King Saud University
College of Sciences
Mathematics Department

Academic Year (G) 2020–2021
Academic Year (H) 1442
Bachelor AFM: M. Eddahbi

March 7, 2021 from 3 to 5 PM, ACTU-464-474 (two pages)
(For numerical values keep 5 digits after dot)

Midterm Exam I, Risk theory

Exercise 1

If N be a discrete non–negative random variable with p.m.f. p_k, a zero–modified distribution is of the form: $p_k^M = \frac{1-p_0^M}{1-p_0} p_k$ where $p_0^M \in [0,1)$.
Consider the zero–modified geometric distribution: $p_0^M = \frac{1}{2}$, $p_k^M = \frac{1}{6}\left(\frac{2}{3}\right)^{k-1}$, $k = 1, 2, 3, \ldots$

1. Find the parameter $p = p_0$ of the initial geometric distribution p_k of N. (recall that $p_k = p(1-p)^k$, $k \geq 0$).
2. Let N^M be a r.v. whose distribution is the zero–modified geometric distribution p_k^M given above. Find moment generating function of N^M.
3. Find the exponential premium $\Pi_{\exp}(\alpha) = \dfrac{\ln\left(M_{N^M}(\alpha)\right)}{\alpha}$, for $\alpha = 0.2$.

<u>**Solution**</u>:

1. Recall that a zero–modified distribution is of the form: $p_k^M = \frac{1-p_0^M}{1-p_0} p_k$.We know that for any $k \geq 1$, $p_k^M = \frac{1-p_0^M}{1-p_0} p_k$ then
$$p_k = \frac{1-p_0}{1-p_0^M} p_k^M = 2\left(1-p_0\right)\frac{1}{6}\left(\frac{2}{3}\right)^{k-1} = \frac{1-p_0}{3}\left(\frac{2}{3}\right)^{k-1} = p\left(1-p\right)^k \text{ for any } k \geq 1$$
Notice that $p_0 = p$ thus in particular for $k = 1$ we have $\frac{1-p}{3} = p\left(1-p\right)$ then $p = \frac{1}{3}$.
2. The m.g.f. of N^M is
$$\begin{aligned} M_{N^M}(\alpha) &= \sum_{k=0}^{\infty} e^{k\alpha} p_k^M = \frac{1}{2} + \sum_{k=1}^{\infty} e^{k\alpha}\frac{1}{6}\left(\frac{2}{3}\right)^{k-1} = \frac{1}{2} + \frac{1}{6} \times \frac{3}{2}\sum_{k=1}^{\infty}\left(\frac{2e^{\alpha}}{3}\right)^k \\ &= \frac{1}{2} + \frac{1}{4}\sum_{k=1}^{\infty}\left(\frac{2e^{\alpha}}{3}\right)^k = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{1-\frac{2e^{\alpha}}{3}} - 1\right) = \frac{1}{4}\left(1 + \frac{1}{1-\frac{2}{3}e^{\alpha}}\right) \end{aligned}$$
3. So for $\alpha = 0.2$ we get
$$M_{N^M}(0.2) = \frac{1}{4}\left(1 + \frac{1}{1-\frac{2}{3}e^{0.2}}\right) = 1.5960.$$
Thus
$$\Pi_{\exp}(0.2) = \frac{\ln\left(1.5960\right)}{0.2} = \mathbf{2.3375}.$$

Exercise 2

Consider a negative binomial random variable with parameters $\beta = 0.5$ and $r = 2.5$.

1. *Determine the first four probabilities for this random variable.*
2. *Then determine the corresponding probabilities for the zero–modified version (with $p_0^M = 0.6$).*
3. *Then determine the corresponding probabilities for the zero–truncated version.*

(Hint $a = \frac{\beta}{1+\beta}$ and $b = (r-1)a$.)

