King Saud University
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Solution of the Final exam ACTU–464 Fall 2019 (40%) (two pages)

December 18, 2019 (three hours 1–4 PM)

Problem 1. (8 marks)
A random variable X has m.g.f. $M_X(t) = (1-2t)^{-10}$ for $t < 0.5$.

1. **(2 marks)** Use σ^2–loading principle or the variance principle to calculate $\Pi_{\text{Var}}(0.05)$,
2. **(2 marks)** Calculate the exponential premium $\Pi_{\text{Exp}}(0.09655)$.
3. **(2 marks)** Use normal approximation to calculate $\Pi_{0.05}$ and $\Pi_{0.01}$ such that $\mathrm{P}(X > \Pi_\alpha) = \alpha$. What is the interpretation of Π_α if X is a loss of an insurance company?
4. **(2 marks)** Consider a risk whose distribution follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 100$. The c.d.f. of a Pareto distribution is

$$F_{\text{Pareto}}(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha, \quad \mathrm{E}[X] = \frac{\theta}{\alpha - 1} \text{ and } \mathrm{Var}(X) = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}.$$

Calculate the parameter b of the σ–loading premium $\Pi_{\text{sd}}(b)$ such that $\mathrm{P}(X \geq \Pi_{\text{sd}}(b)) = 0.08$.

Solution:

1. We have $\mathrm{E}[X] = M_X'(0) = 20$ and $M_X''(0) = 440$, so $\mathrm{Var}(X) = 440 - (20)^2 = 40$, hence

$$\Pi_{\text{Var}}(0.05) = 20 + 40 \times 0.05 = \mathbf{22}.$$

2. We have $\Pi_{\text{Exp}}(0.09655) = \frac{1}{0.09655} \ln\left((1 - 2 \times 0.09655)^{-9}\right) = \mathbf{20}$.
3. We can write

$$\mathrm{P}(X > \Pi_\alpha) = \mathrm{P}\left(\frac{X-20}{2\sqrt{10}} > \frac{t_\alpha - 20}{2\sqrt{10}}\right) = \mathrm{P}\left(Z > \frac{\Pi_\alpha - 20}{2\sqrt{10}}\right) = \alpha,$$

so $\frac{\Pi_{0.05}-20}{2\sqrt{10}} = 1.644854$, thus

$$\Pi_{0.05} = 20 + 1.644854 \times 2\sqrt{10} = \mathbf{30.403},$$

and $\frac{\Pi_{0.01}-20}{2\sqrt{10}} = 2.326348$, then

$$\Pi_{0.01} = 20 + 2.326348 \times 2\sqrt{10} = \mathbf{34.713}.$$

4. The premium $\Pi_{\text{sd}}(b)$ is given by

$$\left(\frac{100}{100+\Pi_{\text{sd}}(b)}\right)^3 = 0.08 \quad \text{that is} \quad \Pi_{\text{sd}}(b) = 132.08,$$

Remember that

$$\Pi_{\text{sd}}(b) = \mathrm{E}[X] + b\sqrt{\mathrm{Var}(X)} = \frac{\theta}{\alpha-1}\left(1 + b\sqrt{\frac{\alpha}{(\alpha-2)}}\right) = \frac{100}{2}\left(1+b\sqrt{3}\right) = 132.08,$$

hence $b = \mathbf{0.94778}$.

Problem 2. **(8 marks)**

An insurer undertakes a risk X distributed as follows $\mathrm{P}(X=0) = 1 - \mathrm{P}(X=36) = \dfrac{1}{3}$ and after collecting the premium, he owns a capital $W = 100$.

1. What is the maximum premium P^+ the insurer is willing to pay to a reinsurer to take over the complete risk, if his utility function is $u(x) = \ln(x)$?
2. Calculate the net premium denoted by μ of the risk X and its variance σ^2.
3. Find the approximation P_a^+ of P^+ where $P_a^+ = \mu - \dfrac{\sigma^2}{2}\dfrac{u''(W-\mu)}{u'(W-\mu)}$.
4. Assume that the reinsurer's minimum premium to take over the risk of the question 1 equals 24 and that the reinsurer has the same utility function. Determine his capital W.

