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(For numerical values keep 5 digits after dot and round by default)

Model Answer of the Second Midterm Exam, Risk theory

Exercise 1
Consider a 1*–year term life insurance paying an extra benefit in case of accidental death. To be specific, if death is accidental, the benefit amount is* 100, 000*. For other causes of death, the benefit amount is* 50, 000*. Assume that for the age, health, and occupation of a specific individual, the probability of an accidental death within the year is* 0.005*, while the probability of a non–accidental death is* 0.020*. Denote by B the claim amount payment for this life insurance. I indicates the occurrence of the claim for $I = 1$, or non–occurrence of the claim for, $I = 0$ and its distribution is given by $\mathrm{P}(I = 1) = q$ and $\mathrm{P}(I = 0) = 1 - q$.*

1. *Find* $\mathrm{P}(I = 1\ ;\ B = 100,000)$ *and* $\mathrm{P}(I = 1\ ;\ B = 50,000)$.
2. *a. Deduce from* 1. *the distribution of I and b. the conditional distribution of B, given $I = 1$*
3. *Calculate expected value of the claim. (Hint* $\mathrm{E}[X] = q\ \mu$ *where* $\mu = \mathrm{E}[B|I = 1]$*).*

<u>**Solution**</u>:

1. From the given information we have,

$$\mathrm{P}(I = 1\ ;\ B = 100,000) = 0.005$$

and

$$\mathrm{P}(I = 1\ ;\ \ B = 50,000) = 0.02.$$

2. a. Summing over the possible values of B, we have $\mathrm{P}(I = 1) = \mathbf{0.025}$, and then $\mathrm{P}(I = 0) = 1 - \mathrm{P}(I = 1) = \mathbf{0.975}$.
b. The conditional distribution of B, given $I = 1$, is

$$\mathrm{P}(B = 100,000|I = 1) = \frac{\mathrm{P}(B = 100,000;\ I = 1)}{\mathrm{P}(I = 1)} = \frac{0.005}{0.025} = \mathbf{0.2}$$

and

$$\mathrm{P}(B = 50,000|I = 1) = \frac{\mathrm{P}(B = 50,000;\ I = 1)}{\mathrm{P}(I = 1)} = \frac{0.02}{0.025} = \mathbf{0.8}.$$

3. We have to find first

$$\mu = \mathrm{E}[B|I = 1] = 50000 \times 0.8 + 100000 \times 0.2 = 60000.$$

Thus the expected value of the claim is $\mathrm{E}[X] = q\ \mu = 0.025 \times 60000 = \mathbf{1500}$.

Exercise 2 *Consider an automobile insurance providing collision coverage (this indemnifies the owner for collision damage to his car) above a* 2000 *deductible up to a maximum claim of* 10,000. *Let* X *denotes the payment amount of the insurance company. The c.d.f. of* X *is given by*

$$F_X(x) = \begin{cases} 0 & \textit{if} \quad x < 0 \\ 0.97 - 0.12\left(1 - \frac{x}{10000}\right)^2 & \textit{if} \quad 0 \leq x < 10000 \\ 1 & \textit{if} \quad x \geq 10000. \end{cases}$$

1. *Calculate the probability that the payment amount equals exactly* 0
   Calculate the probability that the payment amount equals exactly 10000
2. *Find the density of* X *on the open interval* $(0;\ 10000)$.
3. *Calculate variance loading premium* $\Pi_{\text{Var}}(0.0001)$.

Solution:

1. We have

$$\text{P}(X = 0) = F_X(0+) - F_X(0-) = F_X(0+) = 0.97 - 0.12 = \mathbf{0.85}$$

   and

$$\text{P}(X = 10000) = F_X(10000+) - F_X(10000-) = 1 - 0.97 = \mathbf{0.03}$$

2. The density f_X of X on the open interval $(0;\ 10000)$ is given by the first derivative of the c.d.f. F_X on the interval $(0;\ 10000)$ which is given by

$$f_X(x) = 0.000024\left(1 - \frac{x}{10000}\right) \quad \text{if} \quad 0 < x < 10000$$

