Analysis II

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2023

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DEPARTMENT OF MATHEMATICS

Real Analysis II

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Preface

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1 Definition of The Riemann Integral

Definition 1.1.

1. A finite ordered set $\sigma = \{x_0, \ldots, x_n\}$ is called a partition of the interval $[a, b]$ if $a = x_0 < \ldots < x_n = b$. The interval $[x_j, x_{j+1}]$ is called the j^{th} subinterval of σ.

2. If $\sigma = \{x_0, \ldots, x_n\}$ is a partition of the interval $[a, b]$, we define the norm of σ by:
$$||\sigma|| = \sup_{0 \le j \le n-1} x_{j+1} - x_j.$$

3. A partition $\sigma_n = (x_0, \ldots, x_n)$ of the interval $[a, b]$ is called uniform if $(x_k = a + k\dfrac{b-a}{n})$. In this case $\|\sigma\| = \dfrac{b-a}{n}$.

4. A partition $\sigma_1 = \{x_0, \ldots, x_n\}$ is called finer than a partition $\sigma_2 = \{y_0, \ldots, y_m\}$ if $\{y_0, \ldots, y_m\} \subset \{x_0, \ldots, x_n\}$ and we denote $\sigma_2 < \sigma_1$.

5. If $\sigma_1 = \{x_0, \ldots, x_n\}$ and $\sigma_2 = \{y_0, \ldots, y_m\}$ are two partitions of the interval $[a, b]$, we define the partition $\sigma_1 \cup \sigma_2$ defined by ordering the points $\{y_0, \ldots, y_m, x_0, \ldots, x_n\}$.

Definition 1.2.
Let $f\colon [a, b] \longrightarrow \mathbb{R}$ be a bounded function. Define

$$M_j = \sup_{x \in [x_j, x_{j+1}]} f(x), \qquad m_j = \inf_{x \in [x_j, x_{j+1}]} f(x),$$

$$U(f, \sigma) = \sum_{j=0}^{n-1} M_j(x_{j+1} - x_j), \quad L(f, \sigma) = \sum_{j=0}^{n-1} m_j(x_{j+1} - x_j) \tag{1.1}$$

The sums $U(f, \sigma)$ and $L(f, \sigma)$ are called respectively the upper and the lower sums of f on the partition σ. (Note that $L(f, \sigma) \le U(f, \sigma)$.)

Lemma 1.3.
Let $\sigma_1 = \{x_0, \dots, x_n\}$ be a partition of the interval $[a,b]$, $\sigma_2 = \{a, y, b\}$ with $y \in]a,b[$ and $f\colon [a,b] \longrightarrow \mathbb{R}$ a bounded function, then

$$L(f,\sigma) \leq L(f,\sigma_1) \leq U(f,\sigma_1) \leq U(f,\sigma), \tag{1.2}$$

where $\sigma = \sigma_1 \cup \sigma_2$.

Proof .
The proof obvious if $y \in \sigma_1$. Suppose now that $y \in]x_j, x_{j+1}[$, we have
$L(f,\sigma_1) = \sum_{i=0}^{j-1}(x_{i+1}-x_i)m_i + (x_{j+1}-x_j)m_j + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)m_i$,
$U(f,\sigma_1) = \sum_{i=0}^{j-1}(x_{i+1}-x_i)M_i + (x_{j+1}-x_j)M_j + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)M_i$ and

$$\begin{aligned} L(f,\sigma) &= \sum_{i=0}^{j-1}(x_{i+1}-x_i)m_i + (y-x_j)\inf_{x\in]x_j,y[} f(x) \\ &+ (x_{j+1}-y)\inf_{x\in]y,x_{j+1}} f(x) + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)m_i. \end{aligned}$$

$$\begin{aligned} U(f,\sigma) &= \sum_{i=0}^{j-1}(x_{i+1}-x_i)M_i + (y-x_j)\sup_{x\in]x_j,y[} f(x) \\ &+ (x_{j+1}-y)\sup_{x\in]y,x_{j+1}} f(x) + \sum_{i=j+1}^{n-1}(x_{i+1}-x_i)M_i. \end{aligned}$$

But $m_j \leq \inf_{x\in]x_j,y[} f(x)$, $m_j \leq \inf_{x\in]y,x_{j+1}[} f(x)$, $M_j \geq \sup_{x\in]x_j,y[} f(x)$ and $M_j \geq \sup_{x\in]y,x_{j+1}[} f(x)$. This yields that $L(f,\sigma) \leq L(f,\sigma_1)$ and $U(f,\sigma_1) \leq U(f,\sigma)$. □

Corollary 1.4.
If σ_1 is finer than σ_2 and $f\colon [a,b] \longrightarrow \mathbb{R}$ is a bounded function, then

$$L(f,\sigma_2) \leq L(f,\sigma_1) \leq U(f,\sigma_1) \leq U(f,\sigma_2) \tag{1.3}$$

Proof .

