College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

Final Exam
Academic Year 1442-1443 Hijri- First Semester

Exam Information معلومات الامتحان			
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Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

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Student's Name		اسم الطالب
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Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 5 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 5 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
-

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#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
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8				

EXAM COVER PAGE

Exercise 1 *You are given the losses of two policyholders over a period of four years:*

Policyholder	*Year 1*	*Year 2*	*Year3*	*Year 4*
A	1	0	1	0
B	2	3	3	1

Using the nonparametric empirical Bayes method, determine the Bühlmann credibility premium P^A and P^B for Policyholders A and B.
The correct value for P^A among the following $A)$ 0.1607 $B)$ 1.5421 $C)$ 0.5893 $D)$ 0.9421
The correct value for P^B among the following $A)$ 2.1607 $B)$ 3.4580 $C)$ 2.8893 $D)$ 1.4580

Exercise 2 *XYZ Insurance Company offers a janitorial services policy that is rated on a per employee basis. The two insureds shown in the table below were randomly selected from XYZ's policyholder database. Over a* ***four-year*** *period the following was observed:*

				Year	
Insured		*Y*	*Y+1*	*Y+2*	*Y+3*
A	*Number of Claims*	3	2	3	1
	No. of Employees	2	2	2	1
B	*Number of Claims*	0	1	1	
	No. of Employees	4	4	4	

a) Calculate the unbiased estimates for μ in the Bühlmann-Straub model.

$A)$ 8/19 $B)$ 9/19 $C)$ 10/19 $D)$ 11/19

b) Calculate the unbiased estimates for v in the Bühlmann-Straub model.

$A)$ 0.1607 $B)$ 0.1191 $C)$ 0.3580 $D)$ 0.4151

c) Calculate the unbiased estimates for a in the Bühlmann-Straub model.

$A)$ 2.5893 $B)$ 0.8145 $C)$ 0.3226 $D)$ 0.6127

d) Estimate the expected annual claim frequency per employee for each insured using the empirical Bayes Bühlmann-Straub estimation model.
$\tilde{a} = \frac{1}{m - \frac{\Sigma m_i^2}{m}} \left[\sum m_i(\bar{X}_i - \bar{X})^2 - \tilde{v}(r-1)\right]$

Exercise 3 *You are given:*
(i) The number of claims per auto insured follows a Poisson distribution with mean λ*.*
(ii) The claim frequencies of different insureds are independent.
(iii) The prior distribution for λ *has the following probability density function*

$$f(\lambda) = \frac{(500\lambda)^{50}e^{-500\lambda}}{\lambda\Gamma(50)}$$

A company observes the following claims experience:

	Year 1	*Year 2*
Number of claims	75	210
Number of autos insured	600	900

The company expects to insure 1,100 *autos in Year 3. Determine the Bühlmann-Straub credibility estimate of the number of claims in Year* 3*.*

$A)$ 164 $B)$ 174 $C)$ 184 $D)$ 194

Exercise 4 *You are given:*
(i) The annual number of claims for a policyholder follows a Poisson distribution with mean λ*.*
(ii) The prior distribution of λ *is gamma with probability density function:*

$$f(\lambda) = \frac{(5\lambda)^7 e^{-5\lambda}}{720\lambda}$$

A randomly selected policyholder is observed to have a total of 9 *claims for Year 1 through Year 4.*
For the same policyholder, determine the Bayesian estimate of the expected number of claims in Year 5.

