College of Science.
Department of Statistics & Operations Research

كلية العلوم
قسم الإحصاء وبحوث العمليات

First Midterm Exam
Academic Year 1442-1443 Hijri- SecondSemester

Exam Information معلومات الامتحان			
Course name	**Loss**		**اسم المقرر**
Course Code	**ACTU 466**		**رمز المقرر**
Exam Date	**2021-02-22**	**1442-07-10**	**تاريخ الامتحان**
Exam Time	**10: 00 AM**		**وقت الامتحان**
Exam Duration	**2 hours**	**ساعتان**	**مدة الامتحان**
Classroom No.			**رقم قاعة الاختبار**
Instructor Name			**اسم استاذ المقرر**

Student Information معلومات الطالب		
Student's Name		**اسم الطالب**
ID number		**الرقم الجامعي**
Section No.		**رقم الشعبة**
Serial Number		**الرقم التسلسلي**

General Instructions:

- Your Exam consists of 4 PAGES (except this paper)
- Keep your mobile and smart watch out of the classroom.
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تعليمات عامة:

- عدد صفحات الامتحان 4 صفحة. (بإستثناء هذه الورقة)
- يجب إبقاء الهواتف و الساعات الذكية خارج قاعة الامتحان.
-

هذا الجزء خاص بأستاذ المادة

This section is ONLY for instructor

#	Course Learning Outcomes (CLOs)	Related Question (s)	Points	Final Score
1				
2				
3				
4				
5				
6				
7				
8				

EXAM COVER PAGE

Exercise 1 *Losses from a policy covering emergency room visits are distributed as a Pareto distribution with $\alpha = 3$ and $\theta = 1000$. The insurance company wants to impose a deductible such that the expected cost per emergency room visit under the policy is reduced to 50%. In other words:*

$$E[(X - d)^+] = 0.5E[X]$$

Determine d.

Exercise 2 *A company has 50 employees whose dental expenses are mutually independent. For each employee, the company reimburses 100% of dental expenses. The dental expense for each employee is distributed as follows:*

Expense	*Probability*
0	0.5
100	0.3
400	0.1
900	0.1

Using the normal approximation, calculate the 95th percentile of the cost to the company.
Note that the 95th percentile of the standard normal is 1.645.

$A)$ 22453 $B)$ 7634 $C)$ 4534 $D)$ 11173 $E)$ 9354

Exercise 3 *The losses under an insurance policy follow a negative binomial with $r = 2$ and $\beta = 1$. Losses are subject to a deductible of 1. Calculate the expected cost per loss.*

$A)$ 1/4 $B)$ 2/4 $C)$ 3/4 $D)$ 5/4 $E)$ 7/4

Exercise 4 *The number of dental claims per insured is distributed as a geometric distribution with $\beta = 2$. The amount of each dental claim is distributed as a Gamma distribution with $\alpha = 1$ and $\theta = 100$. Weller Dental Insurance Company has 1000 insureds. Assuming a normal distribution, calculate the 85th percentile of aggregate claims for Weller Dental.*
Note that the 85th percentile of the standard normal is 1.036.

$A)$ 132,867 $B)$ 209,266 $C)$ 42,659 $D)$ 87,456 $E)$ 453,756

Exercise 5 *The random variable N is the number of failures per 1000 iPads in a given year. N is distributed as a negative binomial with $r = 2$ and β . Further, β is distributed as a Gamma distribution with $\alpha = 2$ and $\theta = 3$. Calculate the var $[N]$.*

A.2.3.4 Paralogistic—α, θ

This is a Burr distribution with $\gamma = \alpha$.

$$\begin{aligned}
f(x) &= \frac{\alpha^2 (x/\theta)^\alpha}{x[1+(x/\theta)^\alpha]^{\alpha+1}} \qquad F(x) = 1 - u^\alpha, \quad u = \frac{1}{1+(x/\theta)^\alpha} \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)}, \quad -\alpha < k < \alpha^2 \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1]^{1/\alpha} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(1+k/\alpha)\Gamma(\alpha - k/\alpha)}{\Gamma(\alpha)} \beta(1+k/\alpha, \alpha - k/\alpha; 1-u) + x^k u^\alpha, \quad k > -\alpha \\
\text{mode} &= \theta \left(\frac{\alpha - 1}{\alpha^2 + 1} \right)^{1/\alpha}, \quad \alpha > 1, \text{ else } 0
\end{aligned}$$

A.2.3.5 Inverse paralogistic—τ, θ

This is an inverse Burr distribution with $\gamma = \tau$.