3. To calculate $\Pi_{\text{Var}}(0.0001)$ we need to find $\text{E}[X]$ and $\text{Var}[X]$. We have

$$\begin{aligned} \text{E}[X] &= (10000)\text{P}\left(X = 10000\right) + \int_0^{10000} x f_X(x)dx \\ &= 10000 \times 0.03 + 0.000024 \int_0^{10000} x\left(1 - \frac{x}{10000}\right)dx = 700. \end{aligned}$$

   and

$$\begin{aligned} \text{E}[X^2] &= (10000)^2\text{P}\left(X = 10000\right) + \int_0^{10000} x^2 f_X(x)dx \\ &= (10000)^2 \times 0.03 + 0.000024 \int_0^{10000} x^2\left(1 - \frac{x}{10000}\right)dx = 5 \times 10^6. \end{aligned}$$

   Hence

$$\text{Var}(X) = 5 \times 10^6 - (700)^2 = 4510000.$$

   Finally $\Pi_{\text{Var}}(0.0001) = \text{E}[X] + 0.0001\text{Var}[X] = 700 + 0.0001 \times 4510000 = \mathbf{1151}$.

Exercise 3 *The cumulative distribution of an aggregate individual model S is given as follows*

$$F_S(s) = \begin{cases} 0 & \text{if } s \le 0 \\ \dfrac{s^2}{12} & \text{if } 0 < s \le 2 \\ \dfrac{s-1}{3} & \text{if } 2 < s \le 3 \\ 1 - \dfrac{(5-s)^2}{12} & \text{if } 3 < s \le 5 \\ 1 & \text{if } s > 5 \end{cases}$$

Calculate the quantile premiums Π_α for $\alpha = 0.75$, $\alpha = 0.5$ and $\alpha = \frac{1}{12}$. (Π_α is the positive solution of the equation $\mathrm{P}(S > \Pi_\alpha) = \alpha$ or equivalently $F_S(\Pi_\alpha) = 1 - \alpha$).

<u>**Solution**</u>: We shall solve the equation

$$\mathrm{P}(S > \Pi_\alpha) = \alpha \text{ or equivalently } F_S(\Pi_\alpha) = 1 - \alpha$$

1. We have $F_S(2) = \frac{1}{3} = 0.33333$ so the solution of $F_S(\Pi_{0.75}) = 0.25$ belongs to the interval $0 < s < 2$ hence $\Pi_{0.75}$ is the positive solution of the equation $\dfrac{s^2}{12} = 0.25$, which gives $\Pi_{0.75} = \mathbf{1.73205}$.
2. We have $F_S(3) = \frac{2}{3} = 0.66667$ so the solution of $F_S(\Pi_{0.5}) = 0.5$ belongs to the interval $2 < s < 3$ hence $\Pi_{0.5}$ is the unique solution of the equation $\dfrac{s-1}{3} = 0.5$, which gives $\Pi_{0.5} = \mathbf{2.5}$.
3. We have $F_S(5) = 1$ so the solution of $\mathrm{P}(S > \Pi_{\frac{1}{12}}) = \frac{1}{12}$ belongs to the interval $3 < s < 5$ hence $\Pi_{\frac{1}{12}}$ is the unique solution in the interval $3 < s < 5$ of the equation $\dfrac{(5-s)^2}{12} = \frac{1}{12}$, which gives $\Pi_{\frac{1}{12}} = \mathbf{4}$.

Exercise 4 *For an insurance you are given the following information :*
(i) The number of losses per year has a Poisson distribution with mean 100.
(ii) Loss amounts are gamma distributed with density $f(x) = \dfrac{1}{2}(0.005)^3 x^2 e^{-0.005x}$, $0 < x < \infty$.
(iii) Loss amounts and the number of losses are mutually independent.
(iv) There is an ordinary deductible of 1000 *per loss and*

$$\begin{aligned} \int_{1000}^{\infty} (x-1000)\frac{1}{2}(0.005)^3 x^2 e^{-0.005x} \mathrm{d}x &= 34.36353. \\ \int_{1000}^{\infty} (x-1000)^2\frac{1}{2}(0.005)^3 x^2 e^{-0.005x} \mathrm{d}x &= 18057.69796. \end{aligned}$$

