King Saud University
College of Sciences
Mathematics Department

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Solution of the final exam ACTU-464-474 Fall 2020 (40%)

December 16, 2020 (three hours: from 1 to 4 PM)

Problem P1. **(8 marks)**

1. **(2 marks)** The probability that a property **will not be damaged** in the next year is 30%. The probability density function (p.d.f.) f of a positive loss is proportional to $e^{-0.05x}$ for $x > 0$. Find f.

2. **(2 marks)** The owner of the property has a utility function given by $u(x) = -e^{-0.02x}$. Calculate the expected loss and the maximum insurance premium the property **owner** will pay for complete insurance.

3. **(2 marks)** The property owner in the Q2 is offered an insurance policy that will pay, $X - 100$ for $X > 100$ only, during the next year. Calculate the maximum premium that the property **owner** will pay for this insurance.

4. **(2 marks)** Find also the minimum premium that the insurer will ask for this reinsurance with retention level equals to 100. Is there will be a deal ?

Solution:

1. The p.d.f. is of the form $f(x) = ce^{-0.05x}$ for $x > 0$ such that $\int_0^\infty ce^{-0.05x} = 0.7$. This implies that $c = 0.7 \times 0.05 = 0.035$. Hence $f(x) = 0.035e^{-0.05x}$ for $x > 0$.

2. The expected loss is given by

$$\mathrm{E}[X] = 0.3(0) + 0.7\int_0^\infty x(0.05e^{-0.05x})dx = \frac{0.7}{0.05} = 14.$$

The maximum premium is given as the solution of the following equation:

$$\begin{aligned} u(W - P^+) &= 0.3u(W) + \int_0^\infty u(W-x)f(x)dx \text{ hence} \\ -e^{-0.02(W-P^+)} &= -0.3e^{-0.02W} - 0.7\int_0^\infty e^{-0.02(W-x)}(0.05e^{-0.05x})dx \\ &= -e^{-0.02W}\left(0.3 + 0.7\int_0^\infty 0.05e^{-0.03x}dx\right). \end{aligned}$$

Simplify with $-e^{-0.02W}$ in each term we get

$$e^{0.02P^+} = 0.3 + 0.7 \times \frac{5}{3} = 1.4667 \iff P^+ = \frac{\ln(1.4667)}{0.02} = 50\ln(1.4667) = \mathbf{19.151}.$$

Observe that $P^+ = 19.151 > \mathrm{E}[X] = 14$.

3. The maximum premium that the property owner will pay for this partial insurance or excess loss insurance / reinsurance is given by the equation

$$\begin{aligned} u(W-P^+) &= 0.3u(W)+\int_0^\infty u\left(W-\max(x-100;0)\right)f(x)dx \text{ hence} \\ -e^{-0.02\left(W-P^+\right)} &= -0.3e^{-0.02W}-0.7\int_0^\infty e^{-0.02(W-\max(x-100;0))}(0.05e^{-0.05x})dx. \end{aligned}$$

After simplifications by $-e^{-0.02W}$ and the fact that

$$\begin{aligned} \int_0^\infty e^{0.02(\max(x-100;0))}(0.05e^{-0.05x})dx &= \int_0^{100} 0.05e^{-0.05x}dx+\int_{100}^\infty e^{0.02(x-100)}(0.05e^{-0.05x})dx \\ &= 1-e^{-5}+e^{-2}\int_{100}^\infty 0.05e^{-0.03x}dx = 1.0045. \end{aligned}$$

therefore,

$$e^{0.02P^+} = 0.3+(0.7)\,(1.0045) = 1.0032 \iff P^+ = 50\ln\left(1.0032\right) = \mathbf{0.15974}.$$

Remark that

$$\mathrm{E}\left[\max(X-100;0)\right] = 0.035\int_{100}^\infty (x-100)e^{-0.05x}dx = 0.094331 < P^+ = \mathbf{0.15974}.$$

4. The minimum premium P^- is given by

$$\begin{aligned} u(W) &= 0.3u(W+P^-)+\int_0^\infty u\left(W+P^- -\max(x-100;0)\right)f(x)dx \text{ hence} \\ -e^{-0.02W} &= -0.3e^{-0.02\left(W+P^-\right)}-0.7\int_0^\infty e^{-0.02\left(W+P^- -\max(x-100;0)\right)}(0.05e^{-0.05x})dx. \end{aligned}$$

After simplifications by $e^{-0.02\left(W+P^-\right)}$ and the fact that

$$\int_0^\infty e^{0.02(\max(x-100;0))}(0.05e^{-0.05x})dx = 1.2076$$

therefore,

$$e^{0.02P^-} = 0.3+(0.3)(1.2076) \iff P^- = \mathbf{0.15974}.$$

Remark that $P^+ = P^-$.

