College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

First Midterm Exam
Academic Year 1442-1443 Hijri- First Semester

Exam Information معلومات الامتحان			
Course name	نظرية المصداقية		اسم المقرر
Course Code	465 ريك		رمز المقرر
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Exam Time	10: 00 AM		وقت الامتحان
Exam Duration	2 hours	ساعتان	مدة الامتحان
Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

Student Information معلومات الطالب		
Student's Name		اسم الطالب
ID number		الرقم الجامعي
Section No.		رقم الشعبة
Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 4 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 4 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
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هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
3				
4				
5				
6				
7				
8				

EXAM COVER PAGE

Explain your reasoning for why you have answered a certain value.

Exercise 1 *You are given:*
(i) The number of claims incurred in a year by any insured has a Poisson distribution with mean λ*.*
(ii) The claim frequencies of different insureds are independent.
(iii) The prior distribution of λ *is given by the probability density function:*

$$f(\lambda) = \frac{(20\lambda)^4}{6\lambda}e^{-20\lambda}$$

(iv)

Year	*Number of insured*	*Number of claims*
1	40	23
2	50	34
3	60	?

Calculate the Bühlmann-Straub credibility estimate of the number of claims in Year 3*.*

Exercise 2 *You are given four classes of insureds, each of whom may have zero or one claim, with the following probabilities:*

Class	*Number of Claims*	
	0	1
I	0.7	0.3
II	0.6	0.4
III	0.8	0.2

A class is selected at random (with probability 1/3*), and five insureds are selected at random from the class. The total number of claims is one.*
If 15 *insureds are selected at random from the same class, estimate the total number of claims using Bühlmann-Straub credibility.*

Exercise 3 *An insurance company sells two types of policies with the following characteristics:*

Type of Policy	*Proportion of Total Policies*	*Poisson Annual Claim Frequency*
I	θ	$\lambda = 0.50$
II	$1-\theta$	$\lambda = 1.5$

A randomly selected policyholder is observed to have one claim in Year 1*. For the same policyholder, determine the Bühlmann credibility premium factor* Z *for Year* 2*.*

$$A)\ \frac{\theta-\theta^2}{1.5-\theta^2} \quad B)\ \frac{1.5-\theta}{1.5-\theta^2} \quad C)\ \frac{2.25-\theta}{1.5-\theta^2} \quad D)\ \frac{2-\theta^2}{1.5-\theta^2} \quad E)\ \frac{2.25-2\theta^2}{1.5-\theta^2}$$

Exercise 4 *You are given:*
i) Losses in a given year follows a gamma distribution with parameters α *and* θ*, where* θ *does not vary by policyholder.*
ii) The prior distribution of α *has mean* 50*.*
iii) The Bühlmann credibility factor based on two years of experience is 0.25*.*
Calculate $Var(\alpha)$*.*

Exercise 5 *For a portfolio of insurance risks, aggregate losses per year per exposure follow a normal distribution with mean* θ *and standard deviation* 1000*, with* θ *varying by class as follows;*

Class	θ	*Percent of Risks in Class*
X	2000	60%
Y	3000	30%
Z	4000	10%

A randomly selected risk has the following experience over three years:

Year	*Number of Exposures*	*Aggregate Losses*
1	24	24,000
2	30	36,000
3	26	28,000

Calculate the Bühlmann-Straub estimate of the mean aggregate losses per year per exposure in Year 4 *for this risk.*