1. *Calculate the expected value and the variance of aggregate payments in a year.*
2. *Find the standard deviation loading premium with $b = 0.41945$.*
3. *Use normal approximation and calculate the probability that the aggregate loss is greater than $\Pi_{\mathrm{sd}}(0.41945)$. ($\mathrm{P}(\mathcal{N}(0,1) \le 0.41945) = 0.66255$).*

Solution:

Let S be the aggregate claims. Then S is a compound distribution r.v. with a Poisson *primary distribution* for N, the number of claims, and *secondary distribution* is gama distributed with parameters $\alpha = 3$ and $\lambda = 0.005$. Set $Z = \max(X - 1000; 0) = (X - 1000)^+$, the amount paid per loss, where X is the loss amount.

1. We want to calculate $\mathrm{E}[S]$ and $\mathrm{Var}(S)$. In fact

$$\mathrm{E}[Z] = \mathrm{E}[\max(X - 1000; 0)] = \int_{1000}^{\infty} (x - 1000) \frac{1}{2} (0.005)^3 x^2 e^{-0.005x} \mathrm{d}x = 34.36353.$$

Thus

$$\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[Z] = 100 \times 34.36353 = \mathbf{3436.353}.$$

Moreover

$$\begin{aligned} \mathrm{Var}(S) &= \mathrm{E}[N]\,\mathrm{Var}(Z) + \mathrm{Var}(N)\,(\mathrm{E}[Z])^2 = \lambda \left(\mathrm{Var}(Z) + (\mathrm{E}[Z])^2\right) \\ &= \lambda \mathrm{E}\left[Z^2\right] = 100 \int_{1000}^{\infty} (x - 1000)^2 \frac{1}{2} (0.005)^3 x^2 e^{-0.005x} \mathrm{d}x \\ &= 100 \times 18057.69796 = \mathbf{1805769.796} \end{aligned}$$

2. The standard deviation loading premium with $b = 0.41945$ is

$$\Pi_{\mathrm{sd}}(0.41945) = \mathrm{E}[S] + 0.41945\sqrt{\mathrm{Var}(S)} = 3436.353 + 0.41945\sqrt{1805769.796} = 4000$$

3. We have

$$\mathrm{P}\left(S > \Pi_{\mathrm{sd}}(0.41945)\right) = \mathrm{P}(S > 4000) = \mathrm{P}\left(\frac{S - 3436.353}{1343.78934} > \frac{4000 - 3436.353}{1343.78934}\right)$$

Using normal approximation the random variable $T = \frac{S - 3436.353}{1343.78934}$ follows the standard normal distribution $\mathcal{N}(0,1)$. Therefore $\mathrm{P}\left(S > \Pi_{\mathrm{sd}}(0.41945)\right) = \mathrm{P}(T > 0.41945) = \mathbf{0.33745}$.

Exercise 5 *An aggregate collective model can be written as* $S = \sum_{k=1}^{N} X_i$ *where* X_i *are i.i.d. with severity distribution* $f_X(0) = 0.2$, $f_X(1) = 0.5$ *and* $f_X(2) = 0.3$ *and the frequency distribution* N *follows a Negative Binomial distribution with parameters* $r = 4$ *and* $p = 0.3$. *(Hint: Negative binomial* $\mathcal{NB}(r; p)$ *belongs to the class* $C(a, b, 0)$ *with* $a = 1 - p$, $b = (r - 1)(1 - p)$ *and* $p_0 = p^r$. *Moreover* $\mathrm{E}[\mathcal{NB}(r; p)] = \frac{rq}{p}$, $Var(\mathcal{NB}(r; p)) = \frac{rq}{p^2}$, $\mathrm{P}_{\mathcal{NB}}(t) = (\frac{p}{1-qt})^r$*) and the distribution of* S *can be computed recursively by*

$$f_S(n) = \mathrm{P}(S = n) = \frac{\sum_{j=1}^{n} \left(a + \frac{b}{n} j\right) f_X(j) f_S(n - j)}{1 - a f_X(0)} \quad \textit{and } f_S(0) = \mathrm{P}_N(f_X(0))$$

1. *Use Panjer's recursion to calculate the probability that the aggregate loss becomes greater than* 2.25.

