King Saud University
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Solution of the first midterm exam ACTU–464 Fall 2019 (25%) (two pages)

October 14, 2019 (two hours 8–10 AM)

Problem 1. (6 marks)

Assume that the loss X is exponentially distributed with parameter 0.03. Consider partial risk $Z = 0.75X$.

1. **(2 marks)** Calculate the charged premium $\Pi_{\mathrm{SL}}(\theta)$ and θ such that $\mathrm{P}(Z \leq \Pi_{\mathrm{SL}}(\theta)) = 0.9$
2. **(2 marks)** Calculate the charged premium $\Pi_{\mathrm{Var}}(b)$ and b such that $\mathrm{P}(Z > \Pi_{\mathrm{Var}}(b))) = 0.15$
3. **(2 marks)** Calculate the charged premium $\Pi_{\mathrm{sd}}(c)$ and c such that $\mathrm{P}(Z > \Pi_{\mathrm{sd}}(c)) = 0.20$.

Solution:

1. We know that
$$\mathrm{P}\left(Z \leq \Pi_{\mathrm{SL}}(\theta)\right) = F_X\left(\frac{4}{3}\Pi_{\mathrm{SL}}(\theta)\right) = 1 - e^{-0.03\frac{4}{3}\Pi_{\mathrm{SL}}(\theta)} = 0.9$$
that is $e^{-0.03\frac{4}{3}\Pi_{\mathrm{SL}}(\theta)} = e^{-\frac{1}{25}\Pi_{\mathrm{SL}}(\theta)} = 0.1$, then $\Pi_{\mathrm{SL}}(\theta) = -25\ln(0.1) = \mathbf{57.565}$. Moreover
$$57.565 = (1+\theta)\,\mathrm{E}[Z] = (1+\theta)\,0.75\mathrm{E}[X] = (1+\theta)\,\frac{0.75}{0.03}$$
then $\theta = \frac{57.565}{25} - 1 = \mathbf{1.3026}$.
2. We have
$$\mathrm{P}\left(Z > \Pi_{\mathrm{Var}}(b))\right) = 0.15 \iff e^{-\frac{1}{25}\Pi_{\mathrm{Var}}(b)} = 0.15,$$
then $\Pi_{\mathrm{Var}}(b) = -25\ln(0.15) = \mathbf{47.428.}$ But
$$\Pi_{\mathrm{Var}}(b) = \mathrm{E}[Z] + b\mathrm{Var}(Z) = 0.75\frac{1}{0.03} + (0.75)^2\,\frac{b}{0.03^2} = 47.428,$$
hence $b = \mathbf{0.035885}$.
3. We have
$$\mathrm{P}(Z > \Pi_{\mathrm{sd}}(c))) = 0.2 \iff e^{-\frac{1}{25}\Pi_{\mathrm{sd}}(c)} = 0.2,$$
then $\Pi_{\mathrm{sd}}(c) = -25\ln(0.2) = \mathbf{40.236.}$ But
$$\Pi_{\mathrm{sd}}(c) = \mathrm{E}[Z] + c\sqrt{\mathrm{Var}(Z)} = 0.75\frac{1}{0.03} + (0.75)\,\frac{c}{0.03} = 40.236,$$
Thus $c = \mathbf{0.60944}$.

Problem 2. (6 marks)

Assume that the loss X is exponentially distributed with parameter 0.03. Consider partial risk $Z = \max(X - 20; 0)$.

1. **(2 marks)** Calculate the charged premium $\Pi_{\text{SL}}(\theta)$ and θ such that $\mathrm{P}(Z > \Pi_{\text{SL}}(\theta)) = 0.1$
2. **(2 marks)** Calculate the charged premium $\Pi_{\text{Var}}(b)$ and b such that $\mathrm{P}(Z \leq \Pi_{\text{Var}}(b))) = 0.85$
3. **(2 marks)** Calculate the charged premium $\Pi_{\text{sd}}(c)$ and c such that $\mathrm{P}(Z \leq \Pi_{\text{sd}}(c)) = 0.80$.