□

Theorem 1.5.
If $f\colon [a,b] \longrightarrow \mathbb{R}$ is a bounded function and σ_1, σ_2 are two partitions of the interval $[a,b]$, then $L(f,\sigma_1) \leq U(f,\sigma_2)$.

Proof .
$L(f,\sigma_1) \le L(f,\sigma_1 \cup \sigma_2) \le U(f,\sigma_1 \cup \sigma_2) \le U(f,\sigma_2)$. □

Definition 1.6.
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function, $P([a,b])$ the set of partitions of $[a,b]$, then we define respectively the upper and the lower integral of f on the interval $[a,b]$ by:

$$U(f) = \inf_{\sigma \in P([a,b])} U(f,\sigma), \qquad L(f) = \sup_{\sigma \in P([a,b])} L(f,\sigma).$$

$U(f)$ and $L(f)$ are called respectively the upper and the lower Darboux sums of f on the interval $[a,b]$.

Definition 1.7.
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. The function f is called Riemann integrable on the interval $[a,b]$ if $U(f) = L(f)$.

If f is Riemann integrable on the interval $[a,b]$, we denote $\displaystyle\int_a^b f(x)dx = U(f) = L(f)$ and called the integral of f on the interval $[a,b]$.
The set of Riemann integrable functions on the interval $[a,b]$ is denoted by $\mathscr{R}$ $([a,b])$.

Remark 1 :
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. If there exists a partition σ of $[a,b]$ such that $U(f,\sigma) = L(f,\sigma)$, then f is Riemann integrable on $[a,b]$ and $\displaystyle\int_a^b f(x)dx = U(f,\sigma)$.
This is because $L(f,\sigma) \le U(f)$ and $L(f) \le U(f,\sigma)$.

Example 1.1 :

1. Any step function on an interval $[a,b]$ is Riemann integrable. Indeed let $\sigma = (x_0 = a, \ldots, x_n = b)$ be thepartition of $[a,b]$ associated to f. If $f(x) = c_j$ on $]x_j, x_{j+1}[$, then $M_j = m_j = c_j$ and $U(f,\sigma) = L(f,\sigma)$ and f is is Riemann integrable.

2. Let f be the caracteristic function of $\mathbb{Q} \cap [0,1]$. For any partition σ of $[0,1]$, $L(f,\sigma) = 0$ and $U(f,\sigma) = 1$. Then f is not Riemann integrable.

1.1 Criterions for the Riemann Integrability

Theorem 1.8. [Riemann's Criterion]
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent

1. f is Riemann-integrable.

2. $\forall \varepsilon > 0$; there exists a partition σ such that $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$.

Proof .
NC: If $U(f) = L(f)$, then $\forall\ \varepsilon > 0$, there exists a partition σ such that $0 \leq L(f) - L(f,\sigma) \leq \frac{\varepsilon}{2}$ and there exists a partition σ' such that $0 \leq U(f,\sigma') - U(f) \leq \frac{\varepsilon}{2}$. Then $0 \leq U(f,\sigma\cup\sigma') - U(f) \leq U(f,\sigma') - U(f) \leq \frac{\varepsilon}{2}$. Also $0 \leq L(f) - L(f,\sigma\cup\sigma') \leq L(f) - L(f,\sigma) \leq \frac{\varepsilon}{2}$. Then $U(f,\sigma\cup\sigma') - L(f,\sigma\cup\sigma') \leq \varepsilon$.
SC: $L(f,\sigma) \leq L(f) \leq U(f,\sigma)$ and $L(f,\sigma) \leq U(f) \leq U(f,\sigma)$, then $0 \leq U(f) - L(f) \leq U(f,\sigma) - L(f,\sigma) \leq \varepsilon$, for all $\varepsilon > 0$. Hence $U(f) = L(f)$. ☐

Theorem 1.9. [Darboux's Criterion]
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a bounded function. The following statements are equivalent

1. f is Riemann-integrable,
2. For all $\varepsilon > 0$; there exists $\delta > 0$ such that for all partition of the interval $[a,b]$ such that if $||\sigma|| \leq \delta$ then $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$.

Recall the notion of oscillation of a function on an interval.

