King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2020–2021
Academic Year (H) 1442
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**Final Exam Spring 2021 ACTU-362-372 (20%) (two pages)**

**May 5, 2021 (three hours 9:00 –12:00 AM)**

## Exercise 1

1. *You are given* $\mu_x = \frac{2x}{10000-x^2}$ *for* $0 \leq x < 100$. *Determine the probability that an individual aged* 50 *dies within ONE YEAR.* ***(Hint*** $\int \frac{-2t}{a^2-t^2}\mathrm{d}t = \ln(a^2-t^2)$ *for* $0 \leq t < a$*).*

2. *The force of mortality is*
$$\mu_x = \frac{1}{4(100-x)}, \quad 0 \leq x \leq 100$$
*Calculate* ${}_{4|5}q_{30}$. *(Hint:* ${}_tp_x = e^{-\int_0^t \mu_{x+u}\mathrm{d}u} = e^{-\int_x^{x+t} \mu_u \mathrm{d}u}$*).*

**Solution**:

1. The required probability is $q_{50}$. We know
$$\begin{aligned} q_x &= 1 - \exp\left(-\int_x^{x+1} \mu_t \mathrm{d}t\right) = 1 - \exp\left(-\int_x^{x+1} \frac{2t}{100^2-t^2}\mathrm{d}t\right) \\ &= 1 - \frac{100^2-(x+1)^2}{100^2-x^2} = \frac{(x+1)^2-x^2}{10,000-x^2} = \frac{2x+1}{100^2-x^2} \end{aligned}$$
hence
$$q_{50} = \frac{101}{100^2-50^2} = \frac{101}{7500} = \mathbf{0.01346}.$$

2. This is beta with $\alpha = \frac{1}{4}$, $\omega = 100$, we have
$${}_{4|5}q_{30} = {}_4p_{30} - {}_9p_{30}$$
but
$${}_4p_{30} = \left(\frac{100-30-4}{100-30}\right)^{\frac{1}{4}} = 0.98540 \text{ and } {}_9p_{30} = \left(\frac{100-30-9}{100-30}\right)^{\frac{1}{4}} = 0.96618.$$
So
$${}_{4|5}q_{30} = 0.98540 - 0.96618 = \mathbf{0.01922}.$$

## Exercise 2

1. *20–year* ***pure*** *endowment* ***insurance*** *of* 5000 *on* $(x)$

2. *Whole life* ***annuity immediate*** *with annual payments of* 5000 *on* $(x)$.

3. *Whole life* ***annuity due*** *with annual payments of* 5000 *on* $(x)$.

4. 20–*year* ***temporary*** *life* ***annuity immediate*** *with annual payments of* 5000 *on* $(x)$.
   ***Write*** *the formulas of the actuarial present value of each option then* ***rank*** *the APV of each option from the* ***largest*** *to the* ***smallest****. (****Hint****: use the number 1., 2., 3., 4. to order the APV)*

**Solution**: APV are given as follows

1. $5000\ {}_{20}E_x = 5000\ v^{20}\ {}_{20}p_x$
2. $5000\ a_x = 5000\ \sum_{k=1}^{\infty} v^k\ {}_kp_x$
3. $5000\ \ddot{a}_x = 5000\ \sum_{k=0}^{\infty} v^k\ {}_kp_x$
4. $5000\ a_{x:\overline{20}|} = 5000\ \sum_{k=1}^{20} v^k\ {}_kp_x$

The annuity–due provides one more payment than the whole life annuity immediate, which provides more payments than a 20–year annuity. The pure endowment provides the least payments, only 2000 at duration 20. The order is 3. $>$ 2. $>$ 4. $>$ 1.

