College of Sciences
Department of
Statistics and Operations
Research

كلية العلوم
قسم الإحصاء
وبحوث العمليات

Mid 1 Exam

Wednesday, March 4, 2020	ACTU 465	Academic year 1441-42H
9:00 – 11:00 AM	Credibility	Second Semester

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Instructions

- Switch off your mobile and place it under your seat.
- Time allowed is 120 minutes.
- For most questions, four responses are given, only one is correct. You are not asked to select correct answers but to provide **detailed explanations** for all answers.

Exercise 1 *You are given:*
(i) The number of claims per exposure follows a Negative Binomial distribution with $r = 2$ *and* $\beta = 0.2$.
(ii) Claim size follows a Pareto distribution with parameters $\alpha = 3$ *and* $\theta = 1,000$.
(iii) The number of claims per exposure and claim sizes are independent.
(iv) The method of limited fluctuation credibility is used, and the full credibility standard has been selected so that total claim dollars per exposure will be within 10% *of expected total claim dollars per exposure* 90% *of the time.*
Find the credibility factor Z
(a) based on $1,000$ *exposures.*
(1)0.97 (2) 0.59 (3) 0.45 (4) 0.14
(b) based on a total claim number of 300
(1)0.12 (2) 0.77 (3) 0.51 (4) 0.23.
(c) based on a total claim amount of $100,000$.
(1)0.36 (2) 0.23 (3) 0.65 (4) 0.42

Exercise 2 *You are given:*
(i) For each individual insured, the number of claims follows a Poisson distribution.(ii) Claim severity is uniformly distributed in the interval $[2000, 3000]$.
(iii) The number of claims is independent of the severity of claims.
Determine the expected number of claims needed for aggregate losses to be within 10% *of expected aggregate losses with* 95% *probability.*
(1)1020 (2) 176 (3) 244 (4) 389

Exercise 3 *The full credibility standard for a company is set so that the total number of claims is to be within* 2.5% *of the true value with probability* p. *This full credibility standard is calculated to be* $5,000$ *claims. The standard is altered so that the total cost of claims is to be within* 9% *of the true value with probability* p. *The claim frequency has a Poisson distribution and the claim severity* Y *has the following distribution:*

$$f(x) = 0.0008(50 - x), \quad 0 \leq x \leq 50$$

(a) Calculate the first moment $E(Y)$.
(1)123.54 (2) 1.56 (3) 16.67 (4) 40.67
(b) Calculate the variance $var(Y)$.

(1)138.78 (2) 15.56 (3) 61.67 (4) 3.99
(c) What is the expected number of claims necessary to obtain full credibility under the new standard?
(1)129 (2) 54 (3) 578 (4) 1040

Exercise 4 *X is the claim-severity random variable that can take values 10, 20 or 30. The distribution of X depends on the risk group defined by parameter Θ, which are labeled 1, 2 and 3. The relative frequencies of risk groups (or prior) with Θ equal to 1, 2 and 3 are, respectively, 0.4, 0.4, and 0.2. The conditional distribution of X given the risk parameter Θ is given in the following Table.*

		$Pr(X = x \mid \theta)$		
θ	$Pr(\Theta = \theta)$	$x = 10$	$x = 20$	$x = 30$
1	0.4	0.2	0.3	0.5
2	0.4	0.4	0.4	0.2
3	0.2	0.5	0.5	0.0

A sample of 3 claims with $x = (20, 20, 30)$ is observed.
(a) Calculate the posterior distribution of Θ ($Pr(\theta = 1 \mid x), Pr(\theta = 2 \mid x), Pr(\theta = 3 \mid x)$).
(b) Deduce the posterior mean of X_4 ($E(X_4 \mid x)$).
(1)20.92 (2) 32.54 (3) 13.67 (4) 25.25

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1] \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1 \\
\mathrm{E}[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
\mathrm{E}[X \wedge x] &= -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
\mathrm{E}[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1} - 1)^{-1/\gamma} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

B.2.1.2 Geometric—β

$$\begin{aligned} p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\ \mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1}. \end{aligned}$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer)

$$\begin{aligned} p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\ p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\ \mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m. \end{aligned}$$

B.2.1.4 Negative binomial—β, r

$$\begin{aligned} p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\ p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\ \mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}. \end{aligned}$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$.For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1 - p_0)$.

Exo 1 $N \sim$ Neg Bin. $r = 2$ $\beta = 0.2$

$Y \sim$ Pareto $\alpha = 3$ $\theta = 1000$ for (classical credibility)

$k = 10\%$ $1 - \alpha = 0.90$

$-k < \frac{S - \mu_S}{\mu_S} < k$ $S = \sum x_i = n\bar{x}$

$$\boxed{n \geq \lambda_0 \frac{\sigma_X^2}{\mu_X^2}} = \lambda_0 \frac{\mu_N \sigma_Y^2 + \sigma_N^2 \mu_Y^2}{(\mu_N \mu_Y)^2}$$

$N \sim$ Neg. Binomial $EN = r\beta = 0.4$

$VN = r\beta(1+\beta) = 0.48$

$Y \sim$ Pareto. $E(Y) = \mu_Y = \frac{\theta}{\alpha - 1} = 500$

$V(Y) = \sigma_Y^2 = \frac{\alpha\theta^2}{(\alpha-1)^2(\alpha-2)} = 3/4 \times 10^6$.