Solution:

1. We know that

$$\begin{aligned} p_0 &= (1+\beta)^{-r} = (1+0.5)^{-2.5} = 0.362887, \\ a &= \frac{\beta}{1+\beta} = \frac{0.5}{1.5} = \frac{1}{3} \quad \text{and} \quad b = (r-1)\frac{\beta}{1+\beta} = \frac{1}{2}. \end{aligned}$$

   The first three recursions are

$$\begin{aligned} p_1 &= p_0\,(a+b) = 0.362887\left(\frac{1}{3}+\frac{1}{2}\right) = 0.302406, \\ p_2 &= p_1\left(a+b\frac{1}{2}\right) = 0.302406\left(\frac{1}{3}+\frac{1}{2}\frac{1}{2}\right) = 0.176404, \\ p_3 &= p_2\left(a+b\frac{1}{3}\right) = 0.176404\left(\frac{1}{3}+\frac{1}{2}\frac{1}{3}\right) = 0.088202. \end{aligned}$$

2. For the zero–modified random variable, $p_0^M = 0.6$ arbitrarily. We have then,

$$\begin{aligned} p_1^M &= \frac{1-p_0^M}{1-p_0}p_1 = \frac{1-0.6}{1-0.362887}0.302406 = 0.189860, \\ p_2^M &= p_1^M\left(a+b\frac{1}{2}\right) = 0.189860\left(\frac{1}{3}+\frac{1}{2}\frac{1}{2}\right) = 0.110752, \\ p_3^M &= p_2^M\left(a+b\frac{1}{3}\right) = 0.110752\left(\frac{1}{3}+\frac{1}{2}\frac{1}{3}\right) = 0.055376. \end{aligned}$$

3. For the zero–truncated random variable, $p_0^T = 0$ by definition. Then,

$$\begin{aligned} p_1^T &= \frac{p_1}{1-p_0} = \frac{0.302406}{1-0.362887} = 0.474651, \\ p_2^T &= p_1^T\left(a+b\frac{1}{2}\right) = 0.474651\left(\frac{1}{3}+\frac{1}{2}\frac{1}{2}\right) = 0.276880, \\ p_3^T &= p_2^T\left(a+b\frac{1}{3}\right) = 0.276880\left(\frac{1}{3}+\frac{1}{2}\frac{1}{3}\right) = 0.138440. \end{aligned}$$

Exercise 3

Given a loss random variable X, under an excess of loss reinsurance arrangement such that the insurer pays $Y = \min(X, M)$ and the re–insurer pays $Z = \max(0, X - M)$.
Recall that if f is the p.d.f. of X then

$$\mathrm{E}\left[Y^n\right] = \int_0^\infty (\min(x, M))^n f(x)\mathrm{d}x,$$

and

$$\mathrm{E}\left[Z^n\right] = \int_0^\infty (\max(x - M; 0))^n f(x)\mathrm{d}x.$$

1. *Calculate the expected payment of the insurer if X is exponentially distributed with mean* 200 *and $M = 100$.*
2. *Calculate the expected payment of the re–insurer if X is exponentially distributed with mean* 200 *and $M = 100$.*
3. *Calculate the sum of the two expected valued found in 1. and 2.*

Solution:

1. The expected payment of the insurer if X is exponentially distributed with mean 200 is given by

$$\begin{aligned}\mathrm{E}\left[Y\right] &= \int_0^\infty \min(x, M)f(x)\mathrm{d}x = \int_0^M \frac{x}{200}e^{-\frac{1}{200}x}\mathrm{d}x + \int_M^\infty M\ f(x)\mathrm{d}x \\ &= 200\left(1 - e^{-\frac{M}{200}}\right) - Me^{-\frac{M}{200}} + Me^{-\frac{M}{200}} = 200\left(1 - e^{-\frac{M}{200}}\right).\end{aligned}$$

   Hence for $M = 100$, we get $\mathrm{E}\left[Y\right] = 200(1 - e^{-\frac{100}{200}}) = \mathbf{78.6939}$.

2. The expected payment of the re–insurer if X is exponentially distributed with mean 200 is given by

$$\begin{aligned}\mathrm{E}\left[Z\right] &= \int_0^\infty \max(x - M; 0)f(x)\mathrm{d}x = \int_M^\infty (x - M)\frac{1}{200}e^{-\frac{1}{200}x}\mathrm{d}x \\ &= e^{-\frac{M}{500}}\int_M^\infty (x - M)\frac{1}{200}e^{-\frac{(x-M)}{200}}\mathrm{d}x = e^{-\frac{M}{200}}\int_0^\infty z\frac{1}{200}e^{-\frac{z}{200}}\mathrm{d}z = 200e^{-\frac{M}{200}}.\end{aligned}$$

   Hence for $M = 100$, we get $\mathrm{E}\left[Z\right] = 200e^{-\frac{100}{200}} = \mathbf{121.3061}$.