Solution:

1. We know that
$$\mathrm{P}(Z > \Pi_{\text{SL}}(\theta)) = S_X(\Pi_{\text{SL}}(\theta) + 20) = e^{-0.03(\Pi_{\text{SL}}(\theta)+20)} = 0.1$$
that is $e^{-0.03(\Pi_{\text{SL}}(\theta)+20)} = 0.1$, then $\Pi_{\text{SL}}(\theta) = -\frac{1}{0.03}\ln(0.1) - 20 = \mathbf{56.753}$. Moreover
$$56.753 = (1+\theta)\,\mathrm{E}[Z] = (1+\theta)\int_{20}^{\infty}(x-20)0.03e^{-0.03x}\mathrm{d}x = 18.294\,(1+\theta)$$
then $\theta = \frac{56.753}{18.294} - 1 = \mathbf{2.1023}$.

2. We have
$$\mathrm{P}(Z \leq \Pi_{\text{Var}}(b)) = 0.85 \Longleftrightarrow e^{-0.03(\Pi_{\text{Var}}(b)+20)} = 0.15,$$
then $\Pi_{\text{Var}}(b) = -\frac{1}{0.03}\ln(0.15) - 20 = \mathbf{43.237}$. On the other hand
$$\Pi_{\text{Var}}(b) = 43.237 = \mathrm{E}[Z] + b\mathrm{Var}(Z) = \mathrm{E}[Z] + b\left(\mathrm{E}\left[Z^2\right] - (\mathrm{E}[Z])^2\right),$$
but $\int_{20}^{\infty}(x-20)^2 0.03e^{-0.03x}\mathrm{d}x = 1219.6$, hence the equation in terms of b becomes $18.294 + b\left(1219.6 - (18.294)^2\right) = 43.237$, thus $b = \mathbf{0.028186}$.

3. From the relationship between X and Z we can write
$$\mathrm{P}(Z \leq \Pi_{\text{sd}}(c)) = 0.8 \Longleftrightarrow e^{-0.03(\Pi_{\text{sd}}(c)+20)} = 0.2,$$
then $\Pi_{\text{sd}}(c) = -\frac{1}{0.03}\ln(0.2) - 20 = \mathbf{33.648}$. We know also that
$$\Pi_{\text{sd}}(c) = 43.237 = \mathrm{E}[Z] + c\sqrt{\mathrm{Var}(Z)} = \mathrm{E}[Z] + c\sqrt{\mathrm{E}[Z^2] - (\mathrm{E}[Z])^2},$$
hence the equation in terms of c becomes $18.294 + c\sqrt{\left(1219.6 - (18.294)^2\right)} = 33.648$, thus $c = \mathbf{0.51614}$.

Problem 3. (6 marks)

1. The approximated premiums P_a^+ and P_a^- of a given loss X with mean μ and standard deviation σ and utility function u are given by
$$P_a^+ = \mu + \frac{1}{2}\sigma^2 r(W-\mu) \quad \text{and} \quad P_a^- = \mu + \frac{r(W-\mu)}{2}\left(\sigma^2 + (P_a^-)^2\right)$$
where W is a given wealth and $r(x) = \frac{-u''(x)}{u'(x)}$ is the risk aversion coefficient.
(2 marks) Calculate P_a^+ for $u(x) = -0.02e^{-0.02x}$, $\mu = 20$; $\sigma = 4$.

2. **(2 marks)** Calculate P_a^- and explain if there will be a deal between the insurer and the insured.

3. **(2 marks)** Given a loss random variable X the premium $\Pi_X(h)$ with parameter h corresponding to this loss is given by the Esscher principle that is

$$\Pi_X(h) = \frac{\mathrm{E}\left[Xe^{hX}\right]}{\mathrm{E}\left[e^{hX}\right]}.$$

Calculate $\Pi_X(0.25)$ when X is exponentially distributed with parameter 0.5.