Problem P2. (8 marks)

A group life insurance contract covering independent lives is rated in the three age groups as given in the table below.

Age group	Number in age group	Probability of claim per life	claim amounts
$20-35$	400	0.02	50
$36-50$	300	0.05	20
$51-70$	200	0.15	30

1. **(2 marks)** Find the mean and the variance of the aggregate claims.

2. **(2 marks)** Using the normal approximation to determine the **quantile premium** with a risk of 5% that the insurer will collect from all policyholders.

3. **(2 marks)** Complete the table corresponding to the possible payment of the reinsurer for $M = 25$

Age group	Number in age group	Probability of claim per life	claim amounts
$20-35$	400	0.02	
$36-50$	300	0.05	
$51-70$	200	0.15	

4. **(2 marks)** Using the normal approximation to determine the **quantile premium** with a risk of 5% that a reinsurer will collect from all policyholders if single losses are subject to a deductible of 25.

Solution:

1. The mean is given by

$$\mathrm{E}[S] = 400(0.02)(50) + 300(0.05)(20) + 200(0.15)(30) = \mathbf{1600}$$

The variance is given by

$$\mathrm{Var}(S) = 400 \times 50^2 \times 0.02 \times 0.98 + 300 \times 20^2 \times 0.05 \times 0.95 + 200 \times 30^2 \times 0.15 \times 0.85 = \mathbf{48250}.$$

2. We have

$$\begin{aligned} \mathrm{P}\left(S > \Pi_{\mathrm{Qua}}(0.05)\right) &= 0.05 \iff \mathrm{P}\left(S \le \Pi_{\mathrm{Qua}}(0.05)\right) = 0.95 \\ &\iff \mathrm{P}\left(Z \le \frac{\Pi_{\mathrm{Qua}}(0.05) - 1600}{\sqrt{48250}}\right) = 0.95 \end{aligned}$$

which gives $\frac{\Pi_{\mathrm{Qua}}(0.05)-1600}{\sqrt{48250}} = 1.644854$, hence $\Pi_{\mathrm{Qua}}(0.05) = 1600 + 1.644854\sqrt{48250} = \mathbf{1961.307}$.

3. The possible payments of the reinsurer

Age group	Number in age group	Probability of claim per life	claim amounts
$20-35$	400	0.02	$50-25=\mathbf{25}$
$36-50$	300	0.05	0
$51-70$	200	0.15	$30-25=\mathbf{5}$

4. We have

$$\mathrm{E}[S] = 400(0.02)(25) + 200(0.15)(5) = 350$$

and

$$\mathrm{Var}(S) = 400 \times 25^2 \times 0.02 \times 0.98 + 200 \times 5^2 \times 0.15 \times 0.85 = 5537.50$$

$$\begin{aligned} \mathrm{P}\left(S > \Pi_{\mathrm{Qua}}(0.05)\right) &= 0.05 \iff \mathrm{P}\left(S \le \Pi_{\mathrm{Qua}}(0.05)\right) = 0.95 \\ &\iff \mathrm{P}\left(Z \le \frac{\Pi_{\mathrm{Qua}}(0.05) - 350}{\sqrt{5537.50}}\right) = 0.95 \end{aligned}$$

which gives $\frac{\Pi_{\mathrm{Qua}}(0.05)-350}{\sqrt{5537.50}} = 1.644854$, hence $\Pi_{\mathrm{Qua}}(0.05) = 350 + 1.644854\sqrt{5537.50} = \mathbf{472.40}$.

Problem P3. (8 marks)

An insurance portfolio produces N claims with the following distribution:

n	0	1	2
$\mathrm{P}(N = n)$	0.1	0.5	0.4

Individual claim amounts have the following distribution:

x	0	10	20
$f_X(x)$	0.7	0.2	0.1

Individual claim amounts and claim counts are independent.

