| King Saud University | College of Sciences | Department of Mathematics |
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| Final Examination | Math 132 | Semester 2(1443) | Time: 3H |
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**Calculators are not allowed**

**The Examination contains 2 pages**

**Question 1:** (10 marks)

1. Without using truth tables, prove the following logical equivalence: (3 marks)

$$\neg[\neg p \wedge (q \to p)] \equiv p \vee q$$

2. Let $n \in \mathbb{Z}$. Prove that: $3n$ is even if and only if $n+7$ is odd. (3 marks)

3. Consider the sequence $\{a_n\}_{n=0}^{\infty}$ defined as follows:

$$\begin{cases} a_0 = 1 \\ a_1 = 2 \\ a_{n+1} = 5a_n - 6a_{n-1}; \forall n \geq 1 \end{cases}$$

Use mathematical induction to prove that $a_n = 2^n$, $\forall n \in \mathbb{Z}$, with $n \geq 0$. (4 marks)

**Question 2:** (14 marks)

1. Let $R$ be the relation on the set $A = \{0,1,2,3,4\}$ defined by
$R := \{(0,0),(0,2),(0,4),(1,0),(1,3),(2,1),(2,2),(4,3)\}$.
   (a) Is $R$ a function? Justify your answer. (1 mark)
   (b) Find the following relations: $R^{-1} \cap R$ and $R^2$. (2 marks)

2. Let $S$ be the relation from the set of integer $\mathbb{Z}$ defined as follows:

$$\text{for } a, b \in \mathbb{Z},\ [(aSb) \Leftrightarrow (3b - a) \text{ is an even integer }].$$

   (a) Show that $S$ is an equivalence relation on $\mathbb{Z}$. (3 marks)
   (b) Decide whether $-3 \in [6]$, justify your answer. (1 mark)

3. Let $A_1 = \{1\}$, $A_2 = \{-2,-1\}$, $A_3 = \{0,2,3\}$ and $A_4 = \{-3,4\}$ be the equivalence classes of the relation $E$ on the set $B = \{-3,-2,-1,0,1,2,3,4\}$.
   (a) Draw the digraph of $E$. (1 mark)
   (b) List all the ordered pairs in the relation $E$. (1 mark)

4. Let $P$ be the relation defined on the set $\mathbb{Z}^+$ of positive integer by:

$$\text{for } m,n \in \mathbb{Z}^+,\ [(mPn) \Leftrightarrow \frac{n}{m} \text{ is an odd integer}].$$

   (a) Show that $P$ is a partially ordering relation on $\mathbb{Z}^+$. (3 marks)

   (b) Is $P$ a total ordering relation in $\mathbb{Z}^+$? (1 mark)

   (c) Let $C = \{1,2,3,4,6,8,9\}$. Draw the hasse diagram of $P$ on the set $C$. (1 mark)

**Question 3:** (10 marks)

1. Suppose that $g$ is a function from $\mathbb{R}$ to $\mathbb{R}^2$ and f is a function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x,y) = x^2 + y$ and $g(t) = (t, t+1)$. Find $g \circ f$ and $f \circ g$. (2 marks)

2. Consider the sets $A := \{0,1,2,3,4\}$ and $B := \{a,b,c,d,e\}$, and the function $f : A \longrightarrow B$ defined by: $f(0) = a$, $f(1) = f(4) = b$, $f(3) = c$ and $f(2) = d$.

   (a) Find the image of each of the sets $\{1,2,3\}, \{0,3\}$ and $\{4\}$. (3 marks)

   (b) Find the inverse image of each of the sets $\{a,b,c\}, \{d,e\}$ and $\{b,c,d\}$. (3 marks)

   (c) For the function $f$, determine whether it is one-to-one, and whether it is onto $B$. (Justify your answer). (2 marks)

**Question 4:** (6 marks)

1. Give the cardinal of the set $X$ in each of the following cases.

   (a) $X = \{k \in \mathbb{N}; k \text{ is even }\}$. (1 mark)

   (b) $X = (-\infty, 2] \cup [1442, 2020]$. (1 mark)

2. Determine whether each of the following statements is true or false.

   (a) $|\mathbb{Q} \cap (-\infty, 2]| = \aleph_0$. (1 mark)

   (b) $|\mathcal{P}(\mathbb{Z})| = c$. (1 mark)

3. Show that the set of odd negative integers is a countable set.(2 marks)

Answer Final examination

math 132, semester 2, 1443

## Exercise 1:

1) $\neg[\neg p \wedge (q \longrightarrow p)] \equiv p \vee \neg(q \longrightarrow p)$

$\equiv p \vee (q \wedge \neg p)$

$\equiv (p \vee q) \wedge (p \vee \neg p)$

$\equiv p \vee q$ 3

2) * if $3n$ is even then $n+7$ is odd.

