King Saud University
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Solution of the second midterm exam ACTU–464 Fall 2019 (25%) (two pages)

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Problem 1. (6 marks)

Consider an insurance policy against a liability loss S when there is a deductible of 1 and a maximum payment of 10. Let X denotes the payment from the insurer's point of view. Set $X = \text{IB}$, The probability of a positive claim is 10% and the probability of $X = 10$ is 2%. Given $1 < S < 11$, S has a uniform$(1, 11)$ distribution.

1. **(2 marks)** Calculate $\mathbb{E}[X]$ using the formula $\mathbb{E}[X] = \mu q$, where $\mu = \mathbb{E}[B|I = 1]$
2. **(2 marks)** Calculate $\text{Var}(X)$ using the formula $\text{Var}(X) = \mu^2 q(1-q)+\sigma^2 q$, where $\sigma^2 = \text{Var}(B|I = 1)$
3. **(2 marks)** Use normal approximation to calculate the percentile premium $P_{0.95}$ of the risk X. that is $P(X \leq P_{0.95}) = 0.95$. (The solution to the equations $P(\mathcal{N}(0, 1) > t) = 0.05$ is $t = 1.644854$)

Solution:

1. We have $X = \text{IB}$ where $I = 1$ when there is a payment and 0 otherwise and B represents the amount paid, if any. We know that
$$\mathbb{P}(B = 10|I = 1) = \frac{\mathbb{P}(B = 10;\ I = 1)}{\mathbb{P}(I = 1)}.$$
Therefore $\mathbb{P}(I = 1) = 0.1$ (probability of a positive claim) and
$$\mathbb{P}(B = 10;\ I = 1) = \mathbb{P}(X = 10) = \mathbb{P}(\text{ probability of a large loss }) = 0.02,$$
hence $\mathbb{P}(B = 10|I = 1) = \frac{0.02}{0.1} = 0.2$. Moreover $f_{B|I}(x|1) = c$ for $0 < x < 10$.
So c is determined by using the property
$$\mathbb{P}(B = 10|I = 1) + \int f_{B|I}(x|1)dx = 1 \iff 0.8 = \int_0^{10} c dx \text{ that is } 10c = 0.8$$
this yields $c = 0.08$. The conditional distribution function of B, given $I = 1$, is neither discrete, nor continuous. We have $f_{B|I}(x|1) = 0.08$ on $(0, 10)$ and $\mathbb{P}(B = 10|I = 1) = 0.2$. Hence $\mu = \mathbb{E}[B|I = 1] = 10 \times 0.2 + \int_0^{10} 0.08x dx = 6$. So$\mathbb{E}[X] = 6 \times 0.1 = \mathbf{0.6}$.
2. Moreover
$$\mathbb{E}[B^2|I = 1] = 10^2 \times 0.2 + \int_0^{10} 0.08x^2 dx = 46.667$$
and then $\sigma^2 = 46.667 - (0.6)^2 = 46.307$. Thus $\text{Var}(X) = 5^2 \times 0.1(1 - 0.1) + 46.307 \times 0.1 = \mathbf{6.8807}$.
3. $P_{0.95}$ is given by
$$P(X \leq P_{0.95}) = P\left(\frac{X - \mathbb{E}[X]}{\sqrt{\text{Var}(X)}} \leq \frac{P_{0.95} - \mathbb{E}[X]}{\sqrt{\text{Var}(X)}}\right) = P\left(\frac{X - 0.6}{2.6231} \leq \frac{P_{0.95} - 0.6}{2.6231}\right) = 0.95$$
Thus $\frac{P_{0.95}-0.6}{2.6231} = 1.644854$, $P_{0.95} = 0.6 + 2.6231 \times 1.644854 = \mathbf{4.9146}$.

Problem 2. (6 marks)

A portfolio consists of two types of contracts. For type k, $k = 1, 2$, the claim probability is q_k and the number of policies is n_k. If there is a claim, then its size is x with probability $f_k(x)$:

	n_k	q_k	$f_k(1)$	$f_k(2)$	$f_k(3)$
Type I	1000	0.01	0.5	0	0.5
Type II	2000	0.02	0.5	0.5	0

Assume that the contracts are independent. Let S_k denote the total claim amount of the contracts of type k and let $S = S_1 + S_2$.