$A)$ 15/9 $B)$ 16/9 $C)$ 17/9 $D)$ 18/9

Exercise 5 (Bonus) *You are given:*
(i) The annual number of claims for each policyholder has a Poisson distribution with mean θ.
(ii) The distribution of θ across all policyholders has probability density function:

$$f(\theta) = \theta e^{-\theta},\ \theta > 0$$

(iii) $\int_0^\infty \theta e^{-n\theta} d\theta = \dfrac{1}{n^2}$
A randomly selected policyholder is known to have had at least one claim last year. Determine the posterior probability that this same policyholder will have at least one claim this year.

$$A)\ 0.70 \quad B)\ 0.75 \quad C)\ 0.78 \quad D)\ 0.81 \quad E)\ 0.86$$

Exercise 6 *You are given the following information about a commercial auto liability book of business:*
(i) Each insured's claim count has a Poisson distribution with mean λ , where λ has a gamma distribution with $\alpha = 1.5$ and $\theta = 0.2$.
(ii) Individual claim size amounts are independent and exponentially distributed with mean $5,000$.
(iii) The full credibility standard is for aggregate losses to be within 5% of the expected with probability 0.90.
Using limited fluctuated credibility, calculate the expected number of claims required for full credibility.

$$A)\ 2165 \quad B)\ 2381 \quad C)\ 3514 \quad D)\ 7216 \quad E)\ 7938$$

Hint: if X is a Poisson distribution with mean λ and λ has a Gamma distribution with parameters α and θ then X is a negative binomial distribution with parameters $r = \alpha$ and $\beta = \theta$.

Table of the Standard Normal Distribution Function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{(2\pi)^{1/2}} \exp\left(-\frac{1}{2}u^2\right) du$$

