King Saud University
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Academic Year (G) 2019–2020
Academic Year (H) 1441
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Solution of the final exam ACTU–464 Spring 2020 (20%)

April 30, 2020 (three hours + 30 minutes for submission)

Problem 1. (5 marks)
A random loss X has m.g.f. $M_X(t) = (1-2t)^{-5}$ for $t < 0.5$.

1. Calculate the expected loss
2. Calculate $\Pi_{\text{Var}}(0.05)$ of the random loss.
3. Calculate the exponential premium $\Pi_{\text{Exp}}(0.05)$ of the loss X.
4. Use normal approximation to calculate $\Pi_{0.05}$ such that $P(X > \Pi_\alpha) = \alpha$.
5. Consider a risk whose distribution follows a Pareto distribution with parameters $\alpha = 2.5$ and $\theta = 50$. The c.d.f. of a Pareto distribution is

$$F_{\text{Pareto}}(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^\alpha, \quad E[X] = \frac{\theta}{\alpha - 1} \quad \text{and} \quad \text{Var}(X) = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)}.$$

Calculate the σ–loading premium Π_{sd} such that $P(X \geq \Pi_{\text{sd}}) = 0.08$.

Solution:

1. We have $M'_X(t) = 10(1-2t)^{-6}$ then $E[X] = M'_X(0) = \mathbf{10}$.
2. Also we have $M''_X(t) = 120(1-2t)^{-6}$, and $M''_X(0) = 120$, so $\text{Var}(X) = 120 - (10)^2 = 20$, hence

$$\Pi_{\text{Var}}(0.05) = 10 + 20 \times 0.05 = \mathbf{11}.$$

3. We have $\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05} \ln\left((1 - 2 \times 0.05)^{-5}\right) = \mathbf{10.536}$.
4. We can write

$$P(X > \Pi_{0.05}) = P\left(\frac{X - 10}{2\sqrt{5}} > \frac{\Pi_{0.05} - 10}{2\sqrt{5}}\right) = P\left(Z > \frac{\Pi_{0.05} - 10}{2\sqrt{5}}\right) = \alpha,$$

so $\frac{\Pi_{0.05} - 10}{2\sqrt{5}} = 1.644854$, thus

$$\Pi_{0.05} = 10 + 1.644854 \times 2\sqrt{5} = \mathbf{17.356},$$

5. The premium Π_{sd} is given by

$$\left(\frac{50}{50 + \Pi_{\text{sd}}}\right)^{2.5} = 0.08 \quad \text{that is} \quad \Pi_{\text{sd}} = \mathbf{87.32},$$

Problem 2. (5 marks)
A portfolio of independent insurance policies has three classes of policies:

Class	Number in Class	Probability of Claim per Policy	Claim Amount
1	2000	0.01	4
2	1000	0.02	6
3	600	0.06	8

1. Is this a collective risk model or an individual risk model ? Justify your answer.
2. Calculate the expectation of the aggregate loss S
3. Calculate the variance of the aggregate loss S
4. Use normal approximation to calculate θ such that the probability of that the aggregate loss is less than the safety loading premium is equal to 0.95.
5. Find $\Pi_{\mathrm{SL}}(\theta)$.

Solution:

1. This is an individual model since the number of claims are nor random.
2. We have $E[S] = \sum_{i=1}^{3} n_k\ b_k\ q_k = 2000 \times 4 \times 0.01 + 1000 \times 6 \times 0.02 + 600 \times 8 \times 0.06 = \mathbf{488}$.
3. We have
$$\begin{aligned} \sigma_S^2 &= \mathrm{Var}(S) = \sum_{i=1}^{3} n_k\ b_k^2\ q_k\,(1-q_k) \\ &= 2000 \times 4^2 \times 0.01 \times 0.99 + 1000 \times 6^2 \times 0.02 \times 0.98 + 600 \times 8^2 \times 0.06 \times 0.94 \\ &= \mathbf{3188.160} \end{aligned}$$
4. Under normal approximation the r.v. $T = \frac{S-E[S]}{\sigma_S}$ follows a standard normal distribution, therefore
$$P\left(S \le \Pi_{\mathrm{SL}}(\theta)\right) = P\left(\frac{S-E[S]}{\sigma_S} \le \frac{\Pi_{\mathrm{SL}}(\theta)-160}{\sqrt{369.20}} = \theta\frac{160}{\sqrt{369.20}}\right) = 0.95,$$
hence
$$\theta = \frac{1.644854 \times \sqrt{3188.16}}{488} = \mathbf{0.19032}.$$
5. The safety loading premium is $\Pi_{\mathrm{SL}}(0.19753) = 1.19032 \times 388 = \mathbf{461.8442}$.

