King Saud University
جامعة الملك سعود

College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

Second Midterm Exam
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Classroom No.			رقم قاعة الاختبار
Instructor Name			اسم استاذ المقرر

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Serial Number		الرقم التسلسلي

General Instructions:

- Your Exam consists of 6 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 6 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
-

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
3				
4				
5				
6				
7				
8				

EXAM COVER PAGE

Exercise 1 *A company has determined that the limited fluctuation full credibility standard so that the t**otal number of claims** is to be within* 5% *of the true value with probability* 99%. *If the insurance company receives* $2,890$ *claims this year from the risk group and the manual list of expected claim is* $3,000$, *what is the updated expected number of claims next year? Assume the claim-frequency distribution is Poisson and the normal approximation applies. Note that for* $1-\alpha = 0.99$, $z = 2.576$.

Exercise 2 *A portfolio of risks is divided into three classes. The characteristics of the annual claim distributions for the three risk classes is as follows:*

	Class I	Class II	Class III
Annual Claim	Poisson	Poisson	Poisson
Number Distribution	mean 1	mean 2	mean 5

50% *of the risks are in Class I,* 30% *are in Class II, and* 20% *are in Class III.*
A risk is chosen at random from the portfolio and is observed to have 2 *claims in the first year and* 2 *claims in the second year. Find the expected number of claims for the risk in the third year.*

A) 1.30 *B*) 1.50 *C*) 1.70 *D*) 1.90 *E*) 2.10

Exercise 3 *A risk class is made up of three **equally** sized groups of individuals. Groups are classified as Type A, Type B and Type C. Any individual of any type has probability of* 0.5 *of having no claim in the coming year and has a probability of* 0.5 *of having exactly* 1 *claim in the coming year. Each claim is for amount 1 or 2 when a claim occurs. Suppose that the claim distributions given that a claim occurs, for the three types of individuals are*

$$Pr(\text{claim of amount } x \mid \text{Type A and a claim occurs}) = \begin{cases} 2/3 & x=1 \\ 1/3 & x=2 \end{cases}$$

$$Pr(\text{claim of amount } x \mid \text{Type B and a claim occurs}) = \begin{cases} 1/2 & x=1 \\ 1/2 & x=2 \end{cases}$$

$$Pr(\text{claim of amount } x \mid \text{Type C and a claim occurs}) = \begin{cases} 5/6 & x=1 \\ 1/6 & x=2 \end{cases}$$

An insured is chosen at random from the risk class and is found to have a claim of amount 2.
(a) Find the probability that the insured is Type A.

A) 1/6 *B*) 1/3 *C*) 1/2 *D*) 2/3 *E*) 5/6

(b) Find the Bayesian premium.

A) 25/36 *B*) 3/4 *C*) 29/36 *D*) 31/36 *E*) 11/12

Exercise 4 *A portfolio of risks is divided into two classes. The characteristics of the loss amount distributions for the two risk classes is as follows:*

	Class I	Class II
Loss Amount	Exponential	Pareto
Distribution	mean 1000	$\theta = 1000,\ \alpha = 3$

The portfolio is evenly divided between Class I and Class II risks.
(a) A risk is chosen at random from the portfolio and is observed to have a loss of 2000*. Find the expected value of the next loss from the same risk.*

$A)$ 593 $B)$ 693 $C)$ 793 $D)$ 893 $E)$ 993

(b) A risk is chosen at random from the portfolio and is observed to have a first loss of 2000 *and a second loss of* 1000*. Find the probability that the risk was chosen from Class I.*

$A)$ 0.578 $B)$ 0.678 $C)$ 0.778 $D)$ 0.878 $E)$ 0.978

(c) Find the expected value of the third loss from the same risk.

$A)$ 789 $B)$ 839 $C)$ 889 $D)$ 939 $E)$ 989

Exercise 5 (Bonus) *You are given:*
(i) The annual number of claims on a given policy has the geometric distribution with parameter β.
ii) One-third of the policies have $\beta = 2$, and the remaining two-thirds have $\beta = 5$.
A randomly selected policy had two claims in Year 1.
Calculate the Bayesian expected number of claims for the selected policy in Year 2.

