King Saud University
College of Sciences
Mathematics Department

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Model Answer of the final exam ACTU–362 Fall 2019 (40%) (two pages)

January 1, 2020 (three hours 8–11 AM)

Problem 1. **(8 marks)**

1. **2 marks** You are given $\mu_x = 0.01$ for all $x \geq 0$. Find a. $P(K_x = 10)$ and b. $P(K_x \leq 10)$, where $K_x = \lfloor T_x \rfloor$ is the curtate function of the future life time T_x.
2. **2 marks** You are given $\int_{70}^{75} \mu_u du = 0.22314$. Calculate ${}_5q_{70}$.
3. **2 marks** The force of mortality of live 50 is given by $\mu_x = 0.05$ for $50 \leq x < 60$ and $\mu_x = 0.04$ for $60 \leq x < 70$. Calculate ${}_{15|5}q_{50}$.
4. **2 marks** The force of mortality of live (x) is $\mu_x = \frac{1}{100-x}$ for $0 \leq x < 100$, calculate ${}_{10}p_{50}$.

Solution:

1. a. $P(K_x = 10) = {}_{10|}q_x = {}_{10}p_x q_{x+10} = {}_{10}p_x (1 - p_{x+10}) = e^{-0.01\times 10}(1 - e^{-0.01}) = \mathbf{0.0090033}$.
   b. $P(K_x \leq 10) = {}_{10+1}q_x = 1 - {}_{11}p_x = 1 - e^{-0.01\times 11} = \mathbf{0.10417}$.
2. We know ${}_5q_{70} = 1 - {}_5p_{70} = 1 - e^{-\int_{70}^{75}\mu_u du} = 1 - e^{-0.22314} = \mathbf{0.20}$.
3. We have

$$\begin{aligned} {}_{15|5}q_{50} &= {}_{15}p_{50} - {}_{20}p_{50} = {}_{10}p_{50}\ {}_5p_{60} - {}_{10}p_{50}\ {}_{10}p_{60} = {}_{10}p_{50}\left({}_5p_{60} - {}_{10}p_{60}\right) \\ &= e^{-\int_{50}^{60}\mu_u du}\left(e^{-\int_{60}^{65}\mu_u du} - e^{-\int_{60}^{70}\mu_u du}\right) = e^{-10\times 0.05}\left(e^{-5\times 0.04} - e^{-10\times 0.04}\right) = \mathbf{0.090016}. \end{aligned}$$

4. ${}_{10}p_{50} = e^{-\int_{50}^{60}\mu_u du} = e^{-\int_{50}^{60}\frac{1}{100-u}du} = e^{[\ln(100-u)]_{50}^{60}} = e^{\ln\left(\frac{100-60}{100-50}\right)} = \frac{40}{50} = \frac{4}{5} = \mathbf{0.8}$ or directly by short cuts for De Moire's law ${}_{10}p_{50} = 1 - \frac{10}{50} = \mathbf{0.8}$.

Problem 2. **(8 marks)**

1. **2 marks** Assume that Deaths are uniformly distributed between integral ages and $\mu_{50.4} = 0.01$. Calculate ${}_{0.6}q_{50.4}$
2. **4 marks** Given the following 2–year Select–and–Ultimate Table

$[x]$	$1000q_{[x]}$	$1000q_{[x]+1}$	$1000q_{x+2}$	$\ell_{[x]}$	$\ell_{[x]+1}$	ℓ_{x+2}	$x+2$
30	0.222	0.330	0.422	9906.7380	9904.5387	9901.2702	32
31	0.234	0.352	0.459	9902.8941	9900.5769	9897.0919	33
32	0.250	0.377	0.500	9898.7547	9896.2800	9892.5491	34
33	0.269	0.407	0.545	9894.2903	9891.6287	9887.6028	35
34	0.291	0.441	0.596	9889.4519	9886.5741	9882.2141	36

   Calculate a. ${}_2p_{[30]}$, b. ${}_5p_{[30]}$, c. ${}_{1|}q_{[31]}$, d. ${}_3q_{[31]+1}$.

3. **2 marks** The index of selection is defined by $I(k,x) = 1 - \frac{q_{[x]+k}}{q_{x+k}}$, calculate a. $I(0,32)$ and b. $I(1,32)$.

Solution:

1. Notice that $\mu_{50.4} = \frac{q_{50}}{1-0.4q_{50}}$ while ${}_{0.6}q_{50.4} = \frac{0.6q_{50}}{1-0.4q_{50}}$,so ${}_{0.6}q_{50.4} = 0.6(0.01) = \mathbf{0.006}$.