Problem 3. (5 marks)
Let the frequency distribution N of an aggregate loss S follows a geometric distribution with mean 4, and the severity distribution modelling the claim size X has the following c.d.f. $F_X(x) = 1 - \frac{1}{x^3}$ for $x > 1$. The N and X_i's $i \ge 1$ are independent. (The p.m.f. of the geometric distribution is given by $P(N=n) = \frac{4^n}{5^n}\frac{1}{5}$ for $n \ge 0$.)

1. Find the variance of the aggregate loss.
2. Calculate $\Pi_{\mathrm{sd}}(0.05)$ corresponding to the aggregate loss.
3. Use normal approximation to find the minimum capital $C_{\min}$ such that $P(S > C_{\min}) = 0.05$.

4. Now, assume that the c.d.f. $F_S(x) = 1 - \frac{3^2}{x^2}$ for $x > 3$. Calculate $E[S \wedge 5]$.

5. Calculate the net premium P to cover the the aggregate loss with deductible 5.

Solution:

1. We know $E[S] = E[N]E[X]$ and $\text{Var}(S) = E[N]\text{Var}(X) + (E[X])^2 \text{Var}(N)$. Let us first calculate $E[X]$ and $\text{Var}(X)$. We have

$$E[X] = \int_1^\infty x\frac{3dx}{x^4} = \frac{3}{2} \quad \text{and} \quad E[X^2] = \int_1^\infty x^2\frac{3dx}{x^4} = 3,$$

Thus

$$\text{Var}(X) = 3 - \left(\frac{3}{2}\right)^2 = \frac{3}{4} = 0.75.$$

Hence $E[S] = 4\frac{3}{2} = 6$ and $\text{Var}(S) = 4\frac{3}{4} + \left(\frac{3}{2}\right)^2\left(\frac{1-5^{-1}}{5^{-2}}\right) = 3 + \frac{9}{4}(20) = \mathbf{48}$.

2. $\Pi_{\text{Exp}}(0.05) = E[S] + 0.05\sqrt{\text{Var}(S)} = 6 + 0.05\sqrt{48} = \mathbf{6.3464}$.

3. We have

$$P(S > C_{\min}) = 0.05 \iff P(S \le C_{\min}) = 0.95 \iff P\left(\frac{S-6}{\sqrt{48}} \le \frac{C_{\min}-6}{\sqrt{48}}\right) = 0.95$$

$C_{\min} = 6 + 1.644854\sqrt{48} = \mathbf{17.39588}$.

4. By definition

$$E[S \wedge 5] = \int_3^5 x\frac{18}{x^3}dx + 5\int_5^\infty f_S(x)dx = \frac{12}{5} + 5\left(\frac{3^2}{5^2}\right) = \frac{21}{5} = \mathbf{4.2}.$$

5. The net premium $P = E\left[(S-5)^+\right] = E[S] - E[S \wedge 5] = 6 - \frac{21}{5} = \frac{9}{5} = \mathbf{1.8}$.

<u>Problem 4.</u> (5 marks)

A collective model with aggregate loss $S = \sum_{k=1}^N X_i$ where X_i are i.i.d. with common distribution $f_X(0) = 0.5$, $f_X(1) = 0.3$ and $f_X(2) = 0.2$ and N follows a Negative Binomial distribution with parameters $r = 5$ and $p = 0.7$.

1. Find $f_S(x)$, $x = 0, 1$ using Panjer's recursion.

2. Find $f_S(x)$, $x = 2, 3$ using Panjer's recursion.

3. Calculate the probability that the aggregate loss becomes greater than 3.75.

4. Calculate the mean and variance of the aggregate claim.

5. Use normal approximation to calculate quantile premium Π_{Qua} such that

$$P(S > \Pi_{\text{Qua}}) = 0.15.$$

Solution:

Recall that the Negative binomial $\mathcal{NB}(r;p)$ belongs to the class $C(a,b,0)$ with $a = 1-p = 0.3$, $b = (r-1)(1-p) = 4 \times 0.3 = 1.2$ and $p_0 = \dfrac{1}{(1+\beta)^r} = p^r = (0.7)^5 = 0.16807$.