Definition 1.10. [Oscillation of a function]
The Oscillation of a function $f\colon I \longrightarrow \mathbb{R}$ at a point $a \in I$ is defined by

$$w_a(f) = \lim_{r\to 0} \sup\{|f(y) - f(z)|;\ y,z \in]a-r, a+r[\cap I\}.$$

If f is bounded, the oscillation of f on the interval $[a,b]$ denoted by $O(f,[a,b])$ is defined by $\sup\limits_{x\in[a,b]} f(x) - \inf\limits_{x\in[a,b]} f(x)$.

Note that $w_a(f) \geq 0$ and f is continuous at a if and only if $w_a(f) = 0$. Moreover, if f is bounded then $w_a(f) \leq O(f,[a,b])$.

Proof .
The condition is obviously sufficient.
NC: Let f be a Riemann integrable function (we assume that f is not constant), so $\forall\ \varepsilon > 0$ there is a partition $\sigma = (x_0 = a, \ldots, x_n = b)$ such that $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$. We set $M = O(f,[a,b])$ the oscillation of f on the interval $[a,b]$, $\alpha_1 = \dfrac{\varepsilon}{nM}$, $\alpha_2 = \inf\limits_{0\leq j\leq n-1}(x_{j+1} - x_j)$ and $\alpha = \min(\alpha_1,\alpha_2)$. That is $\sigma' = (y_0 = a, \ldots, y_m = b)$ a partition of $[a,b]$ such that $|\sigma'| < \alpha$. There are at most n intervals $]y_{j-1}, y_j[$ which contain x_i. The others are contained in intervals $]x_{k-1}, x_k[$. We denote

$$\begin{aligned} M'_j &= \sup_{x\in]y_j,y_{j+1}[} f(x), \quad M_j = \sup_{x\in]x_j,x_{j+1}[} f(x), \\ m'_j &= \inf_{x\in]y_j,y_{j+1}[} f(x) \quad \text{et } m_j = \inf_{x\in]x_j,x_{j+1}[} f(x). \end{aligned}$$

$$\begin{aligned} U(f,\sigma') - L(f,\sigma') &= \sum_{]y_j,y_{j+1}[\subset]x_i,x_{i+1}[} (y_{j+1}-y_j)(M'_j - m'_j) \\ &+ \sum_{x_i\in]y_j,y_{j+1}[} (y_{j+1}-y_j)(M'_j - m'_j) \end{aligned}$$

It follows that

$$\begin{aligned} U(f,\sigma') - L(f,\sigma') &\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)(M_i - m_i) + n\alpha M \\ &= U(f,\sigma) - L(f,\sigma) + n\alpha M \leq 2\varepsilon. \end{aligned}$$

□

Proposition 1.11.

Let f be a Riemann integrable function and $I = \displaystyle\int_a^b f(x)dx$. Then $\forall\, \varepsilon > 0$ there exists $\alpha > 0$ such that for all partition σ of $[a,b]$ with $\|\sigma\| < \alpha$, $|U(f,\sigma) - I| \leq \varepsilon$ and $|L(f,\sigma) - I| \leq \varepsilon$.

Theorem 1.12.
Any monotone function on an interval $[a,b]$ is Riemann integrable.

Proof .
Suppose that f is increasing. Let $\sigma = (x_0 = a, \ldots, x_n = b)$ be a partition of $[a,b]$ and $\alpha = \|\sigma\| = \sup_{0\leq j\leq n-1}(x_{j+1}-x_j)$.
$U(f,\sigma) - L(f,\sigma) \leq \alpha[(M_0 - m_0) + \ldots + (M_{n-1} - m_{n-1})]$.
$M_j = \sup_{x\in]x_j,x_{j+1}[} f(x) \leq f(x_{j+1})$ and $m_j = \inf_{x\in]x_j,x_{j+1}[} f(x) \geq f(x_j)$. Then

$$U(f,\sigma) - L(f,\sigma) \leq \alpha \sum_{j=0}^{n-1}(f(x_{j+1}) - f(x_j)) \leq \alpha(f(b) - f(a)).$$

For $\varepsilon > 0$, we take a partition $\sigma = (x_0 = a, \ldots, x_n = b)$ of $[a,b]$ such that

$$(f(b) - f(a)) \sup_{0\leq j\leq n-1}(x_{j+1} - x_j) \leq \varepsilon.$$

We get: $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$. Then f is Riemann integrable. □

Theorem 1.13.
Any continuous function on an interval $[a,b]$ is Riemann-integrable.