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.00	0.5000	0.60	0.7257	1.20	0.8849	1.80	0.9641	2.40	0.9918
0.01	0.5040	0.61	0.7291	1.21	0.8869	1.81	0.9649	2.41	0.9920
0.02	0.5080	0.62	0.7324	1.22	0.8888	1.82	0.9656	2.42	0.9922
0.03	0.5120	0.63	0.7357	1.23	0.8907	1.83	0.9664	2.43	0.9925
0.04	0.5160	0.64	0.7389	1.24	0.8925	1.84	0.9671	2.44	0.9927
0.05	0.5199	0.65	0.7422	1.25	0.8944	1.85	0.9678	2.45	0.9929
0.06	0.5239	0.66	0.7454	1.26	0.8962	1.86	0.9686	2.46	0.9931
0.07	0.5279	0.67	0.7486	1.27	0.8980	1.87	0.9693	2.47	0.9932
0.08	0.5319	0.68	0.7517	1.28	0.8997	1.88	0.9699	2.48	0.9934
0.09	0.5359	0.69	0.7549	1.29	0.9015	1.89	0.9706	2.49	0.9936
0.10	0.5398	0.70	0.7580	1.30	0.9032	1.90	0.9713	2.50	0.9938
0.11	0.5438	0.71	0.7611	1.31	0.9049	1.91	0.9719	2.52	0.9941
0.12	0.5478	0.72	0.7642	1.32	0.9066	1.92	0.9726	2.54	0.9945
0.13	0.5517	0.73	0.7673	1.33	0.9082	1.93	0.9732	2.56	0.9948
0.14	0.5557	0.74	0.7704	1.34	0.9099	1.94	0.9738	2.58	0.9951
0.15	0.5596	0.75	0.7734	1.35	0.9115	1.95	0.9744	2.60	0.9953
0.16	0.5636	0.76	0.7764	1.36	0.9131	1.96	0.9750	2.62	0.9956
0.17	0.5675	0.77	0.7794	1.37	0.9147	1.97	0.9756	2.64	0.9959
0.18	0.5714	0.78	0.7823	1.38	0.9162	1.98	0.9761	2.66	0.9961
0.19	0.5753	0.79	0.7852	1.39	0.9177	1.99	0.9767	2.68	0.9963
0.20	0.5793	0.80	0.7881	1.40	0.9192	2.00	0.9773	2.70	0.9965
0.21	0.5832	0.81	0.7910	1.41	0.9207	2.01	0.9778	2.72	0.9967
0.22	0.5871	0.82	0.7939	1.42	0.9222	2.02	0.9783	2.74	0.9969
0.23	0.5910	0.83	0.7967	1.43	0.9236	2.03	0.9788	2.76	0.9971
0.24	0.5948	0.84	0.7995	1.44	0.9251	2.04	0.9793	2.78	0.9973
0.25	0.5987	0.85	0.8023	1.45	0.9265	2.05	0.9798	2.80	0.9974
0.26	0.6026	0.86	0.8051	1.46	0.9279	2.06	0.9803	2.82	0.9976
0.27	0.6064	0.87	0.8079	1.47	0.9292	2.07	0.9808	2.84	0.9977
0.28	0.6103	0.88	0.8106	1.48	0.9306	2.08	0.9812	2.86	0.9979
0.29	0.6141	0.89	0.8133	1.49	0.9319	2.09	0.9817	2.88	0.9980
0.30	0.6179	0.90	0.8159	1.50	0.9332	2.10	0.9821	2.90	0.9981
0.31	0.6217	0.91	0.8186	1.51	0.9345	2.11	0.9826	2.92	0.9983
0.32	0.6255	0.92	0.8212	1.52	0.9357	2.12	0.9830	2.94	0.9984
0.33	0.6293	0.93	0.8238	1.53	0.9370	2.13	0.9834	2.96	0.9985
0.34	0.6331	0.94	0.8264	1.54	0.9382	2.14	0.9838	2.98	0.9986
0.35	0.6368	0.95	0.8289	1.55	0.9394	2.15	0.9842	3.00	0.9987
0.36	0.6406	0.96	0.8315	1.56	0.9406	2.16	0.9846	3.05	0.9989
0.37	0.6443	0.97	0.8340	1.57	0.9418	2.17	0.9850	3.10	0.9990
0.38	0.6480	0.98	0.8365	1.58	0.9429	2.18	0.9854	3.15	0.9992
0.39	0.6517	0.99	0.8389	1.59	0.9441	2.19	0.9857	3.20	0.9993
0.40	0.6554	1.00	0.8413	1.60	0.9452	2.20	0.9861	3.25	0.9994
0.41	0.6591	1.01	0.8437	1.61	0.9463	2.21	0.9864	3.30	0.9995
0.42	0.6628	1.02	0.8461	1.62	0.9474	2.22	0.9868	3.35	0.9996
0.43	0.6664	1.03	0.8485	1.63	0.9485	2.23	0.9871	3.40	0.9997
0.44	0.6700	1.04	0.8508	1.64	0.9495	2.24	0.9875	3.45	0.9997
0.45	0.6736	1.05	0.8531	1.65	0.9505	2.25	0.9878	3.50	0.9998
0.46	0.6772	1.06	0.8554	1.66	0.9515	2.26	0.9881	3.55	0.9998
0.47	0.6808	1.07	0.8577	1.67	0.9525	2.27	0.9884	3.60	0.9998
0.48	0.6844	1.08	0.8599	1.68	0.9535	2.28	0.9887	3.65	0.9999
0.49	0.6879	1.09	0.8621	1.69	0.9545	2.29	0.9890	3.70	0.9999
0.50	0.6915	1.10	0.8643	1.70	0.9554	2.30	0.9893	3.75	0.9999
0.51	0.6950	1.11	0.8665	1.71	0.9564	2.31	0.9896	3.80	0.9999
0.52	0.6985	1.12	0.8686	1.72	0.9573	2.32	0.9898	3.85	0.9999
0.53	0.7019	1.13	0.8708	1.73	0.9582	2.33	0.9901	3.90	1.0000
0.54	0.7054	1.14	0.8729	1.74	0.9591	2.34	0.9904	3.95	1.0000
0.55	0.7088	1.15	0.8749	1.75	0.9599	2.35	0.9906	4.00	1.0000
0.56	0.7123	1.16	0.8770	1.76	0.9608	2.36	0.9909		
0.57	0.7157	1.17	0.8790	1.77	0.9616	2.37	0.9911		
0.58	0.7190	1.18	0.8810	1.78	0.9625	2.38	0.9913		
0.59	0.7224	1.19	0.8830	1.79	0.9633	2.39	0.9916		

"Table of the Standard Normal Distribution Function" from HANDBOOK OF STATISTICAL TABLES by Donald B. Owen. © 1962 by Addison-Wesley.

