King Saud University
College of Sciences
Mathematics Department

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Solution of the first midterm exam ACTU–464 SPRING 2020 (25%) (two pages)

February 24, 2020 (two hours 8–10 AM)

Problem 1. (6 marks)

Consider a loss X having a c.d.f.

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ \dfrac{x^2 - 2x + 2}{2} & \text{if} \quad 1 \le x < 2 \\ 1 & \text{if} \quad x \ge 2 \end{cases}$$

1. **(2 marks)** Calculate the safety loading premium $\Pi_{\text{SL}}(0.05)$ and $P(X \le \Pi_{\text{SL}}(0.05))$
2. **(2 marks)** Calculate the σ–loading premium $\Pi_{\text{sd}}(0.05)$ and $P\left(X > \Pi_{\text{sd}}(0.05)\right)$
3. **(2 marks)** Calculate the exponential premium $\Pi_{\text{Exp}}(0.05)$ and $P(X \le \Pi_{\text{Exp}}(0.05)) = 0.20$.

Solution:

1. We know that $\Pi_{\text{SL}}(0.05) = 1.05E[X]$. We need first to find the distribution of X

$$f_X(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ F_X(1) - F_X(1-) = \dfrac{1}{2} & \text{if} \quad x = 1 \\ x - 1 & \text{if} \quad 1 < x < 2 \\ 0 = F_X(2) - F_X(2-) & \text{if} \quad x \ge 2. \end{cases}$$

Thus

$$\Pi_{\text{SL}}(0.05) = 1.05E[X] = 1.05(0.5 + \int_1^2 x(x-1)\,dx) = 1.05 \times 1.3333 = \mathbf{1.4}.$$

And

$$P(X \le \Pi_{\text{SL}}(0.05)) = P(X \le 1.4) = \frac{(1.4)^2 - 2 \times 1.4 + 2}{2} = \mathbf{0.58}.$$

2. By definition $\Pi_{\text{sd}}(0.05) = E[X] + 0.05\sqrt{\text{Var}(X)}$, $E[X^2] = 0.5 + \int_1^2 x^2(x-1)\,dx) = 1.9167$ and $\text{Var}(X) = 1.9167 - (1.3333)^2 = 0.13901$. Hence

$$\Pi_{\text{sd}}(0.05) = 1.3333 + 0.05\sqrt{0.13901} = \mathbf{1.3519}.$$

And

$$\begin{aligned} P\left(X > \Pi_{\text{sd}}(0.05)\right) &= 1 - P(X \le 1.3519) \\ &= 1 - \frac{(1.3519)^2 - 2 \times 1.3519 + 2}{2} \\ &= 1 - 0.56192 = \mathbf{0.43808}. \end{aligned}$$

3. By definition $\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05}\ln\left(M_X(0.05)\right)$, and

$$M_X(0.05) = \frac{1}{2}e^{0.05} + \int_1^2 e^{0.05x}\,(x-1)\,dx = 1.0691,$$

therefore

$$\Pi_{\text{Exp}}(0.05) = \frac{1}{0.05}\ln\left(1.0691\right) = \mathbf{1.3363}.$$

And

$$\begin{aligned} P(X &> \Pi_{\text{Exp}}(0.05)) = 1 - P(X \leq 1.3363) \\ &= 1 - \frac{(1.3363)^2 - 2\times 1.3363 + 2}{2} \\ &= 1 - 0.55655 = \mathbf{0.44345}. \end{aligned}$$

<u>Problem 2.</u> (6 marks) Consider a stop–loss with retention level $M = 1.2$ (partial insurance cover with payoff Z) of loss considered in the Problem 1.

1. **(2 marks)** Calculate the σ^2–loading premium Π_{Var} such that $P\left(Z > \Pi_{\text{Var}}\right) = 0.2$
2. **(2 marks)** Calculate the quantile premium Π_{Qua} such that $P\left(Z \leq \Pi_{\text{Qua}}\right) = 0.75$.
3. **(2 marks)** Calculate the safety loading parameter θ assuming that Z is normally distributed such that $P\left(Z \leq \Pi_{\text{SL}}(\theta)\right) = 0.95$.