$3n$ is even $\Rightarrow n$ is even $\Rightarrow n = 2k$; $k \in \mathbb{Z}$.

$\Rightarrow n+7 = 2k+7 = 2(k+3)+1$ is odd.

* if $n+7$ is odd, then $3n$ is even.

by contraposition, if $3n$ is odd, then $n+7$ is even 3

$\Rightarrow 3n$ is odd $\Rightarrow n$ is odd $\Rightarrow n = 2k+1$, $k \in \mathbb{Z} \Rightarrow$

$n+7 = 2k+8 = 2(k+4)$ even.

3) $P(n)$: $a_n = 2^n$; $n \geq 0$,

B.S.: $P(0)$: $a_0 = 2^0 = 1$; True

$P(1)$: $a_1 = 2^1 = 2$; True.

I.S.: Let $k \geq 1$, we prove that $(P(0) \wedge P(1) \wedge \dots \wedge P(k)) \longrightarrow P(k+1)$.

we assume that $P(0), P(1), \dots, P(k)$ are true, we prove that $P(k+1)$ is true.

$P(k+1)$: $a_{k+1} = 2^{k+1}$

$a_{k+1} = 5a_k - 6a_{k-1}$ 4

$\left.\begin{array}{l} P(k) \text{ true} \Rightarrow a_k = 2^k; \\ P(k-1) \text{ true} \Rightarrow a_{k-1} = 2^{k-1} \end{array}\right\} \Rightarrow a_{k+1} = 5 \times 2^k - 6 \times 2^{k-1}$

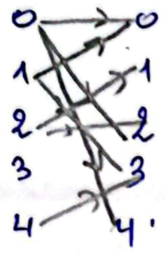
$= 5 \times 2^k - 3 \times 2^k$

$= 2^k(5-3) = 2^{k+1}$

$\Rightarrow P(k+1)$ is true $\Rightarrow \forall n \geq 0$; $a_n = 2^n$

Exercise 2:

1)



a) R is not a function; because 0 have two images. 1

b)

$R = \{(0,0),(0,2),(0,4),(1,0),(1,3),(2,1),(2,2),(4,3)\}$.

$R^{-1} = \{(0,0),(2,0),(4,0),(0,1),(3,1),(1,2),(2,2),(3,4)\}$.

$R^{-1} \cap R = \{(0,0),(2,2)\}$. 1

$R^2 = R \circ R = \{(0,0),(0,2),(0,4),(0,1),(0,3),(1,0),(1,2),(1,4),(2,0),(2,3),(2,1),(2,2)\}$.

1

2).

a) • Reflexive: Let $a \in \mathbb{Z}$; $3a - a = 2a$ is even $\Rightarrow a S a$

So S is reflexive. 1

• symmetric: Let $a, b \in \mathbb{Z}$; $a S b \Rightarrow 3b - a = 2k$; $k \in \mathbb{Z}$.

$\Rightarrow 9b - 3a = 6k \Rightarrow 3a - 9b = -6k$

$\Rightarrow 3a - b = 8b - 6k = 2(4b - 3k)$ is even

$\Rightarrow b S a$ 1

$\Rightarrow$ S is symmetric.

• transitive: Let $a, b, c \in \mathbb{Z}$;

$a S b$; $b S c \Rightarrow 3b - a = 2k$ ; $k \in \mathbb{Z}$.

$3c - b = 2p$ $p \in \mathbb{Z}$.

$\Rightarrow 3c - b + 3b - a = 2k + 2p$

$\Rightarrow 3c - a = 2k + 2p - 2b = 2(k + p - b)$ is even

$\Rightarrow a S c$ 1

$\Rightarrow$ S is transitive

$\Rightarrow$ S is an equivalence relation on $\mathbb{Z}$.

