Cust.(n−1)
(c)	**6.** From the table, the idle time of the server when customer (n) arrive is defined as: **(a)** If Cust.(n) **no wait** then idle time = 0 **(b)** If Cust.(n) **wait** then idle time = Dep. Time Cust.(n−1)-Arrival time Cust(n) **(c)** Both (a) and (b)
(c)	**7.** Col. # 8[Cust. Wait] is computed by the following condition: **(a)** If service start = arrival time then (Cust. Wait?) = 0 **(b)** If service start > arrival time then (Cust. Wait?) = 1 **(c)** Both (a) and (b)
(a)	**8.** To compute the average waiting time from simulation table we use: **(a)** SUM(Col. 9)/10 **(b)** **SUM**(Col. 9)/(total simulation time) **(c)** **SUM**(Col. 8)/10
(c)	**9.** The value of average waiting time in line is = 1.901 **(a)** average waiting time in line is = 0.6 min **(b)** average waiting time in line is = 0.4804 min **(c)** average waiting time in line is = 1.901 min
(b)	**10.** From simulation, the percentage of customers who wait is **(a)** Percentage of customers who wait in line = 40% (b) Percentage of customers who wait in line = 60% **(c)** Percentage of customers who wait in line = 100%
(b)	**11.** The average number of arrivals in one hour is computed by; **(a)** 60 (total arrivals)/10 (b) 60 (total arrivals)/ (total simulation time) **(c)** 60 (1 / λ)
(c)	**12.** The Service time for each customer in the table is computed by: **(a)** 1 – Exp(Rand()/5) **(b)** (−60/5)* ln(1−RAND( )) **(c)** (−5)* ln(1−RAND( ))
(b)	**13.** To commute probability that the cashier is IDLE we use: **(a)** **SUM**(Col.11)/10 **(b)** SUM(Col.11)/ (total simulation time) **(c)** **1 / SUM**(Col.11)
(a)	**14.** From simulation table, the probability that the cashier is IDLE is equal to: (a) probability cashier IDLE = 0.2995 **(b)** probability cashier IDLE = 1.185 **(c)** probability cashier IDLE = 0.6

Question #3 : (20 points)

Consider the following probability random density functions for:

$$f(x) = \frac{1}{3.11} e^{-\frac{1}{4}(x-4)}; \qquad 4 \le x \le 10$$

$$f(y) = \frac{2}{\beta^2}\, y\, e^{\left(\frac{y}{\beta}\right)^2};\ \beta = 5;\ y > 0$$

30% → Correct Logic for arrivals
30% → Correct Logic for Services
20% → Correct Calculation
20% → Ave. result

(a) Compute the CDF of the function *f*(x) and derive an inverse transform for this distribution.

(b) Compute the CDF of the function *f*(y) and derive an inverse transform for this distribution.

(c) Let X (minutes) be the time between customers' arrival to a bank (integer values). Each customer takes a random amount of time (Y) minutes between [4,15] until he leaves the bank with his service competed (integer values). Using U[0,1] random streams (take U1 for X and U2 for Y), estimate average number of customer in bank per hour.

4 points

	1	2	3	4	5	6	7	8	9	10
U1	0.668	0.993	0.736	0.829	0.052	0.849	0.362	0.722	0.131	0.554
U2	0.446	0.007	0.986	0.878	0.190	0.115	0.782	0.798	0.605	0.813
X	**6**	**9**	**7**	**8**	**4**	**8**	**5**	**7**	**4**	**6**
AT	**6**	**15**	**22**	**30**	**34**	**42**	**47**	**54**	**58**	**64**
w	**0.708**	**0.477**	**0.992**	**0.935**	**0.573**	**0.533**	**0.885**	**0.893**	**0.792**	**0.901**
Y(w)	**5**	**4**	**11**	**8**	**4**	**4**	**7**	**7**	**6**	**7**
DT	**11**	**19**	**33**	**38**	**38**	**46**	**54**	**61**	**64**	**71**

Solution:

(a) the CDF of the function *f*(x)

$$F(x) = \int_4^x \frac{1}{3.11} e^{-\frac{1}{4}(t-4)} dt = \frac{4}{3.12}\left[-e^{-\frac{1}{4}(t-4)}\right]_4^x = \frac{4}{3.12}\left(1 - e^{-\frac{1}{4}(x-4)}\right)$$ **2 points**