1. **(2 marks)** Calculate the expected value and the variance of a contract of type k, $k = 1, 2$.
2. **(2 marks)** Then, calculate the expected value and the variance of S.
3. **(2 marks)** Use the normal approximation to determine the minimum capital that covers all claims with probability 95%.

Solution:

1. First we have $\mu_1 = \frac{1}{2} + \frac{3}{2} = 2$ and $\mu_2 = \frac{1}{2} + \frac{2}{2} = \frac{3}{2} = 1.5$, then $2000 \times 1.5 \times 0.02 = 60.0$

$$\begin{aligned}\mathbb{E}[S_1] &= n_1\mu_1 q_1 = 1000 \times 2 \times 0.01 = \mathbf{20}.\\ \mathbb{E}[S_2] &= n_2\mu_2 q_2 = 2000 \times 1.5 \times 0.02 = \mathbf{60}.\end{aligned}$$

Second $\sigma_1^2 = \frac{1}{2} + \frac{3^2}{2} - 2^2 = 1$, then

$$\operatorname{Var}(S_1) = n_1\left(\mu_1^2 q_1(1-q_1) + \sigma_1^2 q_1\right) = 1000\left(2^2 \times 0.01(1-0.01) + 1^2 \times 0.01\right) = \mathbf{49.6}.$$

and $\sigma_2^2 = \frac{1}{2} + \frac{2^2}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4} = 0.25$, then

$$\operatorname{Var}(S_2) = n_2\left(\mu_2^2 q_2(1-q_2) + \sigma_2^2 q_2\right) = 2000\left(1.5^2 \times 0.02(1-0.02) + 0.25^2 \times 0.02\right) = \mathbf{90.7}.$$

2. We know that $\mathbb{E}[S] = \mathbb{E}[S_1] + \mathbb{E}[S_2] = 20 + 60 = 80$ and

$$\operatorname{Var}(S) = \operatorname{Var}(S_1) + \operatorname{Var}(S_2) = 49.6 + 90.7 = \mathbf{140.3}.$$

3. Denote by $C_{\min}$ the minimum capital such that $P(S \le C_{\min}) = 0.95$, then by normal approximation we get

$$C_{\min} = t_{0.95}\sqrt{\operatorname{Var}(S)} + \mathbb{E}[S] = 1.644854\sqrt{140.3} + 80 = \mathbf{99.483}.$$

Problem 3. (6 marks)

1. Assume that $X \sim \text{Uniform}(0, 3)$ and $Y \sim \text{Uniform}(-1, 1)$. Calculate $F_{X+Y}(z)$,
2. Find the premium Π such that $P(X + Y > \Pi) = 0.10$.

Solution:

1. Set $S = X + Y$, observe first that $f_S(s) = 0$ for $s \leq -1$ or $s \geq 4$. Now for $s \leq -1 < s < 4$, we can write

$$\begin{aligned} f_S(s) &= \int_{-\infty}^{\infty} f_X(s-y) f_Y(y) dy = \int_{-\infty}^{\infty} f_Y(s-x) f_X(x) dx \\ &= \int_{-1}^{1} f_X(s-y) f_Y(y) dy = \frac{1}{2} \int_{-1}^{1} f_X(s-y) dy = \frac{1}{2} \int_{-1}^{1} f_X(s-y) dy. \end{aligned}$$

Moreover,

$$f_X(s-y) = \begin{cases} 0 \text{ if } s-y \leq 0 & \Longleftrightarrow & s \leq y \\ \dfrac{1}{3} \text{ if } 0 < s-y < 3 & \Longleftrightarrow & s-3 < y < s \\ 0 \text{ if } s-y \geq 3 & \Longleftrightarrow & s-3 \geq y. \end{cases}$$

then $f_S(s) = \frac{1}{6} \int_{(-1)\vee(s-3)}^{1\wedge s} dy = \frac{1}{6} \left(1 \wedge s - (-1) \vee (s-3)\right)$ and

$$f_S(s) = \begin{cases} \dfrac{s+1}{6} & \text{if } -1 < s < 1 \\ \dfrac{1+1}{6} = \dfrac{1}{3} & \text{if } 1 \leq s < 2 \\ \dfrac{1-(s-3)}{6} = \dfrac{4-s}{6} & \text{if } 2 \leq s < 4 \end{cases}$$

or

$$f_S(s) = \int_{-\infty}^{\infty} f_Y(s-x) f_X(x) dx = \int_{0}^{3} f_Y(s-x) dx = \frac{1}{6} \left(3 \wedge (1+s) - 0 \vee (s-1)\right)$$

consequently

$$f_S(s) = \begin{cases} \dfrac{s+1}{6} & \text{if } -1 < s < 1 \\ \dfrac{s+1-(s-1)}{6} = \dfrac{1}{3} & \text{if } 1 \leq s < 2 \\ \dfrac{3-(s-1)}{6} = \dfrac{4-s}{6} & \text{if } 2 \leq s < 4 \end{cases}$$

