King Saud University	College of Sciences	Department of Mathematics	
First Examination	Math 132	Semester II (1443)	Time:1H30

Exercise 1: (10 pts)

1. Decide whether the following propositions is a tautology or a contradiction or a contingency?

$$[(p \to q) \wedge (q \to p)] \vee [(\neg p \wedge q) \vee (p \wedge \neg q)]. \quad \text{(3 pts)}$$

2. Without using truth tables, prove that the following conditional statement is a Tautology:

$$[(p \wedge \neg q) \vee p)] \vee (\neg p \vee q). \quad \text{(3 pts)}$$

3. Without using truth tables, prove the following logical equivalence:

$$\neg(p \leftrightarrow q) \equiv (p \wedge \neg q) \vee (q \wedge \neg p). \quad \text{(2 pts)}$$

4. Determine the truth value of each of the following statements if the domain consists of all real numbers. (Justify your answer)
   (a) $\forall x \in \mathbb{R};\ (x^2 < x^3)$. (1 pts)
   (b) $\exists x \in \mathbb{R};\ (-x)^3 = -x^3$. (1 pts).

Exercise 2: (11 pts)

1. Prove that if $3n^2 + 3$ is odd then n is even. (2 pts)
2. Let x, y and z be three real numbers. Prove by contradiction that:
   if $(2x^4 + 3y^3 + z = 133)$ then, $(x \leq 2$ or $y \leq 3$ or $z \leq 20)$. (2 pts)
3. Use mathematical induction to prove the following statement:

$$1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2, \quad \text{for each integer } n, \text{ with } n \geq 0. \quad \text{(3 pts)}$$

4. Consider the sequence $\{u_n\}_{n=0}^{\infty}$ defined as follows: $\begin{cases} u_0 = 2 \\ u_1 = 6 \\ u_{n+1} = 3u_n + 10u_{n-1} - 12; \quad n \geq 1 \end{cases}$

   Use mathematical induction to prove the following statement:

$$u_n = 5^n + 1, \quad \text{for each integer } n, \text{ with } n \geq 0. \quad \text{(4 pts)}$$

Exercise 3: (4 pts)

Consider the following three sets:

$C := \{a, b, c, d\}$, $D := \{x, y, z\}$, and $E := \{(a, x), (b, z), (2, 2), (2, 4), (d, z), (z, d), \emptyset\}$.

Find the following sets:

(i) $C \times (C \cap D)$. **(ii)** $(C \times D) \setminus E$. **(iii)** $\{\emptyset\} \times E$. **(iv)** $E \times \emptyset$. (4 pts)

Exercise 1:

1)

p	q	$p\to q$	$q\to p$	$(p\to q)\wedge(q\to p)$	$\neg p$	$\neg q$	$\neg p\wedge q$	$p\wedge\neg q$	$(\neg p\wedge q)\vee(p\wedge\neg q)$	R
T	T	T	T	T	F	F	F	F	F	T
T	F	F	T	F	F	T	F	T	T	T
F	T	T	F	F	T	F	T	F	T	T
F	F	T	T	T	T	T	F	F	F	T

is a tautology.

2) $[(p\wedge\neg q)\vee p]\vee(\neg p\vee q) \equiv p\vee(\neg p\vee q) \equiv (p\vee\neg p)\vee q \equiv T\vee q \equiv T.$

3) $\neg(p\leftrightarrow q) \equiv \neg((p\to q)\vee(q\to p)) \equiv \neg((\neg p\vee q)\wedge(\neg q\vee p))$

$\equiv (p\wedge\neg q)\vee(q\wedge\neg p)$

4) a) false; $x=\frac{1}{2}$; $\left(\frac{1}{2}\right)^2=\frac{1}{4} \not< \left(\frac{1}{2}\right)^3=\frac{1}{8}$.

b). True $x=1$.

Exercise 2:

1) By contraposition, we prove that: if n is odd, then $3n^2+3$ is even.

n is odd $\Rightarrow n=2k+1$; $k\in\mathbb{Z}$.

$3n^2+3 = 3(2k+1)^2+3 = 3(4k^2+4k+1)+3$
$= 12k^2+12k+3+3 = 2(6k^2+6k+3)$ is even.

2) By contradiction, we assume that $x>2$ and $y>3$ and $z>20$

$\Rightarrow$ $x^4>16 \Rightarrow 2x^4>32$

$y^3>27 \Rightarrow 3y^3>81$

$z>20$

$2x^4+3y^3+z>32+81+20$

$\Rightarrow 2x^4+3y^3+z>133$

that contradict the fact $2x^4+3y^3+z=133$.

Scanné avec CamScanner

3) $P(n): 1+3+5+\dots+(2n+1) = (n+1)^2$. $n \geq 0$.

B.S:

$P(0): 1 = (0+1)^2 = 1^2 = 1$. True

I.S: Let $k \geq 0$. we prove that $P(k) \longrightarrow P(k+1)$.

we assume that $1+3+5+\dots+(2k+1) = (k+1)^2$ and we prove that

$1+3+5+\dots+(2k+3) = (k+2)^2$??

$$1+3+5+\dots+(2k+1)+(2k+3) = (k+1)^2 + 2k+3$$
$$= k^2+2k+1+2k+3.$$
$$= k^2+4k+4 = (k+2)^2.$$

$\Rightarrow P(k+1)$ is true.

$\begin{cases} P(0) \text{ is true.} \\ P(k) \rightarrow P(k+1)\ \forall k \geq 0 \end{cases} \Rightarrow \forall n \geq 0\ P(n)$ true.

4) $P(n): u_n = 5^n + 1,\ \forall n \geq 0$.

B.S:

$P(0): u_0 = 5^0+1 = 1+1 = 2$ true.

$P(1): u_1 = 5^1+1 = 6$ - true.

I.S: Let $k \geq 1$, we assume that $P(0), P(1), \dots,$ and $P(k)$ are true

and we prove that $P(k+1)$ is true.

$P(k+1): u_{k+1} = 5^{k+1}+1$?

$u_{k+1} = 3u_k + 10u_{k-1} - 12,$

$P(k)$ true $\Rightarrow u_k = 5^k+1$

$P(k-1)$ true $\Rightarrow u_{k-1} = 5^{k-1}+1$.

$$\Rightarrow u_{k+1} = 3(5^k+1) + 10(5^{k-1}+1) - 12.$$
$$= 3\times5^k+3 + 2\times5\times5^{k-1}+10-12$$
$$= 3\times5^k + 2\times5^k+1$$
$$= 5^k(5)+1 = 5^{k+1}+1.$$

$\Rightarrow P(k+1)$ is true.

$\Rightarrow \forall n \geq 0,\ P(n)$ is true.

Scanné avec CamScanner

Exercice 3 :

(i) $C \cap D = \phi$.

$C \times (C \cap D) = \phi$.

(ii) $C \times D = \{(a,x), (a,y), (a,z), (b,x), (b,y), (b,z), (c,x), (c,y), (c,z), (d,x), (d,y), (d,z)\}$.

$(C \times D) \setminus E = \{(a,y), (a,z), (b,x), (b,y), (c,x), (c,y), (c,z), (d,x), (d,y)\}$.

(iii) $\{\phi\} \times E = \{(\phi,(a,x)), (\phi,(b,z)), (\phi,(c,2)), (\phi,(2,4)), (\phi,(d,z)), (\phi,(z,d)), (\phi,\phi)\}$.

(iv) $E \times \phi = \phi$.

Scanné avec CamScanner