1. **(2 marks)** Find the expected aggregate claims.
2. **(4 marks)** Find $\mathrm{P}(S > 20)$.
3. **(2 marks)** Calculate the probability that the **ratio** of aggregate claim amounts to expected aggregate claim amounts will exceed 4.

Solution:

1. Aggregate claim amount S has a compound distribution with expected value

$$\mathrm{E}[S] = \mathrm{E}[N]\mathrm{E}[X] = (0.5 + 2 \times 0.4)(10 \times 0.2 + 20 \times 0.1) = 5.2$$

2. Since each claim is either 0 or a multiple of 10, we see that the only way that S can be greater than 20 is if there are $N = 2$ claims, and at least one of them is of size 20. If there are 2 claims, there are three combinations that result in $S > 20.8$. These are
(i) $X_1 = 10$ and $X_2 = 20$ (with probability $0.2 \times 0.1 = 0.02$
(ii) $X_1 = 20$ and $X_2 = 10$ (with probability $1 \times 0.2 = 0.02$), and
(iii) $X_1 = 20$ and $X_2 = 20$ (with probability $0.1 \times 0.1 = 0.01$). Hence the total probability that two claims total more than 20.8 is $0.02 + 0.02 + 0.01 = 0.05$. Therefore

$$\begin{aligned} \mathrm{P}(S > 20) &= \mathrm{P}(N = 2;\ X_1 + X_2 > 20) \\ &= \mathrm{P}(X_1 + X_2 > 20 \mid N = 2)\,\mathrm{P}(N = 2) = (0.05)(0.4) = \mathbf{0.02}. \end{aligned}$$

3. The ratio of aggregate claims to expected aggregate claims is $\frac{S}{5.2}$. Then

$$\mathrm{P}\left(\frac{S}{5.2} > 4\right) = \mathrm{P}(S > 20.8) = \mathrm{P}(S > 20) = \mathbf{0.02}.$$

Problem P4. (9 marks)

For an insured portfolio, you are given:
(i) the number of claims has a geometric distribution with $\beta = \frac{1}{3}$,
(ii) individual claim amounts can take on values 3, 4 or 5, with equal probability,
(iii) the number of claims and claim amounts are independent, and
(iv) the premium charged equals expected aggregate claims plus the variance of aggregate claims.
The p.m.f. of a geometric distribution with β is given by $p_n = \mathrm{P}(N = n) = \frac{\beta^n}{(1+\beta)^{n+1}}$ for $n \geq 0$.

1. **(2 marks)** Find $F_X^{*n}(8) = \mathrm{P}(X_1 + X_2 + \ldots + X_n \le 8)$ for $n \ge 3$.
2. **(2 marks)** Find $F_X^{*2}(8)$.
3. **(2 marks)** Calculate $F_S(8)$ (Hint $F_S(x) = \sum_{n=0}^{\infty} F_X^{*n}(x)p_n$).
4. **(2 marks)** Calculate the expected value and the variance of the aggregate claims.
5. **(1 marks)** Determine the exact probability that aggregate claims exceeds the premium.

Solution:

1. We have $F_X^{*n}(8) \ge F_X^{*3}(8) = 0$ for $n \ge 3$ (each claim is at least amount 3, so 3 or more claims must total at least 9).

2. By definition

$$\begin{aligned}
F_X^{*2}(8) &= \mathrm{P}(X_1 + X_2 \le 8) \\
&= \mathrm{P}(X_1 + X_2 \le 8; X_2 = 3) + \mathrm{P}(X_1 + X_2 \le 8; X_2 = 4) + \mathrm{P}(X_1 + X_2 \le 8; X_2 = 5) \\
&= \mathrm{P}(X_1 \le 5 \mid X_2 = 3)\,\mathrm{P}(X_2 = 3) + \mathrm{P}(X_1 \le 4 \mid X_2 = 4)\,\mathrm{P}(X_2 = 4) \\
&\quad + \mathrm{P}(X_1 \le 3 \mid X_2 = 5)\,\mathrm{P}(X_2 = 5) \\
&= \mathrm{P}(X_2 = 3) + \mathrm{P}(X_1 \le 4)\,\mathrm{P}(X_2 = 4) + \mathrm{P}(X_1 \le 3)\,\mathrm{P}(X_2 = 5) \\
&= \frac{1}{3} + \frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3} = \mathbf{\frac{2}{3}} = \mathbf{0.66667}.
\end{aligned}$$