**Exercise 3** *For a special fully discrete* 3*–year term insurance on* $(x)$ $i = 0.03$ *level net premiums are paid at the beginning of each year. The death benefit in year* $k$ *is* $b_{k+1}$*, as listed below.*

| $k$ | 0 | 1 | 2 |
|---|---|---|---|
| $b_{k+1}$ | 60000 | 40000 | 20000 |
| $q_{x+k}$ | 0.02 | 0.06 | 0.10 |

1. *Calculate the* ***net premium***
2. *Calculate the* ***initial net premium*** *reserve for year* 2*. (initial reserve includes premium)*

**Solution**:

1. The APV of future benefits is

$$A = \frac{60000(0.02)}{1.03} + \frac{40000(0.98)(0.06)}{(1.03)^2} + \frac{20000(0.98)(0.94)(0.10)}{(1.03)^3} = 5068.0911$$

   and APV of future annuities of 1 is

$$\ddot{a}_{x:\overline{3}|} = 1 + \frac{0.98}{1.03} + \frac{(0.98)(0.94)}{(1.03)^2} = 2.8197$$

   thus the net premium is $P = \frac{5068.0911}{2.8197} = \mathbf{1797.3866}$.

2. We will calculate the net premium reserve at time 1 sung recursion

$$({}_0V + P)(1+i) = q_x\ b_1 + \ {}_1V\ p_x$$

$${}_1V = \frac{({}_0V + P)(1+i) - q_x\ b_1}{p_x} = \frac{(1797.3866)\,(1.03) - 0.02 \times 60000}{1 - 0.02} = 664.6002$$

   So the initial net premium reserve for year 2 is

$${}_1V + P = 664.6002 + 1797.3866 = \mathbf{2461.9868}.$$

**Exercise 4** *For a fully continuous whole life insurance of* 50000 *on* (40), *you are given*

$$(i) \quad \mu_{x+t} = \begin{cases} 0.004 & t < 30 \\ 0.004e^{0.25(t-30)} & t \geq 30 \end{cases}$$

*(ii)* $\delta = 0.03$ *(iii) The net premium is* $P_1$ *for the first* 30 *years and* $P_2$ *after* 30 *years. (iv) The net premium reserve at time* $t = 30$ *is* 16250.

1. *Find* $P_1$ *using retrospective formula for the reserve.*
2. *Calculate the net premium reserve at time* $t = 20$.

**Solution**:

1. Let us calculate $P_1$ using the net premium reserve at $t = 30$ We have

$${}_{30}\overline{V} = \frac{P_1 \bar{a}_{40:\overline{30}|} - 50000 \bar{A}^{\,1}_{40:\overline{30}|}}{{}_{30}E_{40}}$$

but ${}_{30}E_{40} = e^{-0.034(30)} = 0.36055$,

$$\bar{A}^{\,1}_{40:\overline{30}|} = \frac{0.004}{0.034}\left(1 - e^{-0.034(30)}\right) = 0.075224 \quad \text{and} \quad \bar{a}_{40:\overline{30}|} = \frac{1 - e^{-0.034(30)}}{0.034} = 18.80603$$

therefroe

$$16250 = \frac{18.80603 P_1 - 50000 \times 0.075224}{0.36055}.$$

Solving for $P_1$ we get $P_1 = \mathbf{511.54537}$

2. Now let us calculate the net premium reserve at $t = 20$ with the retrospective formula also.

$$\bar{A}^{\,1}_{40:\overline{20}|} = \frac{0.004}{0.034}\left(1 - e^{-0.034(20)}\right) = 0.058045 \quad \text{and} \quad \bar{a}_{40:\overline{20}|} = \frac{1 - e^{-0.034(20)}}{0.034} = 14.51126$$

hence

$${}_{20}\overline{V} = \frac{P_1 \bar{a}_{40:\overline{20}|} - 50000 \bar{A}^{\,1}_{40:\overline{20}|}}{{}_{20}E_{40}} = \frac{511.54537 \times 14.51126 - 50000 \times 0.058045}{e^{-0.034(20)}} = \mathbf{8923.7391}.$$

**Exercise 5**
*For a special* 20*–year term life insurance of* 80000 *on* (40), *you are given: (i) The death benefit is payable at the moment of death. (ii) During the* $5^{\text{th}}$ *year the gross premium is* 1200 *paid continuously at a constant rate (iii) The force of mortality follows Gompertz's law* $\mu_x = Bc^x$ *with* $B = 0.0004$ *and* $c = 1.12$. *(iv) The force of interest is* 4%. *(v) Expenses are: -* 5% *of premium payable continuously -* 200 *payable at the moment of death (vi) At the end of the* $5^{\text{th}}$ *year the expected value of the present value of future losses random variable is* 8000.