Inverse : let u = F(x) → $u = \frac{4}{3.11}\left(1 - e^{-\frac{1}{4}(x-4)}\right)$ → $e^{-\frac{1}{4}(x-4)} = 1 - \frac{3.11}{4} u$

→ $-\frac{1}{4}(x-4) = \ln\left(1 - \frac{3.11}{4} u\right)$ → $X = 4 - 4\, ln\left(1 - \frac{3.11}{4} u\right)$ **3 points**

(b) Y ~ Weibull Dist → $CDF(y) = 1 - e^{-\left(\frac{x}{5}\right)^2}$ **2 points**

Inverse : let $u = 1 - e^{-\left(\frac{y}{5}\right)^2}$ → $Y = 5\sqrt{-\,ln(1 - u)}$ **3 points**

be the time between customer's arrival to a bank

c) Algorithm

1. Get u~ U(0,1)
2. Generate X be the time between customer's arrival to a bank
   $X = 4 - 4\, ln\left(1 - \frac{3.11}{4} u\right)$
3. Let AT(i): arrival time of customer (i)
4. AT(i) = AT(i-1) + X ; AT(0) = 0 **2 points**
5. Generate Y time in the bank truncated f(y)
   - F(Y=4) = 0.473 , F(Y=15) = 0.999
   - Let w(u) = F(Y=4) + (F(Y=15) - F(Y=4))u = 0.473 + (0.526)u
   - Get u~ U(0,1)
   - Y∈[4,15] → $Y = \sqrt{-\,ln(1 - (0.473 + (0.526)u))}$ **2 points**

Average number of customer in bank per hour =

from	to	change	#N	DT	DT*N
0	6	0	0	6	0
6	11	1	1	5	5
11	15	-1	0	4	0
15	19	1	1	4	4
19	22	-1	0	3	0
22	30	1	1	8	8
30	33	1	2	3	6
33	34	-1	1	1	1
34	38	1	2	4	8
38	38	-1	1	0	0
38	42	-1	0	4	0
42	46	1	1	4	4
46	47	-1	0	1	0
47	54	1	1	7	7
54	54	1	2	0	0
54	58	-1	1	4	4
58	61	1	2	3	6
61	64	-1	1	3	3
64	64	1	2	0	0
64	71	-1	1	7	7
71	71	-1	0	0	0
Average number of customer in bank per hour =					**0.887**

2 points

Question # 4 : (15 points)

Consider the pdf function for the random variable X:

$$f(x) = \begin{cases} 2\beta^{-2} x e^{(-(x/\beta)^2)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let the parameter β = 3. Answer the following:

(a) Derive an inverse transform algorithm for this distribution.

(b) Let X be the time between arrival of airplanes in an international an airport. There are three types of airplanes that land on the airport: ASA-100 with 100 seats, ASA-300 with 300 seats or ASA-500 with 500 seats. The airplanes can be any of the tree types with equal probabilities. It is expected that number of empty seats in any airplane is randomly with discrete uniform between 5 and half of max. number of seats on airplane. Explain the simulation steps for Arrival Time of Airplane, Airplane Type, Empty Seats, Number of Passengers. Using the U[0,1] random number in the following table, using the inverse transform in part **(a)** to determine the arrival time of the 1st five airplanes and the number of passengers on each one.

		1	2	3	4	5	6	7	8
	U[0,1] → X	0.013	0.117	0.681	0.951	0.202	0.362	0.722	0.131
	U[0,1]→ Type	0.370	0.543	0.978	0.067	0.732	0.782	0.798	0.605
	U[0,1]→Seats	0.174	0.208	0.327	0.533	0.332	0.190	0.115	0.782
1 points	**X**	**0.34**	**1.06**	**3.21**	**5.21**	**1.43**	**2.01**	**3.39**	**1.12**
	Arrival Time of Airplane	**0.34**	**1.40**	**4.61**	**9.82**	**11.24**	**13.25**	**16.65**	**17.77**
2 points	Airplane Type	**ASA-300**	**ASA-300**	**ASA-500**	**ASA-100**	**ASA-500**	**ASA-500**	**ASA-500**	**ASA-300**
2 points	Empty Seats	**30**	**35**	**85**	**29**	**86**	**51**	**33**	**119**
2 points	Number of Passengers	**270**	**265**	**415**	**71**	**414**	**449**	**467**	**181**

$$F(x) = -e^{-x^2/\beta^2} + 1$$

$$U = -e^{-x^2/\beta^2} + 1$$

$$\ln(1-U) = \frac{-x^2}{\beta^2} \Rightarrow -x^2 = \beta^2 \ln(1-U) \Rightarrow x = \sqrt{-\beta^2 \ln(1-U)}$$