Solution:

1. The the maximum amount P^+ is given by the equation

$$\begin{aligned} u(W-P^+) &= \mathrm{E}\left[u\left(W-X\right)\right] \iff u\left(100-P^+\right) = \frac{u(100)}{3} + \frac{2u(64)}{3} \\ &\iff 3\ln\left(100-P^+\right) = \ln(100) + 2\ln(64) = \ln\left((100)\left(64^2\right)\right) \end{aligned}$$

hence $\left(100-P^+\right)^3 = (100)\left(64^2\right) = 409600$, then $P^+ = 100 - (409600)^{\frac{1}{3}} = \mathbf{25.735}$.

2. Observe first that $\mu = \frac{2}{3}36 = 24$, $\sigma^2 = \frac{2}{3}(36)^2 = (24)^2 = 576$.

3. We have $\frac{u''(x)}{u'(x)} = -\frac{1}{x}$, therefore

$$P_a^+ = 24 + \frac{576}{2}\frac{1}{100-24} = \mathbf{27.789}.$$

4. The minimum premium form the reinsurer's point of view is given by

$$u(W) = \mathrm{E}\left[u\left(W+P^- - X\right)\right],$$

so for $P^- = 24$ we get that is

$$\ln(W) = \frac{u(W+24)}{3} + \frac{2u(W+24-36)}{3}$$

therefore

$$3\ln(W) = \ln(W+24) + 2\ln(W-12) \iff W^3 = (W+24)(W-12)^2 \iff W = \mathbf{8}.$$

Problem 3. (8 marks)

A portfolio of independent insurance policies has three classes of policies:

Class	Number in Class	Probability of Claim per Policy	Claim Amount b_k
1	1000	0.01	1
2	2000	0.02	1
3	500	0.04	2

1. **(2 marks)** Calculate the expectation of the aggregate loss S
2. **(2 marks)** Calculate the variance of the aggregate loss S
3. **(2 marks)** Use normal approximation to calculate θ such that the probability of that the aggregate loss is less than the $\Pi_{\mathrm{SL}}(\theta)$ is equal to 0.95.
4. **(2 marks)** Find $\Pi_{\mathrm{SL}}(\theta)$.

Solution:

1. We have $\mathrm{E}[S] = \sum_{i=1}^{3} n_k \ b_k \ q_k = 1000 \times 1 \times 0.01 + 2000 \times 1 \times 0.02 + 500 \times 2 \times 0.04 = \mathbf{90}$.
2. We have

$$\begin{aligned}\sigma_S^2 &= \mathrm{Var}(S) = \sum_{i=1}^{3} n_k \ b_k^2 \ q_k \, (1 - q_k) \\ &= 1000 \left(1 \times 1 \times 0.01 \times 0.99 + 2 \times 1 \times 0.02 \times 0.98 + 0.5 \times 2^2 \times 0.04 \times 0.96\right) = \mathbf{125.9}\end{aligned}$$

3. Under normal approximation the r.v. $T = \frac{S - \mathrm{E}[S]}{\sigma_S}$ follows a standard normal distribution, therefore

$$\mathrm{P}\left(S \le \Pi_{\mathrm{SL}}(\theta)\right) = \mathrm{P}\left(\frac{S - \mathrm{E}[S]}{\sigma_S} \le \frac{\Pi_{\mathrm{SL}}(\theta) - 90}{\sqrt{\mathbf{125.9}}} = \theta \frac{90}{\sqrt{\mathbf{125.9}}}\right) = 0.95,$$

hence $\theta = \frac{1.644854 \times \sqrt{125.9}}{90} = \mathbf{0.20507}$.

4. The safety loading premium is $\Pi_{\mathrm{SL}}(0.20507) = 1.20507 \times 90 = \mathbf{108.46}$.

Problem 4. (8 marks)

Let the frequency distribution N of an aggregate loss S follows a geometric distribution with parameter (0.25), and the severity distribution modelling the claim size X is exponentially distributed with parameter 2.