2. a. ${}_2p_{[30]} = \frac{\ell_{[30]+2}}{\ell_{[30]}} = \frac{\ell_{32}}{\ell_{[30]}} = \frac{9901.2702}{9906.7380} = \mathbf{0.99945}$,
   b. ${}_5p_{[30]} = \frac{\ell_{[30]+5}}{\ell_{[30]}} = \frac{\ell_{35}}{\ell_{[30]}} = \frac{9887.6028}{9906.7380} = \mathbf{0.99807}$,
   c. ${}_{1|}q_{[31]} = \frac{\ell_{[31]+1}-\ell_{[31]+2}}{\ell_{[31]}} = \frac{\ell_{[31]+1}-\ell_{33}}{\ell_{[31]}} = \frac{9900.5769-9897.0919}{9902.8941} = \mathbf{0.00035192}$,
   d. ${}_3q_{[31]+1} = 1 - \frac{\ell_{[31]+4}}{\ell_{[31]+1}} = 1 - \frac{\ell_{35}}{\ell_{[31]+1}} = 1 - \frac{9887.6028}{9900.5769} = \mathbf{0.0013104}$.

3. By definition of the index of selection a. $I(0,32) = 1 - \frac{q_{[32]}}{q_{32}} = 1 - \frac{0.000250}{0.000422} = \mathbf{0.40758}$ and
   b. $I(1,32) = 1 - \frac{q_{[32]+1}}{q_{33}} = 1 - \frac{0.377}{0.459} = \mathbf{0.17865}$.

Problem 3. **(8 marks)**

1. **2 marks** A special term insurance policy pays 1000 at the end of the year of death for the first 10 years and 500 at the end of the year of death for the next 10 years. Mortality follows the Illustrative Life Table and $i = 0.06$. Calculate the actuarial present value of a policy on (30).

2. **2 marks** The actuarial present value of an n–year term insurance paying 1000 at the moment of death to (x) is 570.80. Determine n given that $\mu_{x+t} = 0.07$, $t > 0$ and $\delta = 0.05$.

3. **2 marks** A man, age 46, purchases a 3–year endowment insurance with benefits payable at the end of the year of death. The benefit is given by the formula: $b_k = 1000(1.06)^k$ for years $k = 1, 2, 3$. Mortality is described in the following table:

Age	45	46	47	48	49
ℓ_x	600	590	575	550	–

   Calculate the net single premium for this insurance for $i = 0.03$.

4. **2 marks** A continuous whole life insurance is issued to (50). Z is the present value random variable for this insurance. You are given: (i) Mortality follows a uniform distribution with the limiting age $\omega = 100$. (ii) Simple interest with $i = 0.01$ (Simple interest means money accumulates linearly so that 1 becomes $1 + it$ at time t.) and $b_t = 1000 - 0.1t^2$. Calculate $E[Z]$ (*Hint*: write b_t as $1000(1-(0.01t)^2)$ and use the identity $a^2 - b^2 = (a-b)(a+b)$).

Solution:

1. The actuarial present value of the insurance can be written as

$$\begin{aligned} 500A^{1}_{30:\overline{20}|} + 500A^{1}_{30:\overline{10}|} &= 500\left(A^{1}_{30:\overline{20}|} + A^{1}_{30:\overline{10}|}\right) = 500\left(A_{30} - {}_{20}E_{30}A_{50} + A_{30} - {}_{10}E_{30}A_{40}\right) \\ &= 500\left(2A_{30} - {}_{20}E_{30}A_{50} - {}_{10}E_{30}A_{40}\right) \\ &= 500\left(2 \times 0.102\,48 - (0.29374)(0.24905) - (0.54733)(0.16132)\right) = \mathbf{21.754}. \end{aligned}$$

2. We know that $\text{APV(FB)}_0 = 570.80 = 10000\frac{\mu}{\mu+\delta}\left(1 - e^{-n(\mu+\delta)}\right) = \frac{70}{0.07+0.05}\left(1 - e^{-0.12n}\right) = \frac{70}{0.12}\left(1 - e^{-0.12n}\right)$, then $n = \mathbf{32}$.