$$\begin{aligned}
f(x) &= \frac{\tau^2 (x/\theta)^{\tau^2}}{x[1+(x/\theta)^\tau]^{\tau+1}} \qquad F(x) = u^\tau, \quad u = \frac{(x/\theta)^\tau}{1+(x/\theta)^\tau} \\
\mathrm{E}[X^k] &= \frac{\theta^k \Gamma(\tau+k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)}, \quad -\tau^2 < k < \tau \\
\mathrm{VaR}_p(X) &= \theta(p^{-1/\tau} - 1)^{-1/\tau} \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\tau+k/\tau)\Gamma(1 - k/\tau)}{\Gamma(\tau)} \beta(\tau+k/\tau, 1 - k/\tau; u) + x^k [1-u^\tau], \quad k > -\tau^2 \\
\text{mode} &= \theta\,(\tau - 1)^{1/\tau}, \quad \tau > 1, \text{ else } 0
\end{aligned}$$

A.3 Transformed gamma family

A.3.2 Two-parameter distributions

A.3.2.1 Gamma—α, θ

$$\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \qquad F(x) = \Gamma(\alpha; x/\theta) \\
M(t) &= (1-\theta t)^{-\alpha}, \quad t < 1/\theta \qquad \mathrm{E}[X^k] = \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)}, \quad k > -\alpha \\
\mathrm{E}[X^k] &= \theta^k (\alpha + k - 1) \cdots \alpha, \quad \text{if } k \text{ is an integer}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k \Gamma(\alpha+k)}{\Gamma(\alpha)} \Gamma(\alpha+k; x/\theta) + x^k [1 - \Gamma(\alpha; x/\theta)], \quad k > -\alpha \\
&= \alpha(\alpha+1)\cdots(\alpha+k-1)\theta^k \Gamma(\alpha+k; x/\theta) + x^k[1-\Gamma(\alpha; x/\theta)], \quad k \text{ an integer} \\
\text{mode} &= \theta(\alpha - 1), \quad \alpha > 1, \text{ else } 0
\end{aligned}$$

A.2.3 Two-parameter distributions

A.2.3.1 Pareto (Pareto Type II, Lomax)—α, θ

$$
\begin{aligned}
f(x) &= \frac{\alpha\theta^\alpha}{(x+\theta)^{\alpha+1}} \qquad F(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}, \quad -1 < k < \alpha \\
\mathrm{E}[X^k] &= \frac{\theta^k k!}{(\alpha-1)\cdots(\alpha-k)}, \quad \text{if } k \text{ is an integer} \\
\mathrm{VaR}_p(X) &= \theta[(1-p)^{-1/\alpha} - 1] \\
\mathrm{TVaR}_p(X) &= \mathrm{VaR}_p(X) + \frac{\theta(1-p)^{-1/\alpha}}{\alpha-1}, \quad \alpha > 1 \\
\mathrm{E}[X \wedge x] &= \frac{\theta}{\alpha-1}\left[1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha-1}\right], \quad \alpha \neq 1 \\
\mathrm{E}[X \wedge x] &= -\theta \ln\left(\frac{\theta}{x+\theta}\right), \quad \alpha = 1 \\
\mathrm{E}[(X \wedge x)^k] &= \frac{\theta^k\Gamma(k+1)\Gamma(\alpha-k)}{\Gamma(\alpha)}\beta[k+1, \alpha-k; x/(x+\theta)] + x^k\left(\frac{\theta}{x+\theta}\right)^\alpha, \quad \text{all } k \\
\text{mode} &= 0
\end{aligned}
$$

A.2.3.2 Inverse Pareto—τ, θ

$$
\begin{aligned}
f(x) &= \frac{\tau\theta x^{\tau-1}}{(x+\theta)^{\tau+1}} \qquad F(x) = \left(\frac{x}{x+\theta}\right)^\tau \\
\mathrm{E}[X^k] &= \frac{\theta^k\Gamma(\tau+k)\Gamma(1-k)}{\Gamma(\tau)}, \quad -\tau < k < 1 \\
\mathrm{E}[X^k] &= \frac{\theta^k(-k)!}{(\tau-1)\cdots(\tau+k)}, \quad \text{if } k \text{ is a negative integer} \\
\mathrm{VaR}_p(X) &= \theta[p^{-1/\tau} - 1]^{-1} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\tau\int_0^{x/(x+\theta)} y^{\tau+k-1}(1-y)^{-k}dy + x^k\left[1 - \left(\frac{x}{x+\theta}\right)^\tau\right], \quad k > -\tau \\
\text{mode} &= \theta\frac{\tau-1}{2}, \quad \tau > 1, \text{ else } 0
\end{aligned}
$$