2)

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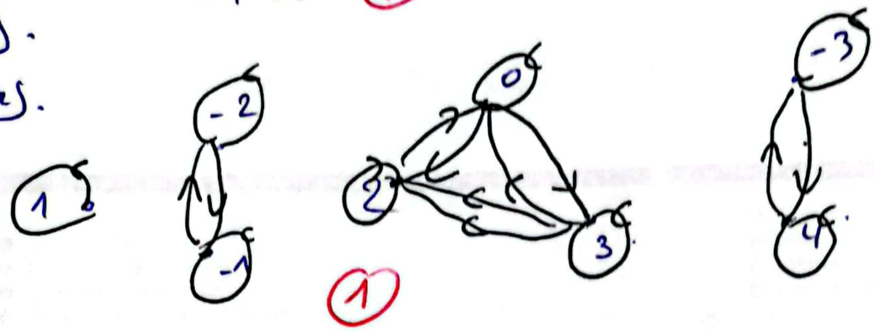
b). $3(-3)-6 = 9-6 = 3$ is odd $\Rightarrow -3 \not{P}\, 6$.

$\Rightarrow -3 \notin [6]$. (1)

3).

a).




(1)

b). $E = \{(1,1), (-1,-1), (-1,-2), (-2,-1), (-2,-2), (0,0), (0,2), (0,3), (2,0), (2,2), (2,3),$
$(3,0), (3,2), (3,3), (-3,-3), (-3,4), (4,-3), (4,4)\}$.

(1)

4).

a) Reflexive: Let $m \in \mathbb{Z}^+$; $\frac{m}{m} = 1$ is odd $\Rightarrow m P m$

$\Rightarrow P$ is reflexive. (1)

Antisymmetric: Let $m, n \in \mathbb{Z}^+$; $m P n$ and $n P m$

$\Rightarrow \frac{n}{m}$ is odd and $\frac{m}{n}$ is odd.

$\Rightarrow \frac{n}{m} = 2k+1$; $\frac{m}{n} = 2r+1$. $k, r \in \mathbb{Z}^+$.

$\Rightarrow \frac{n}{m} \cdot \frac{m}{n} = 1 = 4kr + 2r + 2k + 1$ (1)

$\Rightarrow 4kr + 2r + 2k = 0$. , as $k, r \in \mathbb{Z}^+ \Rightarrow k = r = 0$

$\Rightarrow \frac{n}{m} = 1 \Rightarrow m = n$.

Transitive: Let $m, n, p \in \mathbb{Z}^+$; $m P n$; $n P p$

$\Rightarrow \frac{n}{m}$ odd; $\frac{p}{n}$ is odd

$\Rightarrow \frac{n}{m} \cdot \frac{p}{n}$ is odd $\Rightarrow \frac{p}{m}$ is odd

$\Rightarrow m P p \Rightarrow P$ is transitive (1)

$\Rightarrow P$ is a partially ordering relation on $\mathbb{Z}^+$.

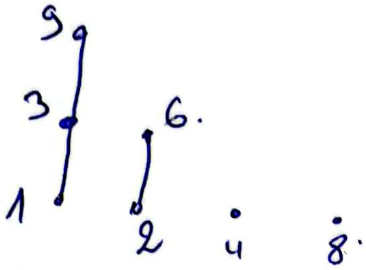
3)

b) P is not a total ordering relation in $\mathbb{Z}^+$;

$3 \not P 6$ and $6 \not P 3$. 1

c) $P = \{(1,1), (1,3), (1,9), (2,2), (2,6), (3,3), (3,9), (4,4), (6,6), (8,8), (9,9)\}$

1




Exercise 3:

1)

$g \circ f = g(f(x)) = (f(x); f(x)+1) = (x^2+y, x^2+y+1)$ 1

$f \circ g = f(g(x)) = f(x, x+1) = x^2+x+1$, 1

2) a) $f(\{1,2,3\}) = \{b, d, c\}$. 1

$f(\{0,3\}) = \{a, c\}$. 1

$f(\{4\}) = b$. 1

b) $f^{-1}(\{a,b,c\}) = \{0,1,4,3\}$. 1

$f^{-1}(\{d,e\}) = \{2\}$. 1

$f^{-1}(\{b,c,d\}) = \{1,4,3,2\}$. 1

c) not one to one because $f(1) = f(4)$. 1

not onto because $f^{-1}(\{e\}) = \phi$. 1

4)

Exercice 4 :

1).

a) $|X| = |\mathbb{Z}^+| = \aleph_0$. 1

b). $|X| = (-\infty, 2] \cup (1442, 2020)$ 1

$= (-\infty, 2020] \subset \mathbb{R}$.

$\Rightarrow |X| = c$.

2)

a) $\mathbb{Q} \cap (-\infty, 2] \subset \mathbb{Q} \Rightarrow$ True. 1

b). $|P(\mathbb{Z})| = c$ True 1

3).

$f(n) = -2n+1$ is a one to one correspondence

from $\mathbb{Z}^+$ to this set.

so is countable set. 2

5)