A) 1100 *B*) 1138 *C*) 1696 *D*) 2462 *E*) 2500

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$
\begin{aligned}
f(x) &= \frac{\alpha^2 (x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1]^{1/\alpha} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha-k/\alpha)}{\Gamma(\alpha)} \beta(1+k/\alpha, \alpha-k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\
\text{mode} &= \theta \left(\frac{\alpha-1}{\alpha^2+1} \right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.5 Inverse paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$
\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau} - 1)^{-1/\tau} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau+k/\tau)\Gamma(1-k/\tau)}{\Gamma(\tau)} \beta(\tau+k/\tau, 1-k/\tau; u) + x^k[1-u^\tau], \quad k > -\tau^2 \\
\text{mode} &= \theta\,(\tau-1)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \qquad F(x) = \Gamma(\alpha; x/\theta) \\
M(t) &= (1-\theta t)^{-\alpha}, \quad t < 1/\theta \qquad \mathrm{E}[X^k] = \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k > -\alpha \\
\mathrm{E}[X^k] &= \theta^k(\alpha+k-1)\cdots\alpha, \quad \text{if } k \text{ is an integer}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \Gamma(\alpha+k; x/\theta) + x^k[1-\Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
&= \alpha(\alpha+1)\cdots(\alpha+k-1)\theta^k \Gamma(\alpha+k; x/\theta) + x^k[1-\Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\
\text{mode} &= \theta(\alpha-1), \quad \alpha > 1, \text{ else } 0
\end{aligned}
$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models fall into three classes. The divisions are based on the algorithm by which the probabilities are computed. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses.

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators which are presented are not intended to be useful estimators but rather for providing starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used [where n_k is the observed frequency at k (if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k) and n is the sample size]:

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} k n_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ and/or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!} \\ \mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N] = \lambda \qquad P(z) = e^{\lambda(z-1)} \end{aligned}$$

Exam Mid 2

ACTU 465.

Ex 1 $f(\lambda) = \frac{(20\lambda)^4}{6\lambda} e^{-20\lambda}$

$\Rightarrow \Lambda \sim$ Gamma with $\alpha = 4$, $\theta = 1/20$

Y: Number of claims by insurer

X = average number of claims by insurer.

$X_i = \frac{\sum_1^{m_i} Y_{ij}}{m_i}$ = average number of claims in year i

$\mu(\lambda) = E(X|\lambda) = E(Y|\lambda) = \lambda$

$V(X|\lambda) = \frac{V(Y|\lambda)}{m} = \frac{\lambda}{m} = \frac{v(\lambda)}{m}$.

$\begin{cases} \mu(\lambda) = \lambda \Rightarrow \mu = E(\mu(\lambda)) = E(\lambda) = \alpha\theta = \frac{4}{20} = 0.2 \\ v(\lambda) = \lambda \Rightarrow v = E(v(\lambda)) = E(\lambda) = 0.2 \end{cases}$

$a = V(\mu(\lambda)) = V(\lambda) = \alpha\theta^2 = \frac{4}{20^2} = 0.01$.

$k = \frac{v}{a} = \frac{0.2}{0.01} = 20.$

$m = \sum_1^2 m_i = 40 + 50 = 90$

$Z = \frac{m}{m+k} = \frac{90}{90+20} = 9/11$

$\bar{X} = \frac{\sum m_i X_i}{\sum m_i} = \frac{\sum_1^2 N_i}{m} = \frac{23+34}{90} = \frac{57}{90}.$

Buhlman Straub estimate of $E(X_3|\lambda)$

is $Z\bar{X} + (1-Z)\mu = \frac{9}{11}\left(\frac{57}{90}\right) + \frac{2}{11}(0.2)$
$= 0.5545.$

