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Model Answer

Midterm Exam 1, Actuarial Mathematical Models 1

Exercise 1

1. *You are given* $q_{x+k} = 0.1(k+1), \quad k = 0,\ 1,\ 2,\ \ldots,\ 9$. *Calculate the following:*
   (a) $\mathrm{P}(K_x = 1)$
   (b)$\mathrm{P}(K_x \leq 2)$.

2. *You are given* $\mu_x = \mu$ *for all* $x \geq 0$.
   (a) Find an expression for $\mathrm{P}(K_x = k)$, *for* $k = 0, 1, 2, \ldots$, *in terms of* μ *and* k.
   (b) Find an expression for $\mathrm{P}(K_x \leq k)$, *for* $k = 0, 1, 2, \ldots$, *in terms of* μ *and* k.
   Now, suppose that $\mu = 0.01$
   (c) Find $\mathrm{P}(K_x = 10)$.
   (d) Find $\mathrm{P}(K_x \leq 10)$.

Solution:

1. (a)$\mathrm{P}(K_x = 1) = {}_{1|}q_x = p_x \times q_{x+1} = (1 - q_x)\, q_{x+1} = (1 - 0.1) \times 0.2 = \mathbf{0.18}$.
   (b) $\mathrm{P}(K_x = 0) = q_x = 0.1$.

$$\begin{aligned}\mathrm{P}(K_x = 2) &= {}_{2|}q_x = {}_2p_x \times q_{x+2} = p_x \times p_{x+1} \times q_{x+2} \\ &= (1 - q_x)(1 - q_{x+1})\, q_{x+2} = 0.9 \times 0.8 \times 0.3 = 0.216\end{aligned}$$

   Hence, $\mathrm{P}(K_x \leq 2) = \mathrm{P}(K_x = 0) + \mathrm{P}(K_x = 1) + \mathrm{P}(K_x = 2) = 0.1 + 0.18 + 0.216 = \mathbf{0.496}$.

2. (a) Given that $\mu_x = \mu$ for all $x \geq 0$, we have ${}_tp_x = e^{-\mu t}, p_x = e^{-\mu}$ and $q_x = 1 - e^{-\mu}$.

$$\mathrm{P}(K_x = k) = {}_{k|}q_x = {}_kp_x\ q_{x+k} = \mathbf{e^{-k\mu}\left(1 - e^{-\mu}\right)}.$$

   (b) $\mathrm{P}(K_x \leq k) = \mathrm{P}(T_x \leq k+1) = {}_{k+1}q_x = 1 - {}_{k+1}p_x = \mathbf{1 - e^{-(k+1)\mu}}$.
   (c) When $\mu = 0.01$, $\mathrm{P}(K_x = 10) = e^{-10\times 0.01}(1 - e^{-0.01}) = \mathbf{0.0090}$.
   (d) When $\mu = 0.01$, $\mathrm{P}(K_x \leq 10) = 1 - e^{-(10+1)\times 0.01} = \mathbf{0.1042}$.

Exercise 2

1. *The probability density function for the future lifetime of a life age 0 is given by*

$$f_0(x) = \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}, \quad \alpha, \lambda > 0$$

   (a) Derive an expression for μ_x.
   (b) Derive an expression for $S_x(t) = {}_tp_x$
   (c) Using (a) and (b), or otherwise, find an expression for $f_x(t)$.

2. *For each of the following equations, determine if it is correct, if it is not give the correct equation.*
   (a) ${}_{t|u}q_x = {}_tp_x \times {}_uq_{x+t}$
   (b) ${}_{t+u}q_x = {}_tq_x \times {}_uq_{x+t}$
   (c) $\frac{\mathrm{d}}{\mathrm{d}t}\left({}_tp_x\right) = {}_tp_x\left(\mu_x - \mu_{x+t}\right)$.