Finally

$$F_S(s) = \int_{-\infty}^{s} f_S(u) du = \begin{cases} 0 & \text{if } s \leq -1 \\ \dfrac{(s+1)^2}{12} & \text{if } -1 < s < 1 \\ \dfrac{1}{3} + \dfrac{s-1}{3} = \dfrac{s}{3} & \text{if } 1 \leq s < 2 \\ 1 - \dfrac{(4-s)^2}{12} & \text{if } 2 \leq s < 4 \\ 1 & \text{if } s \geq 4 \end{cases}$$

2. $P(S > \Pi) = 1 - F_S(\Pi) = 0.10$, that is $F_S(\Pi) = 0.9$, since $F_S(1) = \frac{1}{3} = 0.333$ and $F_S(2) = 0.66667$, so the solution to the equation $F_S(s) = 0.9$ should be in the interval $(2; 4)$, thus $1 - \dfrac{(4-\Pi)^2}{12} = 0.9$, and then $\Pi = \mathbf{2.9046}$.

<u>Problem 4.</u> (6 marks)

1. **(3 marks)** For an aggregate loss the frequency distribution is Poisson with $\lambda = 3$ and individual claim amount distribution

x	1	2	3	4
$f_X(x)$	0.4	0.3	0.2	0.1

Determine the probability that aggregate losses do not exceed 3.

2. **(3 marks)** For an insurance policy you are given:
(i) The number of losses per year has a Poisson distribution with $\lambda = 1$.
(ii) Loss amounts are uniformly distributed on $(0,1)$.
(iii) Loss amounts and the number of losses are mutually independent.
(iv) There is an ordinary deductible of $\frac{1}{4}$ per loss.
Calculate the variance of aggregate payments.

Solution:

1. We have to find $P(S=k)$, $k = 0,1,2,3$

$$\begin{aligned}
P(S=0) &= P(N=0) = e^{-3} = 0.049787 \\
P(S=1) &= P(N=1, X_1=1) = P(N=1)P(X_1=1) \\
&= (3e^{-3})(0.4) = (1.2)e^{-3} = 0.059744 \\
P(S=2) &= P(N=1, X_1=2) + P(N=2, X_1=1, X_2=1) \\
&= P(N=1)P(X_1=2) + P(N=2)P(X_1=1)P(X_2=1) \\
&= (3e^{-3})(0.3) + \left(e^{-3}\frac{3^2}{2}\right)(0.4)^2 = (1.62)e^{-3} = 0.080655 \\
P(S=3) &= P(N=1, X_1=3) + P(N=2, X_1=1, X_2=2) \\
&\quad + P(N=3, X_1=1, X_2=1, X_3=1) + P(N=2, X_1=2, X_2=1) \\
&\quad + P(N=3, X_1=1, X_2=1, X_3=1) \\
&= (3e^{-3})(0.2) + 2(e^{-3}\frac{3^2}{2})(0.4)(0.3) + (e^{-3}\frac{3^3}{3!})(0.4)^3 \\
&= (1.968)e^{-3} = 0.097981.
\end{aligned}$$

Therefore

$$\begin{aligned}
P(S \le 3) &= P(S=0) + P(S=1) + P(S=2) + P(S=3) \\
&= (1 + 1.2 + 1.62 + 1.968)e^{-3} = (5.788)\,e^{-3} = \mathbf{0.28817}.
\end{aligned}$$