Solution:

1. We have $r(x) = \dfrac{-u''(x)}{u'(x)} = 0.02$ for any x then $r(W-\mu) = 0.02$. Hence the premium

$$P_a^+ = 20 + \frac{0.02}{2}4^2 = \mathbf{20.16}.$$

2. For P_a^-, one should solve the second order equation $x = 20 + \frac{0.02}{2}\left(4^2 + x^2\right)$. But there are two solutions 72 and 28, hence $P_a^- = \min(72.0; 28) = \mathbf{28}$. So the **deal cannot be made** since $P_a^- = \mathbf{28} > P_a^+ = \mathbf{20.16}$.

3. Since X is exponentially distributed with parameter 0.5 we can write

$$\Pi_X = \frac{\mathrm{E}\left[Xe^{0.25X}\right]}{\mathrm{E}\left[e^{0.25X}\right]} = \frac{\int_0^\infty xe^{0.25x}\frac{1}{2}e^{-\frac{1}{2}x}\mathrm{d}x}{M_X(0.25)}$$

Moreover $\int_0^\infty xe^{0.25x}\frac{1}{2}e^{-\frac{1}{2}x}\mathrm{d}x = 8$ and $M_X(0.25) = \frac{0.5}{0.5-0.25} = 2$, thus $\Pi_X = \frac{8}{2} = \mathbf{4}$.

Problem 4. **(8 marks)**

1. **(2 marks)** An insurer with net wealth 500 has accepted (and collected the premium for) a risk X with the following probability distribution: $\mathrm{P}(X=0) = \mathrm{P}(X=100) = \mathrm{P}(X=200) = \frac{1}{3}$. What is the maximum amount P^+ it should pay another insurer to accept 80% of this loss? Assume the first insurer's utility function of wealth is $u(x) = \ln(x)$.

2. **(2 marks)** An insurer, with wealth 500 and the same utility function, $u(x) = \ln(x)$, is considering accepting the above risk. What is the minimum amount P^- this insurer would accept as a premium to cover 80% of the loss? The positive solution to the equation $500^3 = (500+x)(420+x)(340+x)$ is **84.267**.

3. **(2 marks)** Verify if the premiums satisfy the natural order such that the deal can be made.

Solution:

1. The the maximum amount P^+ is given by the equation

$$\begin{aligned} u(W-P^+) &= \mathrm{E}\left[u\left(W-X\right)\right] \iff \ln\left(500-P^+\right) = \frac{u\left(500\right)+u\left(500-80\right)+u\left(500-160\right)}{3} \\ &= \frac{u\left(500\right)+u\left(500-80\right)+u\left(500-160\right)}{3} = \frac{\ln\left(500\times 420\times 340\right)}{3} = 6.0279 \end{aligned}$$

Which leads to $\ln\left(500-P^+\right) = 6.0279$, then $P^+ = 500 - e^{6.0279} = \mathbf{85.157}$.

2. The minimum amount P^- this insurer would accept to cover 80% of the loss is given by the equation.

$$u(W) = \mathrm{E}\left[u\left(W + P^- - X\right)\right]$$

that is

$$\ln(500) = \frac{u(500 + P^-) + u(420 + P^-) + u(340 + P^-)}{3}$$

which is equivalent to

$$3\ln(500) = \ln\left(500^3\right) = \ln\left[\left(500 + P^-\right)\left(420 + P^-\right)\left(340 + P^-\right)\right],$$

solving for P^-= **84.267.**

3. The deal can be make since $\mathrm{E}[0.8X] = \mathbf{80} < P^- = \mathbf{84.267} < P^+ = \mathbf{85.157}$.

Problem 5. (6 marks)

1. **(2 marks)** The probability that a property will not be damaged in the next period is 0.40. The probability density function (p.d.f.) of a positive loss is given by $f(x) = 0.6(0.02e^{-0.02x})$ for $x > 0$. The owner of the property has a utility function given by $u(x) = -\frac{1}{100}e^{-\frac{x}{100}}$.
Calculate the expected loss and the maximum insurance premium the property owner will pay for complete insurance.

2. **(2 marks)** The property owner in the question 1 is offered an insurance policy that will pay $Z = \max(X - 200; 0)$ during the next period. Calculate the maximum premium that the property owner will pay for this insurance.

3. **(2 marks)** Find also the minimum premium that the insured will accept to pay for this excess loss insurance.