$A)$ 3.4 $B)$ 3.6 $C)$ 3.8 $D)$ 4.0 $E)$ 4.2

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1] \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1 \\
\mathrm{E}[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
\mathrm{E}[X \wedge x] &= -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
\mathrm{E}[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1}-1)^{-1/\gamma} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

A.3.3 One-parameter distributions

A.3.3.1 Exponential—θ

$$
\begin{aligned}
f(x) &= \frac{e^{-x/\theta}}{\theta} \qquad F(x) = 1 - e^{-x/\theta} \\
M(t) &= (1-\theta t)^{-1} \qquad \mathrm{E}[X^k] = \theta^k \Gamma(k+1), \quad k > -1 \\
\mathrm{E}[X^k] &= \theta^k k!, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= -\theta \ln(1-p) \\
\mathrm{TVaR}_p(X) &= -\theta \ln(1-p) + \theta \\
\mathrm{E}[X \wedge x] &= \theta(1 - e^{-x/\theta}) \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k \Gamma(k+1)\Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k > -1 \\
&= \theta^k k! \Gamma(k+1; x/\theta) + x^k e^{-x/\theta}, \quad k \text{ an integer} \\
\text{mode} &= 0
\end{aligned}
$$

A.3.3.2 Inverse exponential—θ

$$
\begin{aligned}
f(x) &= \frac{\theta e^{-\theta/x}}{x^2} \qquad F(x) = e^{-\theta/x} \\
\mathrm{E}[X^k] &= \theta^k \Gamma(1-k), \quad k < 1 \\
\mathrm{VaR}_p(X) &= \theta(-\ln p)^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k G(1-k; \theta/x) + x^k(1 - e^{-\theta/x}), \quad \text{all } k \\
\text{mode} &= \theta/2
\end{aligned}
$$

A.5 Other distributions

A.5.1.1 Lognormal—μ,σ (μ can be negative)

$$
\begin{aligned}
f(x) &= \frac{1}{x\sigma\sqrt{2\pi}} \exp(-z^2/2) = \phi(z)/(\sigma x), \quad z = \frac{\ln x - \mu}{\sigma} \qquad F(x) = \Phi(z) \\
\mathrm{E}[X^k] &= \exp(k\mu + k^2\sigma^2/2) \\
\mathrm{E}[(X \wedge x)^k] &= \exp(k\mu + k^2\sigma^2/2)\Phi\left(\frac{\ln x - \mu - k\sigma^2}{\sigma}\right) + x^k[1 - F(x)] \\
\text{mode} &= \exp(\mu - \sigma^2)
\end{aligned}
$$

Appendix B

An Inventory of Discrete Distributions

B.1 Introduction

The 16 models fall into three classes. The divisions are based on the algorithm by which the probabilities are computed. For some of the more familiar distributions these formulas will look different from the ones you may have learned, but they produce the same probabilities. After each name, the parameters are given. All parameters are positive unless otherwise indicated. In all cases, p_k is the probability of observing k losses.

For finding moments, the most convenient form is to give the factorial moments. The jth factorial moment is $\mu_{(j)} = \mathrm{E}[N(N-1)\cdots(N-j+1)]$. We have $\mathrm{E}[N] = \mu_{(1)}$ and $\mathrm{Var}(N) = \mu_{(2)} + \mu_{(1)} - \mu_{(1)}^2$.

The estimators which are presented are not intended to be useful estimators but rather for providing starting values for maximizing the likelihood (or other) function. For determining starting values, the following quantities are used [where n_k is the observed frequency at k (if, for the last entry, n_k represents the number of observations at k or more, assume it was at exactly k) and n is the sample size]:

$$\hat{\mu} = \frac{1}{n}\sum_{k=1}^{\infty} k n_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^{\infty} k^2 n_k - \hat{\mu}^2.$$

When the method of moments is used to determine the starting value, a circumflex (e.g., $\hat{\lambda}$) is used. For any other method, a tilde (e.g., $\tilde{\lambda}$) is used. When the starting value formulas do not provide admissible parameter values, a truly crude guess is to set the product of all λ and β parameters equal to the sample mean and set all other parameters equal to 1. If there are two λ and/or β parameters, an easy choice is to set each to the square root of the sample mean.

