جامعة الملك سعود
King Saud University

Math151 (Discrete Mathematics)- Midterm Exam- First Semester - 25/10/2021 - Duration 90 min - Section: ....................
Name: ..........Answer.......................................Dr. Brahem
ID:.............................................................................

1. (2 points) Draw the truth table of the following statement: $(p \to q) \vee (\neg p \to \neg q)$.

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| p | q | ¬p | ¬q | p→q | ¬p→¬q | (p→q)∨(¬p→¬q) |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | T | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | T | T |

We deduce that $(p\to q)\vee(\neg p \to \neg q)$ is a tautology

2. (2 points) Using logic laws, show that $(p \to q) \wedge r \equiv \neg(r \to p) \vee \neg(r \to \neg q)$.

$$\neg(r \to p) \vee \neg(r \to \neg q) \equiv \neg(\neg r \vee p) \vee \neg(\neg r \vee \neg q)$$
$$\equiv (r \wedge \neg p) \vee (r \wedge q) \quad 1$$
$$\equiv r \wedge (\neg p \vee q) \quad 1$$
$$\equiv (\neg p \vee q) \wedge r \equiv (p \to q) \wedge r.$$

3. (2 points) State the inverse and the contrapositive for the following statement: "*If the integer $a+b-c$ is an even , then $a$ is even or $b$ is even or $c$ is even*, where $a, b, c \in \mathbb{Z}$."

1 • The inverse is: if $(a+b-c)$ is odd then a and b and c are all odd.
1 • The contrapositive is: if a, b and c are all odd then $(a+b-c)$ is odd.

4. (2 points) Determine the truth value of each of these statements and justify your answer.

• $\forall x \in \mathbb{R},\ x^2 - 6x + 9 \geq 0.$
1 True, because $x^2-6x+9 = (x-3)^2 \geq 0$ for every $x \in \mathbb{R}$.

• $\exists x \in \mathbb{R},\ x^2 < x.$
1 True, take $x = 1/2$ we have $(\frac{1}{2})^2 = \frac{1}{4} < 1/2$

5. (2 points) Use a direct proof to show that: *the sum of two odd numbers is an even number.*

Let $a$ & $b$ be 2 odd numbers. Then $a = 2k+1$ with $k \in \mathbb{Z}$
Also $b = 2h+1$ with $h \in \mathbb{Z}$

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So $a+b = 2k+1+2h+1 = 2k+2h+2$
$= 2(\underbrace{k+h+1})$
1 $= 2 \quad M$ with $M \in \mathbb{Z}$

we deduce that $(a+b)$ is even.

6. (2 points) Use a proof by contraposition to show that: *if $n^2$ is odd then $n$ is odd.*

(1) – The contrapositive is: if $n$ even then $(n^2)$ is even.

(1) – Proof: Let $n$ even then $n = 2k$ with $k \in \mathbb{Z}$

so $n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2M$ with $M = 2k^2 \in \mathbb{Z}$

we deduce that $n^2$ is even.

7. (3 points) Use mathematical induction to show that:

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \quad \text{for } n \geq 1.$$

Put $P(n)$: $\frac{1}{1\times3} + \frac{1}{3\times5} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

(0,5) • Basis step $n = 1$

$\frac{1}{3} = \frac{1}{(2.1-1)(2.1+1)} \stackrel{?}{=} \frac{1}{2.1+1} = \frac{1}{3}$ ✓ so $P(1)$ is true.

• Inductive step: Let $k \geq 2$, we suppose that $P(k)$ is true

$\left(\frac{1}{1\times3} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}\right)$. Now we prove that

(1) $P(k+1)$ remains true $\left(\frac{1}{1\times3} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \stackrel{?}{=} \frac{k+1}{2k+3}\right)$

(1,5) $\underbrace{\frac{1}{1\times3} + \cdots + \frac{1}{(2k-1)(2k+1)}} + \frac{1}{(2k+1)(2k+3)} = \underbrace{\frac{k}{(2k+1)}} + \frac{1}{(2k+1)(2k+3)}$

$= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)}$ ✓

we deduce that $P(n)$ is true for $n \geq 1$.

8. (5 points) Let $R$ be a relation defined on $A = \{-2, -1, 1, 2\}$ as: $a\,R\,b \iff a.b < 0$.

(a) Write $R$ as a set of ordered pairs.

(1,5) $R = \{(-2,1); (-2,2); (-1,1); (-1,2); (1,-2); (1,-1); (2,-2); (2,-1)\}$

(b) Find the domain and the image of $R$.

(1) The domain of R is $D_R = \{-2,-1,1,2\}$

The image of R is $\text{Im}\,R = \{-2,-1,1,2\}$

(c) Find the matrix $M_R$.

(1)
$$M_R = \begin{array}{c} \\ -2 \\ -1 \\ 1 \\ 2 \end{array} \begin{array}{cccc} -2 & -1 & 1 & 2 \\ \end{array}$$
$$M_R = \begin{array}{r|cccc} & -2 & -1 & 1 & 2 \\ \hline -2 & 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \end{array}$$

(d) Find $R^2$.

(1,5) $R = \{(-2,1); (-2,2); (-1,1); (-1,2); (1,-2); (1,-1); (2,-2); (2,-1)\}$

so $R^2 = R \circ R = \{(-2,-2); (-2,-1); (-1,-2); (-1,-1); (1,1); (1,2); (2,1); (2,2)\}$

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