A.2.3.3 Loglogistic (Fisk)—γ, θ

$$
\begin{aligned}
f(x) &= \frac{\gamma(x/\theta)^\gamma}{x[1+(x/\theta)^\gamma]^2} \qquad F(x) = u, \quad u = \frac{(x/\theta)^\gamma}{1+(x/\theta)^\gamma} \\
\mathrm{E}[X^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma), \quad -\gamma < k < \gamma \\
\mathrm{VaR}_p(X) &= \theta(p^{-1}-1)^{-1/\gamma} \\
\mathrm{E}[(X \wedge x)^k] &= \theta^k\Gamma(1+k/\gamma)\Gamma(1-k/\gamma)\beta(1+k/\gamma, 1-k/\gamma; u) + x^k(1-u), \quad k > -\gamma \\
\text{mode} &= \theta\left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}, \quad \gamma > 1, \text{ else } 0
\end{aligned}
$$

B.2.1.2 Geometric—β

$$
\begin{aligned}
p_0 &= \frac{1}{1+\beta}, \quad a = \frac{\beta}{1+\beta}, \quad b = 0 \qquad p_k = \frac{\beta^k}{(1+\beta)^{k+1}} \\
\mathrm{E}[N] &= \beta, \quad \mathrm{Var}[N] = \beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-1}.
\end{aligned}
$$

This is a special case of the negative binomial with $r = 1$.

B.2.1.3 Binomial—$q, m, (0 < q < 1,\ m$ an integer$)$

$$
\begin{aligned}
p_0 &= (1-q)^m, \quad a = -\frac{q}{1-q}, \quad b = \frac{(m+1)q}{1-q} \\
p_k &= \binom{m}{k} q^k (1-q)^{m-k}, \quad k = 0, 1, \ldots, m \\
\mathrm{E}[N] &= mq, \quad \mathrm{Var}[N] = mq(1-q) \qquad P(z) = [1+q(z-1)]^m.
\end{aligned}
$$

B.2.1.4 Negative binomial—β, r

$$
\begin{aligned}
p_0 &= (1+\beta)^{-r}, \quad a = \frac{\beta}{1+\beta}, \quad b = \frac{(r-1)\beta}{1+\beta} \\
p_k &= \frac{r(r+1)\cdots(r+k-1)\beta^k}{k!(1+\beta)^{r+k}} \\
\mathrm{E}[N] &= r\beta, \quad \mathrm{Var}[N] = r\beta(1+\beta) \qquad P(z) = [1-\beta(z-1)]^{-r}.
\end{aligned}
$$

B.3 The $(a, b, 1)$ class

To distinguish this class from the $(a, b, 0)$ class, the probabilities are denoted $\Pr(N = k) = p_k^M$ or $\Pr(N = k) = p_k^T$ depending on which subclass is being represented. For this class, p_0^M is arbitrary (that is, it is a parameter) and then p_1^M or p_1^T is a specified function of the parameters a and b. Subsequent probabilities are obtained recursively as in the $(a, b, 0)$ class: $p_k^M = (a + b/k)p_{k-1}^M$, $k = 2, 3, \ldots$, with the same recursion for p_k^T There are two sub-classes of this class. When discussing their members, we often refer to the "corresponding" member of the $(a, b, 0)$ class. This refers to the member of that class with the same values for a and b. The notation p_k will continue to be used for probabilities for the corresponding $(a, b, 0)$ distribution.

B.3.1 The zero-truncated subclass

The members of this class have $p_0^T = 0$ and therefore it need not be estimated. These distributions should only be used when a value of zero is impossible. The first factorial moment is $\mu_{(1)} = (a+b)/[(1-a)(1-p_0)]$, where p_0 is the value for the corresponding member of the $(a, b, 0)$ class. For the logarithmic distribution (which has no corresponding member), $\mu_{(1)} = \beta/\ln(1+\beta)$. Higher factorial moments are obtained recursively with the same formula as with the $(a, b, 0)$ class. The variance is $(a+b)[1-(a+b+1)p_0]/[(1-a)(1-p_0)]^2$.For those members of the subclass which have corresponding $(a, b, 0)$ distributions, $p_k^T = p_k/(1-p_0)$.