AT(n) = AT(n-1) + X

Let β = 3 → $X = 3\sqrt{-ln(1-u)}$ **2 points**

Let Y the type of the airplane
Let E be the number of empty seats
Let NP be the number of passengers

U	0~0.333	0.333~0.667	0.667~1.0	
Airplane Type Y	ASA-100	ASA-300	ASA-500	**2 points**
Max seats	100	300	500	
Empty Seats	E = 5+int[50-5+1)*U]	E = 5+ int[150-5+1)*U]	E = 5+ int[250-5+1)*U]	**2 points**
Number of Passengers	NP = 100 – E	NP = 300 – E	NP = 500 – E	**2 points**

Question #5: (15 points)

Consider a fish in a small square aquarium. The fish is monitored by a camera and record the movement of the fish in two dimensions (X,Y). The fish moves randomly on specific points such that X follows binomial distribution with parameters (n = 5 , p = 0.45) with shift 1 and Y follows discrete uniform between 1 and 10 and each point remains a random amount of time follows an Erlang distribution with parameters r= 2 and λ = X per min.

- **(a)** List all random processes in this system and define the simulation method for each processes.
- **(b)** Simulate the movement of the fish for 10 min using the following U[0,1] random streams.
- **(c)** What is the average time that the fish remains in each point and the standard deviation. **Mean =1.260 and S^2 = 0.880**
- **(d)** Give your results in (c), How many simulation moves we have to record in order to have a 99%-C.I. with error less than E≤ 0.01 (Z_{α} = 2.32, $Z_{0.5\alpha}$ = 2.58, $Z_{0.25\alpha}$ = 3.29)

Move #	U1	X	U2	Y	U3	U4	U5	T	CLK
1	0.557	3	0.191	2	0.977	0.224	0.274	1.340	1.340
2	0.138	2	0.480	5	0.659	0.767	0.510	1.267	2.606
3	0.936	5	0.851	9	0.055	0.814	0.747	0.348	2.954
4	0.568	3	0.696	7	0.902	0.074	0.899	0.800	3.754
5	0.233	2	0.580	6	0.804	0.953	0.022	2.343	6.097
6	0.274	3	0.094	1	0.217	0.718	0.741	0.504	6.601
7	0.510	3	0.406	5	0.213	0.911	0.557	0.886	7.487
8	0.747	4	0.575	6	0.874	0.972	0.138	1.409	8.896
9	0.899	5	0.493	5	0.701	0.919	0.936	0.743	9.639
10	0.022	1	0.813	9	0.928	0.581	0.568	3.497	13.136
11	0.741	4	0.219	3	0.734	0.788	0.233	0.719	13.855

1 points 1 points 1 points 2 points

RP#1: X- axis move **3 points**

N	0	1	2	3	4	5
P{N}	0.0503	0.2059	0.3369	0.2757	0.1128	0.0185
P{N<n}	0.0503	0.2562	0.5931	0.8688	0.9815	1.0000
U	0~.0503	~ 0.2562	~ 0.5931	~ 0.8688	~ 0.9815	~ 1.0000
X	1	2	3	4	5	6

RP#2 : Y-axis move → Y = 1 + int[10 -1 +1 u] **2 points**

RP#3 : Time remain in point (X,Y) **2 points**

1. Given generated value of X
2. T = -1/X (ln(1 – u_1) + ln(1 – u_2))

simulation moves we have to record in order to have a 99%-C.I.