Solution:

1. The equation $P\left(Z > \Pi_{\text{Var}}(c)\right) = 0.2 = P\left(X > \Pi_{\text{Var}}(c) + 1.2\right) = 0.2 \Longleftrightarrow P\left(X \leq \Pi_{\text{Var}}(c) + 1.2\right) = 0.8$, that is

$$F_X\left(\Pi_{\text{Var}}(c) + 1.2\right) = 0.8$$

We know that $P(Z \leq 1) = 0.5$ then the solution to the equation $F_X(x) = \dfrac{x^2 - 2x + 2}{2} = 0.8$, in he interval $]1, 2[$ is: 1.7746, thus $\Pi_{\text{Var}}(c) = 1.7746 - 1.2 = \mathbf{0.5746}$.

2. Similarly $F_X\left(\Pi_{\text{Qua}} + 1.2\right) = 0.75$ gives $\Pi_{\text{Qua}} = 1.7071 - 1.2 = \mathbf{0.5071}$.

3. Set $T = \frac{Z - E[Z]}{\sigma_Z}$ then

$$P\left(Z \leq \Pi_{\text{SL}}(\theta)\right) = 0.95 \Longleftrightarrow P\left(T > \theta\frac{E[Z]}{\sigma_Z}\right) = 0.05$$

hence $\theta\frac{E[Z]}{\sigma_Z} = 1.644854$ thus $\theta E[Z] = 1.644854\sigma_Z$ consequently $\Pi_{\text{SL}}(\theta) = E[Z] + 1.644854\sigma_Z$. Now, $E[Z] = \int_{1.2}^2 (x-1.2)\,(x-1)\,dx = 0.23467$ and $E[Z^2] = \int_{1.2}^2 (x-1.2)^2\,(x-1)\,dx = 0.13653$ so $\sigma_Z = \sqrt{0.13653 - (0.23467)^2} = 0.28541$, finally

$$\Pi_{\text{SL}}(\theta) = 0.23467 + 1.644854 \times 0.28541 = \mathbf{0.70413}.$$

<u>Problem 3.</u> (6 marks)

A decision maker's utility function is given by $u(x) = \sqrt{x}$. The decision maker has wealth of $W = 1000$ and faces a random loss X with a uniform distribution on $(0,\ 1000)$.

1. **(1 mark)** Calculate P^+ for complete insurance against the random loss X?
2. **(1 mark)** Calculate P^- for complete insurance against the random loss X?
3. **(2 marks)** Calculate P^+ for stop–loss insurance for a retention level $M = 500$?
4. **(2 marks)** Calculate P^- for stop–loss insurance for a retention level $M = 500$?

Solution:

1. We have

$$\begin{aligned}\sqrt{1000 - P^+} &= \int_0^{1000} \sqrt{1000 - x}\frac{dx}{1000} = \left[-\frac{2}{3000}\left(1000 - x\right)^{\frac{3}{2}}\right]_0^{1000} \\ &= \frac{2}{3000}\left(1000\right)^{\frac{3}{2}} = \frac{2}{3}\sqrt{1000} = \mathbf{21.082}\end{aligned}$$

Thus $1000 - P^+ = \frac{4}{9}1000$ finally $P^+ = 1000 - \frac{4}{9}1000 = \frac{5}{9}1000. = \mathbf{555.56} > E\left[X\right] = 500.$

2. For the premium P^- to be paid by the insurer we have to solve

$$\sqrt{1000} = \int_0^{1000} \sqrt{1000 + P^- - x}\frac{dx}{1000}$$

which leads to $\frac{2}{3000}\left(\left(1000 + P^-\right)^{\frac{3}{2}} - \left(P^-\right)^{\frac{3}{2}}\right) = \sqrt{1000}$, thus $P^- = \mathbf{521.30}$. Clearly

$$E\left[X\right] = 500 < P^- = 521.30 < P^+ = 555.56.$$

3. For stop–loss we have similar equation we just replace X with $Z = (X - 500)^+$, therefore we get

$$\begin{aligned}\sqrt{1000 - P^+} &= \int_0^{1000} \sqrt{1000 - (x - 500)^+}\frac{dx}{1000} \\ &= \int_0^{500} \sqrt{1000}\frac{dx}{1000} + \int_{500}^{1000} \sqrt{1500 - x}\frac{dx}{1000} \\ &= 5\sqrt{2}\sqrt{5} - \frac{10}{3}\sqrt{5} + \frac{20}{3}\sqrt{10} = 29.440.\end{aligned}$$

$\sqrt{1000 - x} = 29.440$, the solution is:

$$P^+ = \mathbf{133.29} > E\left[(X - 500)^+\right] = \int_{500}^{1000} (x - 500)\frac{dx}{1000} = 125.$$