Loss Thd 1 Sol

2021 / S1

Ex 1

$X \sim$ Pareto

$E(X) = \theta/(\alpha-1) = 500$

$E(X \wedge d) = 500\left[1 - \left(\frac{1000}{1000+d}\right)^{2}\right]$

$E(Y^L) = E\left((X-d)^+\right) = E(X) - E(X \wedge d)$

$= 500\left(\frac{1000}{1000+d}\right)^2 = 0.5 \times 500$

$\Rightarrow \frac{1000}{1000+d} = \sqrt{0.5} \Rightarrow d = 414.21$

Ex 2

$Cost = S = \sum_{i=1}^{n=50} X_i$

$\mu_S = E(S) = 50 \times E(X)$

$\sigma_S^2 = V(S) = 50 \times V(X)$

$E(X) = \sum x_i p_i = 0 \times 0.5 + 100 \times 0.3 + 400 \times 0.1 + 900 \times 0.1$

$= 160$

$\sigma_X^2 = E(X^2) - (E(X))^2 = 0^2 \times 0.5 + 100^2 \times 0.3 + 400^2 \times 0.1 + 900^2 \times 0.1$

$- 160^2 = 74\,400$

$P(S \le x) = 0.95$

$\Leftrightarrow P\left(\underbrace{\frac{S-\mu_S}{\sigma_S}}_{\sim N(0,1)} \le \frac{x-\mu_S}{\sigma_S}\right) = 0.95$

$\Rightarrow \frac{x-\mu_S}{\sigma_S} = 1.645 \Rightarrow x = 1.645 \times \sigma_S + \mu_S$

$= 1.645 \times \sqrt{50 \times 74\,400} + 50 \times 160$

$= 11\,173.$

Ex3

$$Y^L = (X-1)^+ = \begin{cases} 0 & X \le 1 \\ X-1 & X > 1 \end{cases}$$

$$E(Y^L) = \sum_{2}^{\infty} (x-1) p_x = \sum_{2}^{\infty} x p_x - \sum_{2}^{\infty} p_x$$

$$= \left[E(X) - \sum_{0}^{1} x p_x\right] - \left[1 - \sum_{0}^{1} p_x\right]$$

$$p_0 = P(X=0) = 1/4$$

$$p_1 = P(X=1) = \frac{r\beta}{(1+\beta)^{r+1}} = 1/4.$$

$$E(X) = r\beta = 2$$

$$E(Y^L) = (2 - 1/4) - (1 - 1/4 - 1/4) = 5/4$$

Ex4

cost per insured $= X_i = \sum_{1}^{N_i} Y_i$ | $\mu_N = 2 \quad \sigma_N^2 = 6$, $\mu_Y = 100, \sigma_Y^2 = 10{,}000$

Aggregate claims $S = \sum_{i=1}^{n=1000} X_i$

$$\mu_X = E(X) = \mu_N \mu_Y = 2 \times 100 = 200$$

$$\sigma_X^2 = V(X) = \sigma_N^2 \mu_Y^2 + \mu_N \sigma_Y^2 = 80{,}000$$

$$\Rightarrow \mu_S = n\mu_X = 200{,}000$$

$$\sigma_S^2 = n\sigma_X^2 = 80 \times 10^6$$

$$P(S \le x) = 0.85 \Rightarrow P\Big(\underbrace{\frac{S-\mu_S}{\sigma_S}}_{N(0,1)} \le \frac{x-\mu_S}{\sigma_S}\Big) = 0.85$$

$$\Rightarrow \frac{x-\mu_S}{\sigma_S} = 1.036$$

$$\Rightarrow x = 1.036\,\sigma_S + \mu_S$$

$$= 1.036\sqrt{80} \times 10^3 + 200{,}000$$

$$= 209{,}266$$

Ex:

$N \sim$ Neg. Binom $r = 2$, β r.v

$\beta \sim$ Gamma $\alpha = 2, \theta = 3$.

$$\text{Var } N = E(\text{Var}(N|\beta)) + \text{Var}(E(N|\beta))$$

$$= E(2\beta(1+\beta)) + \text{Var}(2\beta)$$

$$= 2\{E\beta + E(\beta^2)\} + 4\,\text{Var}(\beta)$$

$$= 2\{E\beta + \text{Var}\,\beta + (E\beta)^2\} + 4\,\text{Var}\,\beta$$

$$= 2E\beta + 2(E\beta)^2 + 6\,\text{Var}\,\beta$$

$$= 2 \times 6 + 2 \times 6^2 + 6 \times 18$$

$$= 192$$