$z = 1.645$. $k = 0.10$

(a) full credibility when

$$n \geq \lambda_0 \frac{\sigma_X^2}{\mu_X^2} = \left(\frac{1.645}{0.10}\right)^2 \frac{\sigma_X^2}{\mu_X^2} = 2841.326$$

$$Z = \sqrt{\frac{n}{n_f}} = \sqrt{\frac{1000}{2841.326}} = 0.59.$$

(b) $E(N_f) = E\left(\sum_1^n N_i\right) = n\mu_N = 2841.326 \times 0.4 = 1136.53$

$$Z = \sqrt{\frac{300}{1136.53}} = 0.51.$$

(c) $E\left(\sum_1^n X_i\right) = n\mu_N\mu_Y = 2841.326 \times 0.4 \times 500$

$= 568\,265.2$

$$Z = \sqrt{\frac{100,000}{568,265.2}} = 0.42$$

Exo 2

$N \sim \mathcal{P}(\lambda)$.

$Y \sim \mathcal{U}[2000, 3000]$

$n \geq \frac{\lambda_0}{\lambda}(1 + C_Y^2)$

$E(\sum_1^n N_i) = n\lambda \geq \lambda_0 (1 + C_Y^2)$

$= \left(\frac{1.96}{0.10}\right)^2 \left(1 + \frac{\frac{(b-a)^2}{12}}{\left(\frac{a+b}{2}\right)^2}\right)$

$= (\quad)^2 \times (1 + 0.0133)$

$= 389.$

Exo 3

first standard. $P\left(-k < \frac{N - \mu_N}{\mu_N} < k\right) \geq p$

$N_f = \lambda_0 = \left(\frac{3p}{0.025}\right)^2 = 5000.$ (1)

second standard $P\left(-k < \frac{S - \mu_S}{\mu_S} < k\right) \geq p$

$n \geq \frac{\lambda_0}{\lambda}(1 + C_Y^2)$

$(\Rightarrow)$ $E(\sum N_i) = n\lambda \geq \lambda_0 (1 + C_Y^2) = \left(\frac{3p}{0.09}\right)^2 (1 + C_Y^2)$

(a) $EY = 0.0008 \int_0^{50} (50x - x^2)\,dx$

$= 0.0008 \left[50\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{50} = 16.67$

(b) $E(Y^2) = 0.0008 \left[50\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{50} = 416.67.$

$V(Y) = 416.67 - (16.67)^2 = 138.78$

(c) (2) $E(\sum_1^n N_i) = \lambda_0 (1 + C_Y^2) = \left(\frac{3p}{0.09}\right)^2 \left(1 + \frac{138.78}{16.67^2}\right)$

(1) $\Rightarrow 3p^2 = 5000 \times 0.025^2$

(2) $\Rightarrow E(\sum_1^n N_i) = 5000 \times \left(\frac{0.025}{0.09}\right)^2 \left(1 + \frac{138.78}{16.67}\right)^2$

$= 578$

Exo 4

$$P(\theta=1|x) = \frac{P(x|\theta=1) \times P(\theta=1)}{P(x)}$$

$$P(x) = \sum_{1}^{3} P(x|\theta) P(\theta) = ?$$

$x = (20, 20, 30)$

$P(x|\theta=1) = 0.3^2 \times 0.5$

$P(x|\theta=2) = 0.4^2 \times 0.2$

$P(x|\theta=3) = 0.5^2 \times 0.0$

$P(x) = 0.3^2 \times 0.5 \times 0.4 + 0.4^2 \times 0.2 \times 0.4 + 0$

$= 0.0308.$

(a) $P(\theta=1|x) = \frac{0.3^2 \times 0.5 \times 0.4}{0.0308} = 0.5844$

$P(\theta=2|x) = \frac{0.4^2 \times 0.2 \times 0.4}{0.0308} = 0.4156$

$P(\theta=3|x) = 0.$

(b) $E(X_4|x) = \sum_{1}^{3} E(X_4|\theta) P(\theta|x)$

$E(X_4|\theta=1) = 10 \times 0.2 + 20 \times 0.3 + 30 \times 0.5$
$= 23$

$E(X_4|\theta=2) = 10 \times 0.4 + 20 \times 0.4 + 30 \times 0.2$
$= 18$

$E(X_4|\theta=3) = 10 \times 0.5 + 20 \times 0.5 = 15$

$E(X|x) = 23 \times 0.5844 + 18 \times 0.4156$
$= 20.92$