3. We have $\mathrm{E}\left[Y\right] + \mathrm{E}\left[Z\right] = 78.6939 + 121.3061 = \mathbf{200}$. In fact this should be equal to $\mathrm{E}\left[X\right] = 200$.

Exercise 4

Assume that the loss X is exponentially distributed with parameter 0.05756. Consider the proportional risk $Z = 0.75X$.

1. Calculate the charged premium Π_{SL} such that $\mathrm{P}(Z \leq \Pi_{\mathrm{SL}}) = 0.9$

2. Calculate the charged premium Π_{Var} such that $\mathrm{P}(Z > \Pi_{\text{Var}})) = 0.15$
3. Calculate the charged premium Π_{sd} such that $\mathrm{P}(Z \leq \Pi_{\text{sd}}) = 0.80$.

Solution:

1. We know that
$$\mathrm{P}(Z \leq \Pi_{\text{SL}}) = F_X(\frac{4}{3}\Pi_{\text{SL}}) = 1 - e^{-0.05756\times\frac{4}{3}\times\Pi_{\text{SL}}} = 0.899981$$
that is $e^{-0.0767467\Pi_{\text{SL}}} = 1 - 0.899981 = 0.100019$, then $\Pi_{\text{SL}} = \mathbf{29.9999} \simeq \mathbf{30}$.

2. We have
$$\mathrm{P}(Z > \Pi_{\text{Var}}) = 0.15 \Longleftrightarrow e^{-0.05756\frac{4}{3}\Pi_{\text{Var}}} = 0.15,$$
then $\Pi_{\text{Var}} = -\frac{\ln(0.15)}{0.05756\times 1.33333} = \mathbf{24.7193.}$

3. We have
$$\mathrm{P}(Z > \Pi_{\text{sd}}) = 0.2 \Longleftrightarrow e^{-0.05756\times\frac{4}{3}\Pi_{\text{sd}}} = 0.2,$$
then $\Pi_{\text{sd}} = -\frac{\ln(0.2)}{0.05756\times 1.33333} = \mathbf{20.9708}$.

Exercise 5

1. An agent has a net wealth 500 has accepted (and collected the premium for) a risk X with the following probability distribution: $\mathrm{P}(X = 0) = \mathrm{P}(X = 100) = \mathrm{P}(X = 200) = \frac{1}{3}$. What is the maximum amount the agent would accept to pay an insurer to accept 80% of this loss? Assume the agent's utility function is $u(x) = \ln(x)$.
2. An insurer, with wealth 500 and the same utility function, $u(x) = \ln(x)$, is considering accepting the above risk. What is the minimum amount this insurer would accept as a premium to cover 80% of the loss? The positive solution to the equation $500^3 = (500 + x)(420 + x)(340 + x)$, is **84.26656**.
3. **(2 marks)** Verify if the premiums satisfy the natural order such that the deal can be made.

Solution:

1. The the maximum amount P^+ is given by the equation
$$\begin{aligned} u(W - P^+) &= \mathrm{E}\left[u\left(W - X\right)\right] \Longleftrightarrow \ln\left(500 - P^+\right) = \frac{u\left(500\right) + u\left(500 - 80\right) + u\left(500 - 160\right)}{3} \\ &= \frac{u\left(500\right) + u\left(500 - 80\right) + u\left(500 - 160\right)}{3} = \frac{\ln\left(500 \times 420 \times 340\right)}{3} = 6.0279 \end{aligned}$$
Which leads to $\ln(500 - P^+) = 6.0279$, then $P^+ = 500 - e^{6.0279} = \mathbf{85.15706}$.

2. The minimum amount P^- this insurer would accept to cover 80% of the loss is given by the equation.
$$u(W) = \mathrm{E}\left[u\left(W + P^- - X\right)\right]$$
that is
$$\ln(500) = \frac{u(500 + P^-) + u(420 + P^-) + u(340 + P^-)}{3}$$
which is equivalent to
$$3\ln(500) = \ln\left(500^3\right) = \ln\left[\left(500 + P^-\right)\left(420 + P^-\right)\left(340 + P^-\right)\right],$$
solving for P^-= **84.26656.**

3. The deal can be make since $E[0.8X] = \mathbf{80} < P^- = \mathbf{84.26656} < P^+ = \mathbf{85.15706}$.