Solution:

1.
   (a) $\mu_x = \frac{f_0(x)}{S_0(x)} = \frac{\alpha}{\lambda+x}$.
   (b) Observe first that

$$S_0(t) = 1 - F_0(t) = 1 - \int_0^t f_0(s)\mathrm{d}s = 1 - \int_0^t \frac{\alpha\lambda^\alpha}{(\lambda+s)^{\alpha+1}}\,\mathrm{d}s = \frac{\lambda^\alpha}{(\lambda+t)^\alpha}$$

therefore

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{\left(\frac{\lambda}{\lambda+x+t}\right)^\alpha}{\left(\frac{\lambda}{\lambda+x}\right)^\alpha} = \left(\frac{\lambda+x}{\lambda+x+t}\right)^\alpha.$$

   (c) We have

$$f_x(t) = {}_tp_x\,\mu_{x+t} = \left(\frac{\lambda+x}{\lambda+x+t}\right)^\alpha \frac{\alpha}{\lambda+x+t} = \left(\frac{\alpha\left(\lambda+x\right)^\alpha}{(\lambda+x+t)^\alpha}\right)$$

2.
   (a) The equation is correct.
   (b) No, the equation is not correct. The correct equation should be ${}_{t+u}p_x = {}_tp_x \times {}_up_{x+t}$
   (c) No, the equation is not correct. The correct equation is $\frac{\mathrm{d}}{\mathrm{d}t}\left({}_tp_x\right) = -f_x'(t) = -{}_tp_x\,\mu_{x+t}$

Exercise 3

1. *For a certain individual new born, you are given:*

$$S_0(t) = \begin{cases} 1 - \frac{t}{100}, & 0 \le t < 30 \\ 0.7e^{-0.02(t-30)}, & t \ge 30 \end{cases}$$

   Calculate $\mathrm{E}\left[T_0\right]$ *for the new born individual.*

2. *You are given:*

$$\mu_x = \frac{2x}{400-x^2}, \quad 0 \le x < 20$$

   Find $\mathrm{Var}\left(T_0\right)$ *(Hint:* $\int_0^a \left(1 - \frac{t^2}{400}\right)\mathrm{d}t = a - \frac{a^3}{1200}$ *and* $\int_0^a \left(t - \frac{t^3}{400}\right)\mathrm{d}t = \frac{a^2}{1600}\left(800 - a^2\right)$*)*

3. *You are given* $\mu_x = 0.02$, $x \ge 0$. *Find* $\mathring{e}_{10:\overline{10}|}$,

Solution:

1. Since the survival function changes at $t = 30$, we need to decompose the integral into two parts.

$$\begin{aligned}
\mathrm{E}\left[T_0\right] &= \int_0^\infty {}_tp_0 \ \mathrm{d}t = \int_0^{30} \left(1 - \frac{t}{100}\right) \mathrm{d}t + \int_{30}^\infty 0.7e^{-0.02(t-30)}\mathrm{d}t \\
&= \left[t - \frac{t^2}{200}\right]_0^{30} + 0.7 \int_0^\infty e^{-0.02u} \ \mathrm{d}u \\
&= 25.5 + \frac{0.7}{0.02} = \mathbf{60.5}
\end{aligned}$$

2. We begin with the calculation of ${}_tp_0$. We have $0.02 \times 30 = 0.6$

$$\begin{aligned}
-\int_0^t \mu_u \ \mathrm{d}u &= \int_0^t \frac{-2u}{400 - u^2}\mathrm{d}u = \left[\ln\left(400 - u^2\right)\right]_0^t \\
&= \ln\left(\frac{400 - t^2}{400}\right) = \ln\left(1 - \frac{t^2}{400}\right).
\end{aligned}$$

hence ${}_tp_0 = 1 - \frac{t^2}{400}$, so

$$\begin{aligned}
\mathrm{E}\left[T_0\right] &= \int_0^{20} {}_tp_0 \ \mathrm{d}t = \int_0^{20} \left(1 - \frac{t^2}{400}\right) \mathrm{d}t \\
&= \left[\left(t - \frac{t^3}{1200}\right)\right]_0^{20} = \frac{40}{3} = \mathbf{13.3333} \\
\mathrm{E}\left[T_0^2\right] &= 2\int_0^{20} t \ {}_tp_0 \ \mathrm{d}t = 2\int_0^{20} \left(t - \frac{t^3}{400}\right) \mathrm{d}t \\
&= \left[2\left(\frac{t^2}{2} - \frac{t^4}{1600}\right)\right]_0^{20} = 200 \\
\mathrm{Var}\left(T_0\right) &= \mathrm{E}\left[T_0^2\right] - \left(\mathrm{E}\left[T_0\right]\right)^2 = 200 - \left(\frac{40}{3}\right)^2 = \mathbf{22.2222}
\end{aligned}$$