1. **(2 marks)** Calculate $\Pi_{\mathrm{Exp}}(0.05)$ corresponding to the aggregate loss S.
2. **(2 marks)** Use the one to one correspondence between c.d.f. and m.g.f. to find the c.d.f. of S?
3. **(2 marks)** Find the minimum capital $C_{\min}$ such that $\mathrm{P}(S \le C_{\min}) = 0.95$.
4. **(2 marks)** Set $S = X_1 + 2X_2 + 3X_3$ and X_j follows a Poisson distribution with parameter j, $j = 1$, 2, 3. Calculate the minimum premium or the net premium $P_{\min}$ to cover the loss $(S - 3)^+$.

Solution:

1. By definition $\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05}\ln\left(M_S(0.05)\right)$, where for $t < 2$ hence
$$M_S(0.05) = 0.25 + 0.75\frac{2\times 0.25}{2\times 0.25 - 0.05} = 1.0833,$$
thus $\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05}\ln(1.0833) = \mathbf{1.6002}$.

2. We know that $M_S(t) = p + q\frac{2p}{2p-t}$, so
$$F_S(x) = p + qF_{\text{Exp}(2p)}(x) = p + q\left(1 - e^{-2px}\right) = 1 - qe^{-2px} = \mathbf{1 - 0.75e^{-0.5x}},$$

3. Therefore $\text{P}(S \le C_{\min}) = F_S(C_{\min}) = 0.95 = 1 - 0.75e^{-0.5C_{\min}}$ which gives $C_{\min} = \mathbf{5.4161}$.

4. By definition $P_{\min} = \text{E}\left[(S-3)^+\right] = \text{E}[S] - \text{E}[S \wedge 3]$, so let us calculate $\text{E}[S]$ and $\text{E}[S\wedge 3]$, clearly $\text{E}[S] = 1 + 4 + 9 = 14$ and
$$\begin{aligned}\text{E}[S\wedge 3] &= \sum_{k=1}^{3} kf_S(k) + 3\left(1 - \text{P}(S\le 3)\right) = f_S(1) + 2f_S(2) + 3f_S(3) + 3\left(1 - \sum_{k=0}^{3} f_S(k)\right)\\ &= 3 - 2f_S(1) - f_S(2) - 3f_S(0) = 3\left(1 - f_S(0)\right) - 2f_S(1) - f_S(2)\\ &= 3\left(1 - e^{-6}\right) - 2e^{-6} - e^{-6}2 = 2.9826.\end{aligned}$$
Finally $P_{\min} = 14 - 2.9826 = \mathbf{11.017}$.

Problem 5. (8 marks)

1. **(2 marks)** Use Panjer's recursion to calculate the p.m.f. $f_S(n)$ for $n = 0, 1, 2, 3, 4$ of $S = \sum_{k=1}^{N} X_i$ where X_i are i.i.d. with common distribution $f_X(1) = 0.7$ and $f_X(2) = 0.3$ and N follows a Negative Binomial distribution with parameters $r = 4.5$ and $p = 0.5$.

2. **(2 marks)** Calculate the probability that the aggregate loss becomes greater than 3.75.

3. **(2 marks)** Calculate the mean and variance of the aggregate claim.

4. **(2 marks)** Use normal approximation to calculate quantile premium Π_{Qua} such that
$$\text{P}(S > \Pi_{\text{Qua}}) = 0.31886.$$

Solution:

1. Recall that the Negative binomial $\mathcal{NB}(r;p)$ belongs to the class $C(a,b,0)$ with $a = 1 - p = 0.5$, $b = (r-1)(1-p) = \frac{3.5}{2} = 1.75$ and $p_0 = \dfrac{1}{(1+\beta)^r} = p^r = 0.5^{4.5} = 0.044194$.
From the Panjer's recursion we have $f_S(0) = \text{P}(N = 0) = p_0 = \mathbf{0.044194}$ and from for any $n \ge 1$, we have
$$\begin{aligned} f_S(n) &= \text{P}(S = n) = \frac{\sum_{j=1}^{n}\left(a + \frac{b}{n}j\right) f_X(j) f_S(n-j)}{1 - af_X(0)}\\ &= \frac{1}{2}\sum_{j=1}^{n}\left(1 + \frac{3.5}{n}j\right) f_X(j) f_S(n-j) \text{ (since } f_X(0) = 0)\end{aligned}$$