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$
\begin{aligned}
f(x) &= \frac{\alpha^2 (x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1]^{1/\alpha} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)} \beta(1+k/\alpha, \alpha - k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\
\text{mode} &= \theta \left(\frac{\alpha - 1}{\alpha^2 + 1} \right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.5 Inverse paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$
\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau} - 1)^{-1/\tau} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau + k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)} \beta(\tau + k/\tau, 1 - k/\tau; u) + x^k [1 - u^\tau], \quad k > -\tau^2 \\
\text{mode} &= \theta \left(\tau - 1 \right)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \qquad F(x) = \Gamma(\alpha; x/\theta) \\
M(t) &= (1-\theta t)^{-\alpha}, \quad t < 1/\theta \qquad \mathrm{E}[X^k] = \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)}, \quad k > -\alpha \\
\mathrm{E}[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha, \quad \text{if } k \text{ is an integer}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha + k)}{\Gamma(\alpha)} \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
&= \alpha(\alpha+1) \cdots (\alpha + k - 1) \theta^k \Gamma(\alpha + k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\
\text{mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models fall into three classes. The divisions are based on the algorithm by which the probabilities are computed. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses.

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators which are presented are not intended to be useful estimators but rather for providing starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used [where n_k is the observed frequency at k (if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k) and n is the sample size]:

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} k n_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ and/or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!} \\ \mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N] = \lambda \qquad P(z) = e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\ \mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1}. \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer$)$

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\ \mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m. \end{aligned}$$

B.2.1.4 Negative binomial—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ \mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}. \end{aligned}$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a+b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$.For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

Tables Gamma, Neg. Binomial, Normal, Poisson

Ex1

$\mu(\theta_A) = E(X_{1j}|\theta_A)$ estimated by $\bar{X}_A = 1/2$

$\mu(\theta_B) = E(X_{2j}|\theta_B)$ " " $\bar{X}_B = 9/4$.

$\mu = E(\mu(\theta))$ " " $\bar{X} = \frac{\bar{X}_A + \bar{X}_B}{2} = 11/8$.

$v(\theta_A) = V(X_{Aj}|\theta_A)$ " " $\hat{v}_A = \frac{(1-0.5)^2+(0-0.5)^2+\dots+(0-0.5)^2}{4-1}$

$= 1/3$

$v(\theta_B) = V(X_{Bj}|\theta_B)$ " " $\hat{v}_B = 11/12$.

$v = E(V(\theta))$ " " $\hat{v} = (\hat{v}_A + \hat{v}_B)/2 = 5/8$

$\underbrace{V(\bar{X}_i)}_{\text{estimated by}} = \underbrace{V(E(\bar{X}_i|\theta))}_{=a} + \underbrace{E(V(\bar{X}_i|\theta))}_{\text{estimated by}}$ $\hat{v}/4 = 0.15625$

$\frac{(\bar{X}_A - \bar{X})^2 + (\bar{X}_B - \bar{X})^2}{2-1} = 1.53125$

$\hat{a} = 1.53125 - 0.15625 = 1.375 = 11/8$

$\hat{k} = \hat{v}/\hat{a} = \frac{5/8}{11/8} = 5/11$.

$\hat{Z} = \frac{n}{n+\hat{k}} = \frac{4}{4+5/11} = \frac{44}{49}$.