The last item presented is the probability generating function,

$$P(z) = \mathrm{E}[z^N].$$

B.2 The $(a, b, 0)$ class

B.2.1.1 Poisson—λ

$$\begin{aligned} p_0 &= e^{-\lambda}, \quad a = 0, \quad b = \lambda \qquad p_k = \frac{e^{-\lambda}\lambda^k}{k!} \\ \mathrm{E}[N] &= \lambda, \quad \mathrm{Var}[N] = \lambda \qquad P(z) = e^{\lambda(z-1)} \end{aligned}$$

B.2.1.2 Geometric—β

$$
\begin{aligned}
p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\
\mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1}.
\end{aligned}
$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer)

$$
\begin{aligned}
p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\
p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\
\mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m.
\end{aligned}
$$

B.2.1.4 Negative binomial—β, r

$$
\begin{aligned}
p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\
p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\
\mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}.
\end{aligned}
$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$.For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

Ex 2 $E[X_3 \mid x_1 = x_2 = 2]$?

$= \sum E[X_3 \mid \lambda_i]\, \pi[\lambda_i \mid x_1 = x_2 = 2]$ (λ_i)

$\lambda_1 = 1,\ \lambda_2 = 2;\ \lambda_3 = 5$

$$\pi(\lambda_1 \mid x_1 = x_2 = 2) = \frac{P(x_1 = x_2 = 2 \mid \lambda_1)\, \pi(\lambda_1)}{\sum P(x_1 = x_2 = 2 \mid \lambda_i)\, \pi(\lambda_i)}$$

$$= \frac{0.5\,(e^{-1}/2)^2}{A}$$

$$A = 0.5\,(e^{-1}/2)^2 + 0.3\left(e^{-2}\frac{2^2}{2}\right)^2 + 0.2\left(e^{-5}\,5^2/2!\right)^2$$

$= 0.42$

$$P(\lambda_2 \mid x_1 = x_2 = 2) = \frac{0.3\,(e^{-2}\,2^2/2)^2}{A}$$

$= 0.54$

$\Rightarrow P(\lambda_3 \mid x_1 = x_2 = 2) = 0.04$

$E[X_3 \mid x_1 = x_2 = 2] = 1 \times 0.42 + 2 \times 0.54 + 5 \times 0.04$

$= 1.70$

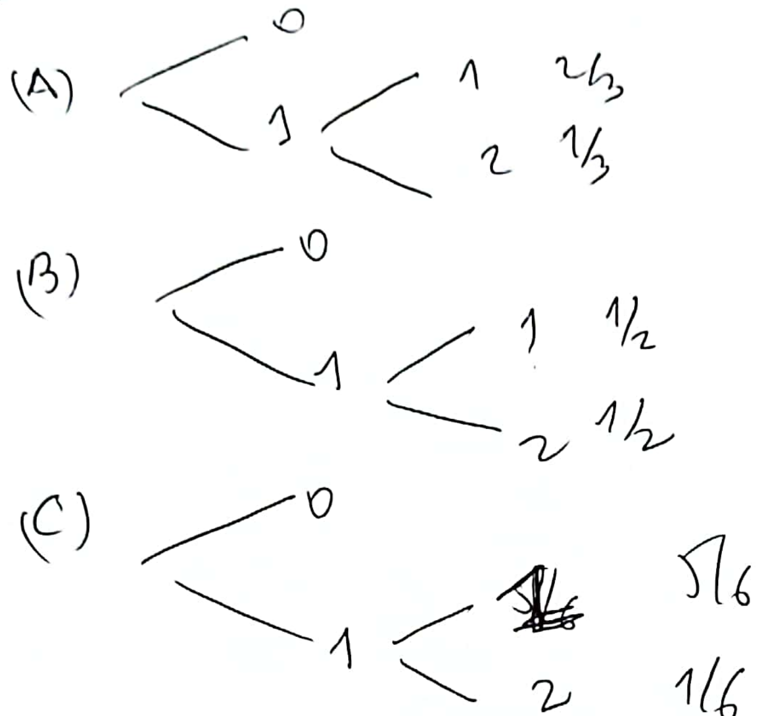
Table: Poisson (9); Expon (6); Pareto (3); Geometric (10)

Ex 1 $N_f = \lambda_0 = \left(\frac{2.576}{0.05}\right)^2 = 2654$

$N = 2890 > N_f \Rightarrow$ full credibility is achieved

we use $N = 2890$ to estimate Expected Value.