2. *Calculate the mean and variance of the aggregate claim. (Hint* $\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[X]$ *and* $\mathrm{Var}(S) = \mathrm{E}[N]\mathrm{Var}(X) + (\mathrm{E}[X])^2\,\mathrm{Var}(N)$*)*

3. *Use normal approximation to calculate quantile premium* Π_{Qua} *such that* $\mathrm{P}(S > \Pi_{\text{Qua}}) = 0.23316$. *(The solution to the equation* $\mathrm{P}(\mathcal{N}(0,1) \le t) = 0.76684$ *is* $t = 0.72848$*).*

Solution:

1. We have $\mathrm{P}(S > 2.25) = 1 - F_S(2) = 1 - (f_S(0) + f_S(1) + f_S(2))$. Recall that the Negative binomial $\mathcal{NB}(r;p)$ belongs to the class $C(a,b,0)$ with $a = 1 - p = 0.7$, $b = (r-1)(1-p) = 3 \times 0.7 = 2.1$ and $p_0 = p^r = (0.3)^4 = 0.0081$.
   We need the values of $f_S(0)$, $f_S(1)$ and $f_S(2)$. From the Panjer's recursion we have

$$f_S(0) = \mathrm{P}_N(f_X(0)) = \left(\frac{0.3}{1 - 0.7 \times 0.2}\right)^4 = 0.01481$$

   and

$$\begin{aligned} f_S(1) &= \frac{100}{86}\left(0.7 + \frac{2.1}{1}\right) f_X(1) f_S(1-1) = \frac{100}{86}\left(0.7 + \frac{2.1}{1}\right) 0.5 \times 0.01481 = 0.02411 \\ f_S(2) &= \frac{100}{86}\left(\left(0.7 + \frac{2.1}{2}\right) f_X(1) f_S(2-1) + \left(0.7 + \frac{2.1}{2}2\right) f_X(2) f_S(2-2)\right) \\ &= \frac{100}{86}\left(\left(0.7 + \frac{2.1}{2}\right) 0.5 \times 0.02411 + \left(0.7 + \frac{2.1}{2}2\right) 0.3 \times 0.01481\right) = 0.03810 \end{aligned}$$

   so

$$\mathrm{P}(S > 2.25) = 1 - (0.01481 + 0.02411 + 0.03810) = \mathbf{0.92298}$$

2. We know

$$\begin{aligned} \mathrm{E}[N] &= \mathrm{E}[\mathcal{NB}(4;0.3)] = \frac{0.7 \times 4}{0.3} = \frac{28}{3} = 9.33333, \\ \mathrm{Var}(N) &= \frac{0.7 \times 4}{(0.3)^2} = \frac{280}{9} = 31.11111 \end{aligned}$$

   and

$$\begin{aligned} \mathrm{E}[X] &= 0.5 + 2 \times 0.3 = 1.1, \mathrm{E}\left[X^2\right] = 0.5 + 2^2 \times 0.3 = 1.7, \\ \mathrm{Var}(X) &= 1.7 - (1.1)^2 = \frac{49}{100} = 0.49. \end{aligned}$$

   Then

$$\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[X] = \frac{28}{3} \times \frac{11}{10} = \frac{154}{15} = \mathbf{10.26667}$$

   and

$$\mathrm{Var}(S) = \mathrm{E}[N]\mathrm{Var}(X) + (\mathrm{E}[X])^2\,\mathrm{Var}(N) = \frac{28}{3} \times \frac{49}{100} + (1.1)^2 \times 31.11111 = \mathbf{42.21778}.$$

3. We have

$$\begin{aligned} \mathrm{P}(S > \Pi_{\text{Qua}}) &= 0.23316 \iff \mathrm{P}(S \le \Pi_{\text{Qua}}) = 1 - 0.23316 = 0.76684 \\ &\iff \mathrm{P}\left(\frac{S - 10.26667}{\sqrt{42.21778}} \le \frac{\Pi_{\text{Qua}} - 10.26667}{\sqrt{42.21778}}\right) = 0.76684 \end{aligned}$$

   thus

$$\Pi_{\text{Qua}} = 10.26667 + 0.72848 \times \sqrt{42.21778} = 14.999984 \simeq \mathbf{15}.$$