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3. Observe that $F_X^{*0}(8) = \mathbf{1}$ and $F_X^{*1}(8) = \mathbf{1}$, hence

$$\begin{aligned}
F_S(8) &= \sum_{n=0}^{\infty} F_X^{*n}(8)p_n = \sum_{n=0}^{2} F_X^{*n}(8)p_n = p_0 + p_1 + \frac{2}{3}p_2 \\
&= (1)\frac{3}{4} + (1)\frac{3}{16} + \frac{2}{3}\frac{3}{64} + 0 = \mathbf{\frac{31}{32}} = \mathbf{0.96875}
\end{aligned}$$

4. $\mathrm{E}[S] = \mathrm{E}[N]\mathrm{E}[X] = \frac{1}{3} \times 4 = \mathbf{\frac{4}{3}} = \mathbf{1.3333}$,

$$\mathrm{Var}(S) = \mathrm{E}[N]\,\mathrm{Var}(X) + \mathrm{Var}(N)\,(\mathrm{E}[X])^2 = \frac{1}{3} \times \frac{2}{3} + \frac{4}{9} \times (4)^2 = \mathbf{\frac{22}{3}} = \mathbf{7.3333}.$$

5. The Premium is $= \frac{4}{3} + \frac{22}{3} = \frac{26}{3} = 8.6667$. Since S is integer–valued we have

$$\mathrm{P}\left(S > \frac{26}{3}\right) = \mathrm{P}(S > 8) = 1 - \mathrm{P}(S \le 8) = 1 - F_S(8) = 1 - \frac{31}{32} = \frac{1}{32} = \mathbf{0.03125}.$$

<u>**Problem P5.**</u> **(7 marks)**

1. Find the **quantile premium** with a risk of 5% for each of the following loss random variables:
   (i) **(2 marks)** the random loss X is exponential with mean $\theta = 500$.
   (ii) **(2 marks)** the random loss X is Pareto with parameters $\alpha = 3$ and $\theta = 500$.

2. **(2 marks)** The c.d.f. of aggregate losses S covered under a policy of stop–loss insurance is given by $F_S(x) = 1 - \frac{4}{x^2},\ x > 2$. Calculate the expected value of **1000** times the insurance payment if the stop–loss parameter is 10.

3. **(1 marks)** The marginal survival function in a frailty model is given to be

$$S_X(x) = 2^{-5} M_\Lambda(x).$$

The frailty random variable Λ is a Gamma random variable with parameters 5 and 200. Determine $A(x)$. (Hint $S_X(x) = M_\Lambda(-A(x))$).

Solution:

1. (i) the equation $S_X(x) = e^{-\frac{x}{\theta}} = 0.05$ implies that $x = -\theta \ln(0.05) = 2.99557\theta$ hence $\Pi_{0.05} = 2.9957 \times 500 = \mathbf{1497.85}$.
(ii) the equation $S_X(x) = \left(\frac{\theta}{x+\theta}\right)^\alpha = 0.05$ implies that $0.05 = \left(\frac{500}{x+500}\right)^3$, hence $\Pi_{0.05} = \mathbf{857.21}$.

2. We have

$$\mathrm{E}\left[(S-10)^+\right] = \int_{10}^{\infty} (x-10)\, f_S(x)dx = \int_{10}^{\infty} (x-10)\, \frac{8}{x^3} dx = \frac{2}{5}$$

thus

$$\mathrm{E}\left[1000(S-10)^+\right] = \frac{2000}{5} = \mathbf{400}.$$

Alternatively we can use

$$\mathrm{E}\left[1000(S-10)^+\right] = 1000 \int_{10}^{\infty} (1 - F_S(x))\, dx = 1000 \int_{10}^{\infty} \frac{4}{x^2} dx = \frac{2000}{5} = \mathbf{400}.$$

3. The moment generating function of Λ is $M_\Lambda(x) = (1-200x)^{-5}$. Thus,

$$2^{-5}(1-200x)^{-5} = S_X(x) = M_\Lambda(-A(x)) = (1+200A(x))^{-5}.$$

Hence,

$$2^{-5}(1-200x)^{-5} = (1+200A(x))^{-5} \iff 1 + 200A(x) = 2(1-5x)$$

therefore

$$A(x) = \frac{1}{200}\left(2(1-200x) - 1\right) = \frac{1 - 2\times 200x}{200} = \frac{1}{200} - 2x.$$