Thus

$$
\begin{aligned}
f_S(1) &= \frac{1}{2}\sum_{j=1}^{1}\left(1+\frac{3.5}{1}j\right) f_X(j) f_S(1-j) = \frac{1}{2}\sum_{j=1}^{1}(4.5)\, f_X(1) f_S(0) \\
&= \frac{1}{2} 4.5 \times 0.7 \times 0.044194 = \mathbf{0.069606} \\
f_S(2) &= \frac{1}{2}\sum_{j=1}^{2}\left(1+\frac{3.5}{2}j\right) f_X(j) f_S(2-j) = \frac{1}{2}\left(1+\frac{3.5}{2}\right) f_X(1) f_S(1) + \frac{1}{2}\left(1+\frac{3.5}{2}2\right) f_X(2) f_S(0) \\
&= \frac{1}{2}\left(1+\frac{3.5}{2}\right) 0.7 \times 0.069606 + \frac{1}{2}\left(1+\frac{3.5}{2}2\right) 0.3 \times 0.044194 = \mathbf{0.096827} \\
f_S(3) &= \frac{1}{2}\sum_{j=1}^{3}\left(1+\frac{3.5}{3}j\right) f_X(j) f_S(3-j) \\
&= \frac{1}{2}\left(1+\frac{3.5}{3}\right) f_X(1) f_S(2) + \frac{1}{2}\left(1+\frac{3.5}{3}2\right) f_X(2) f_S(1) \text{ since } (f_X(3)=0) \\
&= \frac{1}{2}\left(1+\frac{3.5}{3}\right) 0.7 \times 0.096827 + \frac{1}{2}\left(1+\frac{3.5}{3}2\right) 0.3 \times 0.069606 = \mathbf{0.108230}. \\
f_S(4) &= \frac{1}{2}\sum_{j=1}^{4}\left(1+\frac{3.5}{4}j\right) f_X(j) f_S(4-j) \\
&= \frac{1}{2}\left(1+\frac{3.5}{4}\right) f_X(1) f_S(3) + \frac{1}{2}\left(1+\frac{3.5}{3}2\right) f_X(2) f_S(2) \text{ since } (f_X(3)=f_X(4)=0) \\
&= \frac{1}{2}\left(1+\frac{3.5}{4}\right) 0.7 \times 0.108230 + \frac{1}{2}\left(1+\frac{3.5}{4}2\right) 0.3 \times 0.096827 = \mathbf{0.110967}.
\end{aligned}
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2. We have $\mathrm{P}(S > 3.75) = 1 - F_S(3) = 1 - (0.044194 + 0.069606 + 0.096827 + 0.108230) = \mathbf{0.68114}$.

3. We know $\mathrm{E}[N] = \mathrm{E}[\mathcal{NB}(4.5; 0.5)] = 4.5$. (since $q = p$) and $\mathrm{Var}(N) = 2 \times 4.5 = 9$.
and $\mathrm{E}[X] = 0.7 + 2 \times 0.3 = 1.3$, $\mathrm{E}[X^2] = 0.7 + 2^2 \times 0.3 = 1.9$, and $\mathrm{Var}(X) = 1.9 - (1.3)^2 = 0.21$.

$$\mathrm{E}[S] = \mathrm{E}[N]\,\mathrm{E}[X] = 4.5 \times 1.3 = \mathbf{5.85}$$

and $\mathrm{Var}(S) = \mathrm{E}[N]\mathrm{Var}(X) + (\mathrm{E}[X])^2\,\mathrm{Var}(N) = 4.5 \times 0.21 + (1.3)^2 \times 9 = \mathbf{16.155}$.

4. We have

$$\mathrm{P}(S > \Pi_{\text{Qua}}) = 0.31886 \iff \mathrm{P}(S \le \Pi_{\text{Qua}}) = 0.9 \iff \mathrm{P}\left(\frac{S - 5.85}{4.0193} \le \frac{\Pi_{\text{Qua}} - 5.85}{4.0193}\right) = 0.68114$$

thus

$$\Pi_{\text{Qua}} = 5.85 + 4.0193 \times 0.470889 = \mathbf{7.7426}.$$