3. The net single premium is equal to $\mathrm{APV(FB)}_0$ hence

$$\begin{aligned}
\mathrm{APV(FB)}_0 &= \sum_{k=0}^{2} b_{k+1}v^{k+1}\ {}_{k|}q_{46} + b_3 v^3\ {}_3p_{46} \\
&= 1000\left(\frac{1.06}{1.03}\frac{\ell_{46}-\ell_{47}}{\ell_{46}} + \left(\frac{1.06}{1.03}\right)^2\frac{\ell_{47}-\ell_{48}}{\ell_{46}} + \left(\frac{1.06}{1.03}\right)^3\frac{\ell_{48}-\ell_{49}}{\ell_{46}} + \left(\frac{1.06}{1.03}\right)^3\frac{\ell_{49}}{\ell_{46}}\right) \\
&= 1000\left(\frac{1.06}{1.03}\frac{\ell_{46}-\ell_{47}}{\ell_{46}} + \left(\frac{1.06}{1.03}\right)^2\frac{\ell_{47}-\ell_{48}}{\ell_{46}} + \left(\frac{1.06}{1.03}\right)^3\frac{\ell_{48}}{\ell_{46}}\right) \\
&= (1000)\left(\frac{1.06}{1.03}\frac{590-575}{590} + \left(\frac{1.06}{1.03}\right)^2\frac{575-550}{590} + \left(\frac{1.06}{1.03}\right)^3\frac{550}{590}\right) = \mathbf{1087.1}
\end{aligned}$$

4. By definition of the expected present value of Z

$$\begin{aligned}
E\left[Z\right] &= \int_0^{100-50} b_t v^t f_{50}(t)dt = \int_0^{50}\frac{1000-0.1t^2}{1+0.1t}\frac{1}{50}dt \\
&= 1000\int_0^{50}\frac{(1+0.01t)(1-0.01t)}{1+0.01t}\frac{1}{50}dt = 20\int_0^{50}(1-0.01t)\,dt = \mathbf{750}.
\end{aligned}$$

<u>Problem 4.</u> (8 marks)

1. **(2 marks)** Mortality follows the Illustrative Life Table. The interest rate is $i = 0.06$. Calculate $\ddot{a}_{35:\overline{30}|}$.
(*Hint*: use the factorization ${}_{n+m}E_x = {}_nE_x\ {}_mE_{x+n}$)

2. You are given the following:
The probability that a newborn lives to be 25 is 70%
The probability that a newborn lives to be 35 is 50%
The following annuities–due each have an expected present value (EPV or APV) equal to $60,000$
a life annuity–due of $7,500$ on (25)
a life annuity–due of $12,300$ on (35)
a life annuity–due of $9,400$ on (25) that makes at most 10 payments

   (a) **(2 marks)** Find the relationship between these three annuities

   (b) **(2 marks)** What is the interest rate?

3. **(2 marks)** For a 10–year certain–and–life annuity on (x) paying at a continuous rate of 1 per year, you are given: $\mu_{x+t} = 0.01t = \frac{t}{100}$ for $t < 5$ and $\mu_{x+t} = 0.05$ for $t > 5$ and $\delta = 0.02$. Calculate the actuarial present value for this annuity.

Solution:

1. We have $\ddot{a}_{35:\overline{30}|} = \ddot{a}_{35} - {}_{30}E_{35}\ddot{a}_{65} = 15.3926 - {}_{30}E_{35}9.8969$. But

$${}_{30}E_{35} = {}_{10}E_{35}\ {}_{20}E_{45} = (0.54318)(0.25634) = 0.13924.$$

Thus $\ddot{a}_{35:\overline{30}|}{=}15.3926 - (0.13924)(9.8969) = \mathbf{14.0146}$.

2. We have $7500\ddot{a}_{25} = 60000$, $12300\ddot{a}_{35} = 60000$ and $9400\ddot{a}_{25:\overline{10}|} = 60000$.

(a) The 3 annuities–due are given by: then $\ddot{a}_{25} = \frac{60000}{7500} = 8$, $\ddot{a}_{35} = \frac{60000}{12300} = 4.878$ and $\ddot{a}_{25:\overline{10}|} = \frac{60000}{9400} = 6.383$. Now, we use the decomposition: whole life annuity is equal to the sum of a temporary and deferred annuity that is

$$\ddot{a}_{25} = \ddot{a}_{25:\overline{10}|} + {}_{10|}\ddot{a}_{25} = \ddot{a}_{25:\overline{10}|} + {}_{10}E_{25}\ddot{a}_{35}$$

(b) We have ${}_{25}p_0 = S_0(25) = 0.7$ and ${}_{35}p_0 = S_0(35) = 0.5$, so ${}_{10}p_{25} = S_{10}(25) = \frac{S_0(35)}{S_0(25)} = \frac{0.5}{0.7} = \frac{5}{7}$. Therefore ${}_{10}E_{25} = v^{10}\ {}_{10}p_{25} = v^{10}\frac{5}{7}$. Thus

$$8 = 6.383 + v^{10}\frac{5}{7}4.878 \iff 3.4843v^{10} = 1.617$$

$v^{10} = \frac{1.617}{3.4843} = (0.46408)^{\frac{1}{10}} = 0.9261$, finally $i = \frac{1}{0.9261} - 1 = \mathbf{0.079797}$.