2. Let S be the aggregate claims. Then S is a compound distribution r.v. with a Poisson primary distribution for N, the number of claims, and secondary distribution is uniformly distributed on $(0,1)$ $Y = \max(X - 0.25; 0)$, the amount paid per loss, where X is the loss amount. We want

$$\mathrm{Var}(S) = \mathbb{E}[N]\,\mathrm{Var}(Y) + \mathrm{Var}(N)\,(\mathbb{E}[Y])^2 = \lambda\left(\mathrm{Var}(Y) + (\mathbb{E}[Y])^2\right) = \lambda\mathbb{E}\left[Y^2\right] = \mathbb{E}\left[Y^2\right]$$

Thus

$$\mathrm{Var}(S) = \mathbb{E}\left[Y^2\right] = \mathbb{E}\left[\max(X - 0.25; 0)^2\right] = \int_{0.25}^{1} (x - 0.25)^2 dx = \frac{9}{64} = \mathbf{0.14063}.$$

<u>Problem 5.</u> (6 marks)

1. **(3 marks)** Let N_1, N_2 and N_3 are i.i.d. with common distribution Poisson(1). For the retention level or deductible or stop–loss $d = 2.5$, determine $\mathbb{E}[(N_1 + 2N_2 + 3N_3 - d)^+]$.

2. **(3 marks)** Let S be an aggregate loss having a Poisson frequency distribution with parameter $\lambda = 5$. The individual claim amount has the following distribution:

x	100	500	1000
$f_X(x)$	0.8	0.16	0.04

Calculate the probability that aggregate claims will be exactly 600.

Solution:

1. Set $S = N_1 + 2N_2 + 3N_3$, we that

$$\begin{aligned}
\mathbb{E}\left[(S-2.5)^+\right] &= \mathbb{E}[S] - \mathbb{E}[S \wedge 2.5] = 6 - \sum_{k=0}^{2} k f_S(k) - 2.5P(S > 2.5) \\
&= 6 - \left(f_S(1) + 2f_S(2) + 2.5\left(1 - \sum_{k=0}^{2} f_S(k)\right)\right) \\
&= 6 - \left(f_S(1) + 2f_S(2) + 2.5\left(1 - \left(f_S(0) + f_S(1) + f_S(2)\right)\right)\right) \\
&= 3.5 + 2.5f_S(0) + 1.5f_S(1) + 0.5f_S(2)
\end{aligned}$$

Now, let us calculate $f_S(0)$, $f_S(1)$ and $f_S(2)$. By definition

$$\begin{aligned}
f_S(0) &= P(N_1 = 0, N_2 = 0, N_3 = 0) = e^{-3}, \\
f_S(1) &= P(N_1 = 1, N_2 = 0, N_3 = 0) = e^{-1}e^{-2} = e^{-3}, \\
f_S(2) &= P(N_1 = 2, N_2 = 0, N_3 = 0) + P(N_1 = 0, N_2 = 1, N_3 = 0) \\
&= \frac{e^{-1}}{2}e^{-2} + e^{-1}e^{-2} = \frac{3}{2}e^{-3}.
\end{aligned}$$

Therefore

$$\mathbb{E}\left[(S-2.5)^+\right] = 3.5 + 2.5e^{-3} + 1.5e^{-3} + 0.5\frac{3}{2}e^{-3} = \mathbf{3.7365}.$$

2. **Case 1**. $N = 2$ in which we have $X_1 = 100$, $X_2 = 500$ or $X_2 = 100$, $X_1 = 500$.
Case 2. $N = 6$ in which we have $X_i = 100$ for $i = 1, \ldots, 6$. Thus

$$\begin{aligned}
P(S = 600) &= P(N = 2,\ X_1 = 100,\ X_2 = 500) + P(N = 2,\ X_1 = 500,\ X_2 = 100) \\
&\quad + P(N = 6,\ X_i = 100, \text{ for } i = 1, \ldots, 6).
\end{aligned}$$

hence

$$\begin{aligned}
P(S = 600) &= P(N = 2)\,P(X_1 = 100)\,P(X_2 = 500) \\
&\quad + P(N = 2)\,P(X_1 = 500)\,P(X_2 = 100) \\
&\quad + P(N = 6)\prod_{i=1}^{6} P(X_i = 100) \\
&= 2P(N = 2)\,P(X_1 = 100)\,P(X_2 = 500) + P(N = 6)\left(P(X_1 = 100)\right)^6 \\
&= 2e^{-5}\frac{5^2}{2}0.8 \times 0.16 + e^{-5}\frac{5^6}{6!}(0.8)^6 = \mathbf{0.059893}.
\end{aligned}$$