3. Since $\mu_x = 0.02$ for all $x \geq 0$, we immediately have ${}_tp_x = e^{-0.02t}$. Then,

$$\mathring{e}_{10:\overline{10}|} = \int_0^{10} {}_tp_{10} \ \mathrm{d}t = \int_0^{10} e^{-0.02t} \ \mathrm{d}t = -\left[\frac{1}{0.02}e^{-0.02t}\right]_0^{10} = \mathbf{9.063}.$$

Exercise 4

1. *You are given the following select–and–ultimate life table:*

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.01	0.04	0.07	67
66	0.03	0.06	0.09	68
67	0.05	0.08	0.12	69

(a) State the select period. (b) Calculate ${}_{1|2}q_{[65]+1}$ *(c) Calculate* ${}_{0.4}p_{[66]+0.3}$, *assuming constant force of mortality between integer ages.*

2. *You are given the following life table:*

x	91	92	93	94	95	96	97	98	99
ℓ_x	27	21	16	12	8	5	3	1	0

(Under UDD $\mathring{e}_x = \frac{1}{2} + e_x$)
(a) Calculate e_{91}
(b) Calculate $\mathring{e}_{91}$, assuming uniform distribution of deaths between integer ages.
(c) Calculate $\mathring{e}_{91}$, assuming De Moivre's law with $\omega = 99$.

Solution:

1. (a) $n = \mathbf{2}$ years.
   (b)

$$\begin{aligned} {}_{1|2}q_{[65]+1} &= p_{[65]+1} \times {}_2q_{[65]+2} = p_{[65]+1} \times {}_2q_{67} = p_{[65]+1}\left(1 - (1 - q_{67})(1 - q_{68})\right) \\ &= (1 - 0.04)(1 - (1 - 0.07)(1 - 0.09)) = \mathbf{0.147552}. \end{aligned}$$

   (c)

$$ {}_{0.4}p_{[66]+0.3} = \left(p_{[66]}\right)^{0.4} = (1 - 0.03)^{0.4} = \mathbf{0.987890}. $$

2. (a) By definition

$$\begin{aligned} e_{91} &= \sum_{k=1}^{99-91-1} {}_kp_{91} = \sum_{k=1}^{7} {}_kp_{91} = p_{91} + {}_2p_{91} + {}_3p_{91} + \cdots + {}_7p_{91} \\ &= \frac{\ell_{92} + \ell_{93} + \ldots + \ell_{98}}{\ell_{91}} = \frac{21 + 16 + 12 + 8 + 5 + 3 + 1}{27} \\ &= \frac{22}{9} = \mathbf{2.4444} \text{ years.} \end{aligned}$$

   (b) Under UDD $\mathring{e}_{91} = e_{91} + 0.5 = \frac{22}{9} + \frac{1}{2} = \frac{53}{18} = \mathbf{2.9444}$.
   (c) Under De Moivre's law with $\omega = 99$, ${}_tp_{91} = 1 - \frac{t}{\omega - 91} = 1 - \frac{t}{8}$. Therefore,

$$ \mathring{e}_{91} = \int_0^{99-91} {}_tp_{91}\, \mathrm{d}t = \int_0^8 \left(1 - \frac{t}{8}\right) \mathrm{d}t = \left[\left(t - \frac{t^2}{16}\right)\right]_0^8 = 8 - \frac{64}{16} = \mathbf{4} $$

   You can alos use the shortcut $\mathring{e}_{91} = \frac{99-91}{2} = \frac{8}{2} = 4$.

Exercise 5

1. *You are given: (i) $i = 0.05$ (ii) $q_x = 0.04$ and $q_{x+1} = 0.08$ (iii) Deaths are uniformly distributed over each year of age. Calculate $10^5 A^{1\,(12)}_{x:\overline{2}|}$ (Hint $A^{1\,(m)}_{x:\overline{n}|} = \sum_{k=0}^{nm-1} v^{\frac{k}{m}+\frac{1}{m}}\ {}_{\frac{k}{m}|\frac{1}{m}}q_x$ or under UDD $A^{1\,(m)}_{x:\overline{n}|} = \frac{i}{i^{(m)}} A^1_{x:\overline{n}|}$ and $(1 + \frac{i^{(m)}}{m})^m = 1 + i$.).*

2. *You are given: (i) Deaths are uniformly distributed over each year of age. (ii) $i = 0.07$ (iii) $q_{40} = 0.1$ and $q_{41} = 0.2$.*
   (a) Calculate $10^5 \bar{A}_{40:\overline{2}|}$. (Hint under UDD $\bar{A}^1_{x:\overline{n}|} = \frac{i}{\delta} A^1_{x:\overline{n}|}$).
   (b) Calculate $10^5 A^{(3)}_{40:\overline{2}|}$.