$PC^{(A)} = \hat{Z}\,\bar{X}_A + (1-\hat{Z})\,\bar{X} = \frac{44}{49}\left(\frac{1}{2}\right) + \frac{5}{49}\left(\frac{11}{8}\right) = \frac{33}{56}$

$= 0.5893$

$PC^{(B)} = \hat{Z}\,\bar{X}_B + (1-\hat{Z})\,\bar{X} = \frac{44}{49}\left(\frac{9}{4}\right) + \frac{5}{49}\left(\frac{11}{8}\right) = \frac{121}{56}$

$= 2.1607$.

Ex2

X_{ij} = average loss of policy i in Year j,

$= \frac{\sum Y_{ijk}}{m_{ij}}$; m_{ij} = number of policyholders.

$m_i = \sum_{j=1}^{n_i} m_{ij}$ = total number of " ...

n_i = number of years.

$i = 1, 2$; $m_{1j} = (2, 2, 2, 1)$; $m_{2j} = (4, 4, 4)$;

$m_1 = 7$; $m_2 = 12$; $m = \sum m_i = 19$.

$\mu(\theta_i) = E(X_{ij}|\theta_i)$ estimated by $\begin{cases} \bar{X}_1 = 9/7 \\ \bar{X}_2 = 2/12 = 1/6. \end{cases}$

$\mu = E(\mu(\theta_i))$ " " $\bar{X} = \frac{\sum m_i \bar{X}_i}{m} = \frac{9+2}{19} = \frac{11}{19}$

$V(X_{ij}|\theta_i) = \frac{V(Y)}{m_{ij}} = \frac{v(\theta_i)}{m_{ij}}$

$v(\theta_i)$ estimated by $\hat{v}_i = \begin{cases} \left[2\left(3/2 - 9/7\right)^2 + 2\left(2/2 - 9/7\right)^2 + 2\left(3/2 - 9/7\right)^2 + 1\left(1 - 9/7\right)^2\right] / (4-1) = 0.1429 \\ \left[4\left(0/4 - 1/6\right)^2 + 4\left(1/4 - 1/6\right)^2 + 4\left(1/4 - 1/6\right)^2\right] / (3-1) = 0.0833 \end{cases}$

$\hat{v} = \sum w_i \hat{v}_i = \frac{4-1}{(4-1)+(3-1)}(0.1429) + \frac{3-1}{(4-1)+(3-1)} 0.0833 = 0.1191$

$\hat{a} = 0.6127$.

$\hat{k} = \hat{v}/\hat{a} = 0.1944$.

$\hat{Z}_i = \frac{m_i}{m_i + \hat{k}} = \begin{cases} \frac{7}{7+\hat{k}} = 0.9730 \\ \frac{12}{12+\hat{k}} = 0.9841 \end{cases}$

$\hat{P}_1 = \hat{Z}_1 \bar{X}_1 + (1 - \hat{Z}_1)\bar{X} = 1.2667$

$\hat{P}_2 = \hat{Z}_2 \bar{X}_2 + (1 - \hat{Z}_2)\bar{X} = 0.1732$

Ex3

Y = number of claims per insured $\sim \mathcal{P}(\text{mean}=\lambda)$

$\Lambda \sim \text{Gamma}(\alpha = 50;\ \theta = 1/500)$

X_i = average number of claims in year i

$= \dfrac{\sum Y_{ij}}{m_i}$, m_i = number of insured.

$\mu(\lambda) = E(X|\lambda) = E(Y|\lambda) = \lambda.$

$\mu = E(\mu(\Lambda)) = E\lambda = \alpha\theta = 1/10$

$V(X|\lambda) = \dfrac{V(Y|\lambda)}{m} = \dfrac{v(\lambda)}{m}$

$v(\lambda) = V(Y|\lambda) = \lambda.$

$v = E(v(\lambda)) = E\lambda = 1/10.$

$a = V(\mu(\lambda)) = V(\lambda) = \alpha\theta^2 = 1/5000$

$k = v/a = 500;\quad Z = \dfrac{m}{m+k} = \dfrac{1500}{1500+500} = 3/4.$

BS estimate of total number of claims in year 3 is

$$1100\left[Z\bar{X} + (1-Z)\mu\right]$$

$$= 1100\left[\tfrac{3}{4}\left(\frac{75+210}{1500}\right) + \tfrac{1}{4}\left(\tfrac{1}{10}\right)\right]$$