1. *Use Euler's method*

$${}_{t+h}V^g \simeq {}_tV^g + h\left(G_t - (e_t + c_t G_t) + \left(\delta_t + \mu_{x+t}\right) {}_tV^g - (b_t + E_t)\,\mu_{x+t}\right)$$

*with steps of* $h = 0.25$ *to calculate the expected value of the present value of future losses random variable at the end of* 4.5 *years.*

2. *For a fully discrete whole life insurance of* 35000 *on* $(x)$*: (i) Death is the only decrement. (ii) The annual net premium is* 2800*. (iii) The annual* ***contract premium*** *is* 3000*. (iv) Expenses in year* 1*, payable at the start of the year, are* 40% *of* ***contract premiums****. (v)* $i = 0.05$ *(vi)* ${}_1V = 1400$*. Calculate the asset share at the end of the first year assuming* ${}_0AS = 0$*.* ***(Hint:*** *use recursion for net premium* ${}_1V$ *and annual net premium to find the unknown death rate assuming* ${}_0V = 0$*).*

**Solution:**

1. The term "expected value of present value of future losses" is used instead of "**reserve**". The Euler's method leads to

$$ {}_tV^g \simeq \frac{{}_{t+h}V^g - h\left(G_t - (e_t + c_tG_t) - (b_t + E_t)\,\mu_{x+t}\right)}{1 + h\left(\delta_t + \mu_{x+t}\right)} $$

In our situaton we need to compute ${}_{4.75}V^g$ and ${}_{4.5}V^g$ hence have first to compute $\mu_{44.75}$ and $\mu_{44.5}$

$$ \mu_{44.75} = 0.0004(1.12)^{44.75} = 0.06376 \quad \text{and} \quad \mu_{44.50} = 0.0004(1.12)^{44.50} = 0.06198 $$

therefore

$$ \begin{aligned} {}_{4.75}V &= \frac{{}_5V^g - h\left((1-0.05)G - (b+E)\,\mu_{40+4.75}\right)}{1 + h\left(0.04 + \mu_{40+4.75}\right)} \\ &= \frac{8000 - 0.25(0.95 \times 1200 - 80200 \times (0.06376))}{1 + 0.25(0.04 + 0.06376)} = \mathbf{8765.998} \\ {}_{4.50}V &= \frac{8765.998 - 0.25(0.95 \times 1200 - 80200 \times (0.06198)}{1 + 0.25(0.04 + 0.06198)} = \mathbf{9481.9546}. \end{aligned} $$

2. We have

$$ ({}_0AS + G - c_hG - e_h)\,(1+i) = 35000\ q_x + {}_1AS\ p_x $$

hence

$$ \begin{aligned} {}_1AS &= \frac{({}_0AS + G - c_hG - e_h)\,(1+i) - 35000\ q_x}{p_x} \\ &= \frac{(0 + 3000 - 0.4 \times 3000)\,(1.05) - 35000\ q_x}{1 - q_x}. \end{aligned} $$

Thus we need back out $q_x$: the first–year death rate. Set ${}_1V = 1400$, by the recursive relation for net premium reserve,

$$ ({}_0V + P)(1+i) = 35000\ q_x + {}_1V\ p_x = {}_1V + (35000 - {}_1V)\ q_x $$

$$ q_x = \frac{({}_0V + P)(1+i) - {}_1V}{35000 - {}_1V} = \frac{2800 \times 1.05 - 1400}{35000 - 1400} = 0.04583. $$

Then,

$$ {}_1AS = \frac{(0.6 \times 3000)\,(1.05) - 35000 \times 0.04583}{1 - 0.04583} = \frac{(100 - 40)\,(1.1) - 1000\ q_x}{1 - q_x} = \mathbf{299.68454} $$