Ex 3



(a) $P(A \mid x_1=2) = \dfrac{P(x_1=2 \mid A)\,\pi(A)}{\sum P(x_1=2 \mid i)\,\pi(i)}$

$= \dfrac{(1/2)(1/3)(1/3)}{(1/2)(1/3)(1/3)+(1/2)(1/2)(1/3)+\frac{1}{2}(1/6)(1/3)} = 1/3 = 2/6$

(b) $E[X_2 \mid x_1=2]$?

$P(B \mid x_1=2) = \dfrac{(1/2)(1/2)(1/3)}{(1/2)(1/3)(1/3)+\frac{1}{2}\frac{1}{2}(1/3)+\frac{1}{2}(1/6)(1/3)} = 3/6$

$\Rightarrow P(C \mid x_1=2) = 1/6$

$E[X_2 \mid x_1=2] = ?$

$E[X_2 \mid A] = \frac{1}{2}\times 0 + \frac{1}{2}\left[1\times 2/3 + 2(1/3)\right] = 2/3$

$E[X_2 \mid B] = \frac{1}{2}\times 0 + \frac{1}{2}\left[1\times 1/2 + 2\times 1/2\right] = 3/4$

$E[X_2 \mid C] = \frac{1}{2}\times 0 + \frac{1}{2}\left[1\times 5/6 + 2(1/6)\right] = 7/12$

$E[X_2 \mid x_1=2] = 2/3\times 1/3 + 3/4\times 3/6 + \frac{7}{12}\times 1/6$

$= 0.6944 = \boxed{25/36}$

Ex 4

(a) $E[x_2 \mid x_1 = 2000]$

We have $E[X_2 \mid \theta_1] = 1000$

$E(X_2 \mid \theta_2) = \frac{\theta}{\alpha - 1} = 500$

$$\pi[\theta_1 \mid x_1 = 2000] = \frac{f(x_1 = 2000 \mid \theta_1)\,\pi(\theta_1)}{f(x_1 = 2000 \mid \theta_1)\,\pi(\theta_1) + f(x_1 = 2000 \mid \theta_2)\,\pi(\theta_2)}$$

$$= \frac{1/2\left(\frac{1}{1000}e^{-2000/1000}\right)}{1/2\left(\frac{1}{1000}e^{-2000/1000}\right) + \frac{1}{2}\left(\frac{3(1000)^3}{(2000+1000)^4}\right)}$$

$= 0.785$

$E[X_2 \mid x_1 = 2000] = 1000 \times 0.785 + 500 \times (1 - 0.785)$

$= \boxed{892.5}$

(b) $$P(\theta_1 \mid x_1 = 2000, x_2 = 1000) = \frac{P(x_1 = 2000, x_2 = 1000 \mid \theta_1) \times \pi(\theta_1)}{\sum P(x \mid \theta_i)\,\pi(\theta_i)}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{1000}e^{-2000/1000}\right)\left(\frac{1}{1000}e^{-1000/1000}\right)}{\frac{1}{2}(\quad)(\quad) + \frac{1}{2}\left(\frac{3 \times 1000^3}{(2000+1000)^4}\right)\left(\frac{3 \times 1000^3}{(1000+1000)^4}\right)}$$

$= \boxed{0.878}$

(c) $E[X_3 \mid x_1, x_2] = 1000(0.878) + 500(0.122)$

$= \boxed{939}$

Ex5 $X|\beta \sim \text{Geom}(\beta) \Rightarrow E(X|\beta) = \beta$

$$\beta = \begin{cases} 2 & \pi(\beta) = 1/3 \\ 5 & \pi(\beta) = 2/3 \end{cases}$$

$$P(\beta|x_1=2) = \frac{f(x_1=2|\beta)\,\pi(\beta)}{f(x_1=2)}$$

$$P(\beta=2|x_1=2) = \frac{f(x_1=2|\beta=2)\,1/3}{f(x_1=2|\beta=2)\,1/3 + f(x_1=2|\beta=5)\,2/3}$$

$$= \frac{1/3\left(\frac{2^2}{3^3}\right)}{1/3\left(2^2/3^3\right) + 2/3\left(5^2/6^3\right)} = 0.3902$$

$$E[X_2|x_1=2] = \underbrace{E[X_2|\beta=2]}\,P(\beta=2|x_1=2)$$
$$+ E[X_2|\beta=5]\,P(\beta=5|x_1=2)$$

$$= 2 \times 0.3902 + 5 \times 0.6098$$

$$= 3.83$$