3. APV certain and life annuity of 1 on (x) is

$$\bar{a}_{\overline{x:\overline{10}|}} = \bar{a}_{\overline{10}|} + {}_{10|}\bar{a}_x = \bar{a}_{\overline{10}|} + {}_{10}E_x\bar{a}_{x+10}.$$

The 10–year certain annuity has present value $\bar{a}_{\overline{10}|} = \frac{1-e^{-\delta 10}}{\delta} = \frac{1-e^{-0.2}}{0.02} = 9.0635$ and $\bar{a}_{x+10} = \frac{1}{0.05+0.02} = 14.286$. Moreover

$$\begin{aligned} {}_{10}E_x &= v^{10}\ {}_{10}p_x = v^{10}\ {}_5p_x\ {}_5p_{x+5} = e^{-0.2}e^{-\int_0^5 0.01tdt}e^{--\int_0^5 0.05dt} \\ &= e^{-0.2}e^{-\int_0^5 0.01tdt}e^{-\int_0^5 0.05dt} = e^{-0.2}e^{-0.01\frac{5^2}{2}}e^{-0.05\times 5} = 0.5627. \end{aligned}$$

Thus $\bar{a}_{\overline{x:\overline{10}|}} = 9.0635 + 0.5627 \times 14.286 = \mathbf{17.102}$.

<u>Problem 5.</u> (8 marks)

1. **(2 marks)** For a fully continuous whole life policy with benefit 800,000 on a life aged 40, you are given:
(i) Survival follows De Moivre's law with $\omega = 110$. (ii) $i = 7\%$. Calculate the net premium.

2. **(2 marks)** An insured, age 25, purchases a 10–year continuous payment, continuous whole life insurance policy with a benefit of 10^6. You are given that the insured is subject to a constant force of mortality equal to 0.025 and a constant force of interest equal to 0.075. Determine the net annual premium for this policy using equivalence principle.

3. For a life age (35) whose mortality follows $\mu_x = 0.025$ with $\delta = 0.05$. **Find** the 25^{th}–percentile premium for:

   (a) **(2 marks)** a 10–year payment whole life insurance of 50000.

   (b) **(1 mark)** a 20–year term life insurance of 50000.

   (c) **(1 mark)** a 20–year endowment life insurance of 100000.

Solution:

1. By definition $\overline{P}\left(\overline{A}_x\right) = \frac{\delta\overline{A}_x}{1-\overline{A}_x}$. We have $\delta = \ln(1.07) = 0.06766$ and

$$\overline{A}_{40} = \frac{\ddot{a}_{\overline{110-40}|}}{110-40} = \frac{1-(1.07)^{-70}}{70\times 0.06766} = 0.20929.$$

So the premium of this policy is given by

$$800000\overline{P}\left(\overline{A}_{40}\right) = 800000\frac{0.06766\times 0.20929}{1-0.20929} = \mathbf{14327}.$$

2. We know that under CFM

$$\begin{aligned} P &= \frac{10^6 \bar{A}_{25}}{\bar{a}_{25:\overline{10}|}} = 10^6 \frac{\frac{\mu}{\mu+\delta}}{\frac{1-e^{-10(\mu+\delta)}}{\mu+\delta}} = 10^6 \frac{\mu}{1-e^{-10(\mu+\delta)}} \\ &= 10^6 \frac{0.025}{1-e^{-10\times 0.1}} = 10^6 \frac{0.25}{6.3212} = \mathbf{39549}. \end{aligned}$$

3. The distribution of T_{35} is exponential with parameter 0.025. We know that the c.d.f. $F_{35}(t) = 1 - e^{-0.025t}$.

   (a) Solving $F_{35}(t_{0.25}) = 0.25$, we get $t_{0.25} = 11.507$. Then the 25^{th} –percentile premium for a 10–year payment whole life insurance of 50000 on (35) is given by since $t_{0.25} > 10$

$$P_{0.25} = \frac{Se^{-\delta t_{0.25}}}{\bar{a}_{\overline{10}|}} = \frac{50000e^{-0.05\times 11.507}}{\frac{1-e^{-0.05\times 10}}{0.05}} = \mathbf{3574}.$$

   (b) We have ${}_{20}q_{35} = F_{35}(20) = 1 - e^{-0.025\times 20} = 0.393 > 0.25$ that $20 > t_{0.25}$ so the 25^{th}–percentile premium of a 20–year term life insurance of 50000 on (35) the premium is

$$P_{0.25} = \frac{S}{\bar{s}_{\overline{t_{0.25}}|}} = \frac{50000 \times 0.05}{e^{0.05\times 11.507} - 1} = \mathbf{3214.4}.$$

   (c) Since $20 > t_{0.25}$, so the 25^{th}–percentile premium of a 20–year endowment life insurance of $100000 = 2 \times 50000$ is $P_{0.25} = 2(3214.4) = \mathbf{6428.8}$.