Solution:

1. The expected loss is given by

$$\mathrm{E}[X] = 0.4(0) + 0.6\int_0^\infty x(0.02e^{-0.02x})\mathrm{d}x = 30.$$

The maximum premium will be consistent with the property owner's preferences as summarized in the utility function:

$$\begin{aligned} u(W - P^+) &= 0.4u(W) + \int_0^\infty u(W - x)f(x)\mathrm{d}x \text{ hence} \\ -\frac{1}{100}e^{-\frac{W-P^+}{100}} &= -0.4\frac{1}{100}e^{-\frac{W}{100}} - 0.6\int_0^\infty \frac{1}{100}e^{-\frac{W-x}{100}}(0.02e^{-0.02x})\mathrm{d}x. \end{aligned}$$

Simplify with $-\frac{1}{100}e^{-\frac{W}{100}}$ in each term we get

$$e^{0.01P^+} = 0.4 + 0.6 \times 2 = 1.6 \iff P^+ = 100\ln(1.6) = \mathbf{47}.$$

Therefore, in accord with the property owner's preferences, he will pay up to $47 - 30 = 17$ in excess of the expected loss to purchase insurance covering all losses in the next period.

2. The maximum premium that the property owner will pay for this partial insurance or excess loss insurance / reinsurance is given by the equation

$$\begin{aligned} u(W-P^+) &= 0.4u(W) + \int_0^\infty u\left(W - \max(x-200;0)\right) f(x)\mathrm{d}x \text{ hence} \\ -\frac{1}{100}e^{-0.01\left(W-P^+\right)} &= -0.4\frac{1}{100}e^{-0.01W} - 0.6\int_0^\infty \frac{1}{100}e^{-0.01(W-\max(x-200;0))}(0.02e^{-0.02x})\mathrm{d}x. \end{aligned}$$

After simplifications by $-0.01e^{-0.01W}$ and the fact that

$$\begin{aligned} \int_0^\infty e^{0.01(\max(x-200;0))}(0.02e^{-0.02x})\mathrm{d}x &= \int_0^{200} 0.02e^{-0.02x}\mathrm{d}x + \int_{200}^\infty e^{0.01(x-200)}(0.02e^{-0.02x})\mathrm{d}x \\ &= 0.98168 + 0.03661 = \mathbf{1.0183}. \end{aligned}$$

therefore,

$$e^{0.01P^+} = 0.4 + (0.6)(1.0183) = 1.0110 \iff P^+ = 100\ln(1.0110) = \mathbf{1.093994}.$$

Remark that

$$\mathrm{E}\left[\max(X-200;0)\right] = \int_{200}^\infty (x-200)\times 0.6\times 0.02e^{-0.02x}\mathrm{d}x = 0.54947 < P^+ = \mathbf{1.093994}.$$

3. The minimum premium P^- is given by

$$\begin{aligned} u(W) &= 0.4u(W+P^-) + \int_0^\infty u\left(W + P^- - \max(x-200;0)\right) f(x)\mathrm{d}x \text{ hence} \\ -\frac{1}{100}e^{-0.01W} &= -0.4\frac{1}{100}e^{-0.01\left(W+P^-\right)} - 0.6\int_0^\infty \frac{1}{100}e^{-0.01\left(W+P^- -\max(x-200;0)\right)}(0.02e^{-0.02x})\mathrm{d}x. \end{aligned}$$

After simplifications by $\frac{1}{100}e^{-0.01\left(W+P^-\right)}$ and the fact that

$$\begin{aligned} \int_0^\infty e^{0.01(\max(x-200;0))}(0.02e^{-0.02x})\mathrm{d}x &= \int_0^{200} 0.02e^{-0.02x}\mathrm{d}x + \int_{200}^\infty e^{0.01(x-200)}(0.02e^{-0.02x})\mathrm{d}x \\ &= 0.98168 + 0.03661 = \mathbf{1.0183} \end{aligned}$$

therefore,

$$e^{0.01P^-} = 0.4 + (0.6)(1.0183) = 1.0110 \iff P^- = \mathbf{1.093994}.$$

Remark that $P^+ = P^-$.