4. For P^- we have

$$\begin{aligned}\sqrt{1000} &= \int_0^{1000} \sqrt{1000 + P^- - (x - 500)^+}\frac{dx}{1000} \\ &= \int_0^{500} \sqrt{1000 + P^-}\frac{dx}{1000} + \int_{500}^{1000} \sqrt{1500 + P^- - x}\frac{dx}{1000} \\ &= \frac{1}{2}\sqrt{1000 + P^-} + \left[-\frac{2}{3000}\left(1500 + P^- - x\right)^{\frac{3}{2}}\right]_{500}^{1000} \\ &= \frac{1}{2}\sqrt{1000 + P^-} + \frac{2}{3000}\left(1000 + P^-\right)^{\frac{3}{2}} - \frac{2}{3000}\left(500 + P^-\right)^{\frac{3}{2}}\end{aligned}$$

Thus the solution to the

$$\frac{1}{2}\sqrt{1000 + x} + \frac{2}{3000}\left(1000 + x\right)^{\frac{3}{2}} - \frac{2}{3000}\left(500 + x\right)^{\frac{3}{2}} = \sqrt{1000},$$

is $P^- = \mathbf{132.09}$ clearly $E\left[(X - 500)^+\right] = 125 < P^- = 132.09 < P^+ = 133.29.$

Problem 4. (6 marks)

1. For a given loss X with mean and standard deviation equal to 5. When a utility function u is replaced by its quadratic form, the approximated premiums P_a^+ and P_a^- can be expressed as follows

$$P_a^+ = E[X] + \frac{r(W - E[X])}{2}\mathrm{Var}(X) \quad \text{and} \quad P_a^- = E[X] + \frac{r(W - E[X])}{2}\left(\mathrm{Var}(X) + (P_a^-)^2\right)$$

   where W is a given wealth and $r(x) = -\frac{u''(x)}{u'(x)}$ is the risk aversion coefficient.
   (2 marks) Calculate P_a^+ for $u(x) = -5e^{-0.03x}$ and $W = 100,000$.

2. **(2 marks)** Calculate P_a^- for $u(x) = -5e^{-0.003x}$ and $W = 1000$.

3. **(2 marks)** Explain if there will be a deal between the insurer and the insured.

Solution:

1. We have $r(x) = \dfrac{-u''(x)}{u'(x)} = 0.03$ for any x then $r(1000000 - 5) = 0.03$. Hence the premium

$$P_a^+ = 5 + \frac{0.03}{2}(5)^2 = \mathbf{5.375}.$$

2. For P_a^-, one should solve the second order equation $x = 5 + \frac{0.03}{2}\left((5)^2 + x^2\right)$, but there are two solutions 5.8965 and 60.77, hence $P_a^- = \mathbf{5.8965}$.

3. There will not be a **deal** since $P_a^- > P_a^+$.

Problem 5. (6 marks)

Given exponentially distributed loss X with with parameter 0.2.

1. **(2 marks)** Calculate the exact values of P^+ for $u(x) = -5e^{-0.03x}$ and $W = 5000$

2. **(2 marks)** Calculate the exact values of P^- for $u(x) = -5e^{-0.03x}$ and $W = 7000$

3. **(2 marks)** Compare the premiums obtained in the Problems 4 and 5 and conclude.

Solution:

1. We have $\alpha = 0.03$, $\lambda = 0.2$, and $M_X(\alpha) = \frac{\lambda}{\lambda - \alpha}$ since $(\alpha < \lambda)$ thus

$$P^+ = \frac{1}{\alpha}\ln\left(\frac{\lambda}{\lambda - \alpha}\right) = \frac{1}{0.03}\ln\left(\frac{0.2}{0.2 - 0.03}\right) = \mathbf{5.4173}.$$

2. We know that for the exponential utility $P^- = P^+ = \frac{1}{\alpha}\ln\left(\frac{\lambda}{\lambda - \alpha}\right) = \mathbf{5.4173}$.

3. When comparing premiums we observe that $P_a^- = \mathbf{5.8965} > P^- = P^+ = \mathbf{5.4173} > P_a^+ > \mathbf{5.375}$. Moreover

$$P^+ - P_a^+ = 5.4173 - 5.375 = 0.0423 \text{ but } P_a^- - P^- = 5.8965 - 5.4173 = 0.4792$$

   we can see that P^+ is close to P_a^+ but P_a^- and P^- are not close to each other. Finally with the values of P_a^- and P_a^+ there will be no deal between the two parts.