BS estimate of number of claims

is $60 \times 0.5545 = 33.273$

Ex2

$\mu = E(\mu(\theta)) = 0.3$

$a = V(\mu(\theta)) = E(\mu^2(\theta)) - 0.3^2$

$= \frac{0.3^2 + 0.4^2 + 0.2^2}{3} - 0.3^2$

$= 0.0066\ldots = 2/300$

$v = E(v(\theta)) = \frac{0.21 + 0.24 + 0.16}{3}$

$= 0.2233.$

$\mu(\theta) = E(X|\theta) = E(Y|\theta)$

$= q = \begin{cases} 0.3 \\ 0.4 \\ 0.2 \end{cases}$

$v(\theta) = V(Y|\theta) = q(1-q)$

$= \begin{cases} 0.21 \\ 0.24 \\ 0.16. \end{cases}$

$k = v/a = 30.5.$

$Z = \frac{m}{m+k} = \frac{5}{5+30.5} = \frac{5}{35.5}.$

Estimate of numbers of claims per insured

$= \frac{5}{35.5}(1/5) + \left(1 - \frac{5}{35.5}\right)0.3$

$= 0.2858155$

Estimate of number of claims for 15 insured

is $0.2858155 \times 15 = 4.288$

Ex3

$$\lambda_1 = 0.5 \ ; \ \lambda_2 = 1.5$$

$$X|_{\lambda_1} \sim \mathcal{P}(\lambda_1) \qquad X|_{\lambda_2} \sim \mathcal{P}(\lambda_2).$$

$$\mu(\lambda_1) = E(X|\lambda_1) = \lambda_1 \ ; \ \mu(\lambda_2) = E(X|\lambda_2) = \lambda_2$$

$$\mu = E(\mu(\lambda)) = 0.5(\theta) + 1.5(1-\theta) = 1.5 - \theta$$

$$V(\lambda_1) = V(X|\lambda_1) = \lambda_1 \ ; \quad V(\lambda_2) = \lambda_2$$

$$v = E(V(\lambda)) = 1.5 - \theta$$

$$a = Var(\mu(\lambda)) = E(\mu^2(\lambda)) - \mu^2$$

$$= 0.5^2\,\theta + 1.5^2(1-\theta) - (1.5-\theta)^2$$

$$= \theta - \theta^2$$

$$k = v/a = \frac{1.5-\theta}{\theta-\theta^2}.$$

$$Z = \frac{1}{1+k} = \frac{1}{1+\frac{1.5-\theta}{\theta-\theta^2}} = \frac{\theta-\theta^2}{1.5-\theta^2}.$$

Ex4

$$X \sim Gamma(\alpha, \theta)$$

$$E(\alpha) = 50 \ ; \ Z = \frac{2}{2+k} = 0.25 \Rightarrow k = 6$$

$$k = \frac{v}{a} = \frac{E(V(X|\alpha))}{Var(E(X|\alpha))}.$$

$$\mu(\alpha) = E(X|\alpha) = \alpha\theta. \Rightarrow \mu = \theta E(\alpha) = 50\theta$$

$$V(\alpha) = V(X|\alpha) = \alpha\theta^2 \Rightarrow a = V(\mu(\alpha)) = \theta^2 V(\alpha)$$

$$v = E(V(\alpha)) = 50\,\theta^2$$

$$k = \frac{v}{a} = \frac{50\,\theta^2}{\theta^2 V(\alpha)} = \frac{50}{V(\alpha)} = 6 \quad \Rightarrow V(\alpha) = 50/6$$

Ex 5

$$X_i = \bar{Y}_i = \frac{\sum x_{ij}}{m_i}$$

$$Y = \text{loss}|_{\text{exposure}} \sim N(\mu = \theta, \sigma = 1000)$$

$$\mu(\theta) = E(X|\theta) = E(Y|\theta) = \theta$$

$$V(X|\theta) = \frac{v(Y|\theta)}{m} = \frac{v(\theta)}{m} = \frac{1000^2}{m}$$

$$\mu = E(\mu(\theta)) = E(\theta) = 0.6(2000) + 0.3(3000) + 0.1(4000) = 2{,}500$$

$$a = V(\mu(\theta)) = E(\theta^2) - \mu^2 = 0.6(2000^2) + 0.3(3000^2) + 0.1(4000^2) - 2500^2 = 450{,}000$$

$$v = E(v(\theta)) = E(1000^2) = 1000^2$$

$$k = v/a = 10/45 .$$

$$m = \sum m_i = 80 \Rightarrow Z = \frac{80}{80 + 10/45} = 0.9729$$

Estimate of $E(X|\theta) = Z\bar{X} + (1-Z)\mu$

$$= Z\,\frac{24{,}000 + 36{,}000 + 28{,}000}{80} + (1-Z)\,2500$$

$$= 1{,}137.8$$

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