1. Computation of $f_S(0)$ and $f_S(1)$. From the Panjer's recursion we have

$$f_S(0) = P_N\left(P_X(0)\right) = P_N\left(P(X=0)\right) = \left(\frac{0.7}{1-0.3\times 0.5}\right)^5 = \mathbf{0.37879}.$$

and from for any $n \geq 1$, we have

$$\begin{aligned} f_S(n) &= P(S=n) = \frac{\sum_{j=1}^{n}\left(a+\frac{b}{n}j\right) f_X(j) f_S(n-j)}{1-af_X(0)} \\ &= \frac{\sum_{j=1}^{n}\left(0.3+\frac{1.2}{n}j\right) f_X(j) f_S(n-j)}{1-0.3\times 0.5} = \frac{20}{17}\sum_{j=1}^{n}\left(0.3+\frac{1.2}{n}j\right) f_X(j) f_S(n-j) \end{aligned}$$

Thus

$$\begin{aligned} f_S(1) &= \frac{20}{17}\sum_{j=1}^{1}\left(0.3+\frac{1.2}{1}j\right) f_X(j) f_S(1-j)) = \frac{20}{17}\left(0.3+\frac{1.2}{1}\right) f_X(1) f_S(0) \\ &= \frac{20}{17}\left(0.3+\frac{1.2}{1}\right) 0.3\times 0.37879 = \mathbf{0.20054} \end{aligned}$$

2. Computation of $f_S(2)$ and $f_S(3)$

$$\begin{aligned} f_S(2) &= \frac{20}{17}\sum_{j=1}^{2}\left(0.3+\frac{1.2}{2}j\right) f_X(j) f_S(2-j) \\ &= \frac{20}{17}\left(0.3+\frac{1.2}{2}\right) f_X(1) f_S(1) + \frac{20}{17}\left(0.3+\frac{1.2}{2}2\right) f_X(2) f_S(0) \\ &= \frac{20}{17}\left(0.3+\frac{1.2}{2}\right) 0.3\times 0.20054 + \frac{20}{17}\left(0.3+\frac{1.2}{2}2\right) 0.2\times 0.37879 = \mathbf{0.19739} \\ f_S(3) &= \frac{20}{17}\sum_{j=1}^{3}\left(0.3+\frac{1.2}{3}j\right) f_X(j) f_S(3-j) \\ &= \frac{20}{17}\left(0.3+\frac{1.2}{3}\right) f_X(1) f_S(2) + \frac{20}{17}\left(0.3+\frac{1.2}{3}2\right) f_X(2) f_S(1) \text{ since } (f_X(3)=0) \\ &= \frac{20}{17}\left(0.3+\frac{1.2}{3}\right) 0.3\times 0.19739 + \frac{20}{17}\left(0.3+\frac{1.2}{3}2\right) 0.2\times 0.20054 = \mathbf{0.10067}. \end{aligned}$$

3. We have

$$\begin{aligned} P(S>3.75) &= 1-F_S(3.75) = 1-F_S(3) = 1-\sum_{n=0}^{3} P(S=n) \\ &= 1-(0.37879+0.20054+0.19739+0.10067) = \mathbf{0.12261}. \end{aligned}$$

4. We know $E[N] = E[\mathcal{NB}(5;0.7)] = \frac{5\times 0.3}{0.7} = 2.1429$ and $\mathrm{Var}(N) = \frac{5\times 0.3}{(0.7)^2} = 3.0612$ and $E[X] = 0.3+2\times 0.2 = 0.7$, $E\left[X^2\right] = 0.3+2^2\times 0.2 = 1.1$, and $\mathrm{Var}(X) = 1.1-(0.7)^2 = 0.61$.

$$E[S] = E[N]\,E[X] = 2.1429\times 0.7 = \mathbf{1.50003}.$$

and $\mathrm{Var}(S) = \mathbb{E}[N]\mathrm{Var}(X) + (\mathbb{E}[X])^2\,\mathrm{Var}(N) = 2.1429\times 0.61 + (0.7)^2\times 3.0612 = \mathbf{2.8072}$.

5. We have

$$P(S>\Pi_{\text{Qua}}) = 0.15 \Longleftrightarrow P(S\leq \Pi_{\text{Qua}}) = 0.85 \Longleftrightarrow P\left(\frac{S-1.50003}{\sqrt{2.8072}} \leq \frac{\Pi_{\text{Qua}}-1.50003}{\sqrt{2.8072}}\right) = 0.85$$

thus

$$\Pi_{\text{Qua}} = 1.50003 + 1.036433\sqrt{2.8072} = \mathbf{3.2365}.$$