3. *(a) A 5–year deferred whole life insurance of* $200,000$ *on* (x) *is payable at the moment of death. You are given that*

$$\mu_{x+t} = 0.02 \text{ for all } t \geq 0 \quad \text{and } \delta_t = 0.06 \text{ for all } t \geq 0$$

Calculate the expected present value (net single premium) of this insurance.
(b) The expected present value of an n*–year term insurance paying* $10,000$ *at the moment of death to* (x) *is* 1101.34*. You are given: (i)* $\mu_{x+t} = 0.008,\ t > 0$ *(ii)* $\delta = 0.032$*. Determine* n

Solution:

1. We use the relation $A^{1\,(12)}_{x:\overline{2}|} = \frac{i}{i^{(12)}} A^{1}_{x:\overline{2}|}$ under UDD. For $i = 0.1.$ we have

$$\begin{aligned} A^{1}_{x:\overline{2}|} &= v\ q_x + v^2\ p_x\ q_{x+1} = \frac{0.04}{1.05} + \frac{0.96 \times 0.08}{1.05^2} = 0.10776 \\ \text{Also, } i^{(12)} &= 12\left((1.05)^{\frac{1}{12}} - 1\right) = 0.048889. \end{aligned}$$

So, the answer is

$$10^5 A^{1\,(12)}_{x:\overline{2}|} = 10^5 \frac{i}{i^{(12)}} A^{1}_{x:\overline{2}|} = 10^5 \frac{0.05}{0.048889} \times 0.10776 = \mathbf{11021}$$

2. (a) We have $\bar{A}_{40:\overline{2}|} = \bar{A}^{1}_{40:\overline{2}|} + {}_2E_{40} = \frac{i}{\delta} A^{1}_{40:\overline{2}|} + {}_2E_{40}$ and

$$A^{1}_{40:\overline{2}|} = v\ q_{40} + v^2\ p_{40}\ q_{41} = \frac{0.1}{1.07} + \frac{0.9 \times 0.2}{(1.07)^2} = 0.25068$$

and

$${}_2E_{40} = v^2\ {}_2p_{40} = \frac{0.9 \times 0.8}{(1.07)^2} = 0.62888$$

Hence

$$10^5 \bar{A}_{40:\overline{2}|} = 10^5 \left(\frac{0.07}{\ln(1.07)} 0.25068 + 0.62888\right) = \mathbf{88823}$$

(b) We have

$$i^{(3)} = 3\left((1+i)^{\frac{1}{3}} - 1\right) = 3\left((1.07)^{\frac{1}{3}} - 1\right) = 0.06843.$$

$$A^{1\,(3)}_{40:\overline{2}|} = \frac{i}{i^{(3)}} A^{1}_{40:\overline{2}|} = \frac{0.07}{0.06843} \times 0.25068 = 0.25643$$

So,

$$10^5\ A^{(3)}_{40:\overline{2}|} = 10^5\ \left(A^{1\,(3)}_{40:\overline{2}|} + {}_2E_{40}\right) = 10^5\,(0.25643 + 0.62888) = 10^5 \times 0.88531 = \mathbf{88531}.$$

3. (a) The APV of the insurance is given by 200000 $\bar{A}_{[}5|]x$ where

$$\bar{A}_{[}5|]x = \int_5^\infty e^{-\delta t}\ {}_tp_x\ \mu_{x+t}\ dt = \int_5^\infty e^{-\delta t}\ e^{-\mu t}\ \mu\ dt = \frac{\mu}{\mu+\delta} e^{-5(\mu+\delta)} = \frac{0.02}{0.08} e^{-5\times 0.08} = 0.16758$$

so the APV is $200000 \times 0.16758 = \mathbf{33516}$.
(b) We know that $\bar{A}^{1}_{x:\overline{n}|} = \frac{\mu}{\mu+\delta}\left(1 - e^{-n(\mu+\delta)}\right) = \frac{0.008}{0.04}\left(1 - e^{-0.04\times n}\right) = 0.1101.34,$ which gives $n = \mathbf{20}$.