$\left(\frac{Z_{\alpha/2}\ S}{E}\right)^2 \leq n$ Let $Z_{0.5\alpha}$ = 2.58, and S = 0.88 and E = 0.01

$$\left(\frac{(2.58)\ (0.88)}{0.01}\right)^2 \leq n \quad \rightarrow \quad \left(\frac{227.04}{0.01}\right)^2 = 51547.16$$

3 points

Then we must run the simulation for **n > 51,548** simulation run

Question #6:

A bus driver is taking tourists on a tour every day on his own bus to visit 3 major cities: City-A, City-B, City-C. During the tour, the bus may breakdown between cities. If the bus breaks down, it needs a repair time which is a random variable as shown in the following table. The travel time between cities is a exponential. The table below shows the mean travel time, the shift parameter, probability of breakdown and the mean repair time if the bus breakdown on the road.

from-to	**Mean Travel Time**	**Probability of break down**	**Repair Time Dist.** (hours)
A to B	1 hours	0.4	shifted exponential with δ=2 and λ= 0.5
B to C	2.5 hours	0.35	Discrete Uniform[3,5]
C to A	3 hours	0.45	Erlang Dist. k=2 , λ = 0.75

(a) Write the steps and functions to simulate to the total travel time for the tour. (explain your answers fully)
(b) Do the simulation of the tour for five days using the U[0,1] in the table.
(c) From the simulation output, what is the ***average travel time of the tour***
(d) From the simulation output, what is ***average number of breakdowns***.
(e) If you take a tour on that bus and the tour started at 8:00 am. From the simulation output, estimate the probability that you will come back after the tour before 7:00 pm.

Solution: (*use the back of the page for more space*)

2 points **Process #1 :** Let T travel time
TAB → travel time A → B ; get u~U(0,1) TAB = – ln (1– u)
TBC → travel time B → C ; get u~U(0,1) TBC = – 2.5 ln (1– u)
TCA → travel time C → A ; get u~U(0,1) TCA = – 3 ln (1– u)

2 points **Process #2 :** Let Breakdowns
A–B → get u~U(0,1) if u <=0.4 then "YES- bus breakdown between A,B" Else "NO"
B –C → get u~U(0,1) if u <=0.35 then "YES- bus breakdown between B,C" Else "NO"
C–A → get u~U(0,1) if u <=0.45 then "YES- bus breakdown between C,A" Else "NO"

Process #3 : Repair time
2 points A–B → get u~U(0,1) if "YES" then RT-AB = 2 – 2 ln (1– u) Else RT-AB = 0
B –C → get u~U(0,1) if "YES" then RTAB = 3+INT(3 u) Else RT-BC = 0
C–A → get u~U(0,1) if "YES" then RTAB = –(4/3) (ln (1– u_1)+ ln (1– u_2)) Else RT-CA = 0

Write the simulation results in the following table:

4 points

Day#	Trvl Time A-B	Brkdn ??	Repair time	Trvl Time B-C	Brkdn ??	Repair time	Trvl Time C-A	Brkdn ??	Repair time	TOT Tour Time
U	0.328	0.708	0.027	0.653	0.283	0.113	0.662	0.701	0.328	
1	0.40	NO	--	2.65	**YES**	3.00	3.25	NO	--	9.30
U	0.554	0.64	0.11	0.863	0.566	0.306	0.192	0.083	0.554	
2	0.81	NO	--	4.97	NO	--	0.64	**YES**	2.15	8.57
U	0.96	0.047	0.468	0.281	0.332	0.281	0.053	0.974	0.16	
3	3.22	**YES**	3.26	0.82	**YES**	3.00	0.16	NO	--	10.47
U	0.37	0.543	0.978	0.067	0.732	0.852	0.891	0.287	0.37	
4	0.46	NO	--	0.17	NO	--	6.65	**YES**	1.23	8.52
U	0.485	0.389	0.601	0.374	0.808	0.527	0.533	0.532	0.485	
5	0.66	**YES**	3.84	1.17	NO	--	2.28	NO	--	7.96

1 points (b) Average Travel time = Sum(tour times)/5 = 43.6/5 = **8.72 hr/tour**

2 points (c) Average number of breakdowns per tour = (1 + 1 + 2 + 1 + 1) / 5 = **1.2 BRD/tour**

2 points (e) probability that you will come back after the tour before 7:00 pm = (number of days tour time is less than 11 hrs)/5 = **5/5 = 1**