$$= 1100 * 0.1675 = 184.25$$

Ex 4 $X|_\lambda \sim \mathcal{P}(\lambda)$

$\lambda \sim \text{Gamma}(\alpha = 7;\ \theta = 1/5)$

$\sum_1^4 x_i = 9$

$$\pi(\lambda | x_1, \dots, x_n) \propto \prod_{i=1}^{4} e^{-\lambda} \frac{\lambda^{x_i}}{x!} \lambda^6 e^{-5\lambda}$$

$$\propto e^{-4\lambda} \lambda^{\sum x_i} \lambda^6 e^{-5\lambda}$$

$$= \lambda^{15} e^{-9\lambda}$$

$$\Rightarrow \lambda|_{x_1,\dots,x_n} \sim \text{Gamma}(\alpha = 16;\ \theta = 1/9)$$

$$E(X_5 | x_1, \dots, x_n) = \int \underbrace{E(X_5|\lambda)}_{\lambda} \pi(\lambda | x_1, \dots, x_n)\, d\lambda$$

$$= \text{mean of Gamma} = \alpha\theta = \frac{16}{9}.$$

Ex 5 $P(X_2 \geq 1) = \int P(X_2 \geq 1 | \theta)\, \pi(\theta | x_1 \geq 1)\, d\theta$

$$\pi(\theta | x_1 \geq 1) = \frac{P(x_1 \geq 1|\theta) f(\theta)}{\int \text{—} d\theta} = \frac{[1 - P(x_1 = 0|\theta)] f(\theta)}{\int \text{—} d\theta}$$

$$= \frac{[1 - e^{-\theta}]\theta e^{-\theta}}{\int \text{—} d\theta} = \frac{\theta e^{-\theta} - \theta e^{-2\theta}}{\int \theta e^{-\theta} - \int \theta e^{-2\theta}} = \frac{\theta e^{-\theta} - \theta e^{-2\theta}}{1 - 1/4}.$$

$$P(X_2 \geq 1) = \frac{4}{3} \int [1 - e^{-\theta}](\theta e^{-\theta} - \theta e^{-2\theta})\, d\theta$$

$$= \frac{4}{3} \int [\theta e^{-\theta} - 2\theta e^{-2\theta} + \theta e^{-3\theta}]\, d\theta$$

$$= \frac{4}{3}\left[1 - \frac{2}{4} + \frac{1}{9}\right] = \frac{22}{27} = 0.81$$

Ex. 6

$$n \geq \lambda_0 C_X^2 = \lambda_0 \frac{\sigma_X^2}{\mu_X^2}.$$

$$\begin{cases} \mu_X = \mu_N \mu_Y \\ \sigma_X^2 = \sigma_N^2 \mu_Y^2 + \mu_N \sigma_Y^2. \end{cases}$$

$\mu_N = \alpha\theta = 1.5 \times 0.2 = 0.3$

$\sigma_N^2 = \alpha\theta(1+\theta) = 0.36$

$\Rightarrow \mu_X = 0.3 \times 5000 = 1500$

$\sigma_X^2 = 0.36 \times 5000^2 + 0.3 \times 5000^2 = 5000^2 \times 0.66$

$N = \sum_1^n N_i : n \geq \left(\frac{1.645}{0.05}\right)^2 \left(\frac{5000^2 \times 0.66}{1500^2}\right) = 7937.63$

~~$\mu_N = n$~~ Total number of claims is $\sum_1^n N_i$

$\Rightarrow$ Number of expected claims required for full credibility is $n\mu_N = n \times 0.3 \geq 2.381.$