Proof .
Let f be a continuous function on an interval $[a, b]$, then f is uniformly continuous. Hence $\forall\, \varepsilon > 0$, $\exists\, \alpha > 0$ such that $|f(x) - f(x')| < \frac{\varepsilon}{b-a}$ for all $|x - x'| < \alpha$. Let $\sigma = (x_0 = a, \ldots, x_m = b)$ be a partition of $[a, b]$ such that $\sup_{0\leq j\leq n-1}(x_{j+1} - x_j) < \alpha$. As f is continuous on $[a, b]$, there exists x'_j and x''_j in $[x_j, x_{j+1}]$ such that $M_j = f(x'_j)$ and $m_j = f(x''_j)$; $|x'_j - x''_j| \leq |x_{j+1} - x_j| < \alpha$, then $M_j - m_j \leq \frac{\varepsilon}{b-a}$. We deduce that

$$0 \leq U(f, \sigma) - L(f, \sigma) \leq \sum_{j=0}^{n-1}(x_{j+1} - x_j)(M_j - m_j) \leq \frac{\varepsilon}{b-a}\sum_{j=0}^{n-1}(x_{j+1} - x_j) = \varepsilon.$$

$\square$

Definition 1.14.
Let $\sigma = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$. We say that $\alpha = \{\alpha_0, \ldots, \alpha_{n-1}\}$ is a mark of σ if $\forall 0 \leq j \leq n-1$, $\alpha_j \in [x_j, x_{j+1}]$.
We define

$$U(f, \sigma, \alpha) = \sum_{j=0}^{n-1} f(\alpha_j)(x_{j+1} - x_j)$$

called the Riemann sum of f on the partition σ with respect to the mark α.

Remark 2 :

1. Let f be a Riemann integrable function on the interval $[a, b]$. If $\sigma = \{x_0, \ldots, x_n\}$ a partition of $[a, b]$ and $\tau = (\lambda_1, \ldots, \lambda_n)$ a mark on σ, then the sum $R(f, \sigma, \tau) = \sum_{j=0}^{n-1}(x_{j+1} - x_j)f(\lambda_j)$ verifies

$$U(f, \sigma) \leq R(f, \sigma, \tau) \leq L(f, \sigma).$$

   Then $\forall\, \varepsilon > 0$ $\exists \alpha > 0$ such that for all partition σ such that $\|\sigma\| < \alpha$ and for all $\tau = (\lambda_1, \ldots, \lambda_n)$ a mark on σ, we have: $|R(f, \sigma, \tau) - I| \leq \varepsilon$.

2. The same result is obtained if we replace $f(\lambda_j)$ by any constant μ_j, with $m_j \leq \mu_j \leq M_j$.

3. If f is Riemann integrable on the interval $[a, b]$, the sequence $(S_n)_n$ defined by:

$$S_n = \frac{b-a}{n}\sum_{k=1}^{n} f(a + k\frac{b-a}{n})$$

converges to $\displaystyle\int_a^b f(x)dx$.

Proposition 1.15.
A function f is Riemann integrable if and only if
$\forall\, \varepsilon > 0$, there are two step functions on $[a,b]$ f_ε and g_ε such that $f_\varepsilon \leq f \leq g_\varepsilon$ and $\displaystyle\int_a^b (g_\varepsilon - f_\varepsilon)\, dx \leq \varepsilon$.

Proof .

1. If f is Riemann integrable, then $\forall\, \varepsilon > 0$, there exists a partition σ of $[a,b]$ such that $U(f,\sigma) - L(f,\sigma) \leq \varepsilon$. We take $f_\varepsilon = m_i$ and $g_\varepsilon = M_i$ on $]x_i, x_{i+1}[$ and $f_\varepsilon(x_i) = g_\varepsilon(x_i) = f(x_i)$ for all $0 \leq i \leq n-1$.
2. Conversely: Let $\varepsilon > 0$ and σ a partition of $[a,b]$ associated to both f_ε and g_ε. $f_\varepsilon \leq f \leq g_\varepsilon$.

   $0 \leq U(f,\sigma) - L(f,\sigma) \leq U(g_\varepsilon,\sigma) - L(f_\varepsilon,\sigma) = \displaystyle\int_a^b (g_\varepsilon - f_\varepsilon)\, dx \leq \varepsilon$. So f is Riemann integrable.

□

1.2 Properties of the Riemann Integral

Properties 1.16.

1. Linearity: $\displaystyle\int_a^b \alpha(f+\beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$.
2. If $f \geq 0$, then $\displaystyle\int_a^b f(x)dx \geq 0$.
3. If $f \leq g$, then $\displaystyle\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
4. $\displaystyle\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.
5. If $m \leq f(x) \leq M$, for all $x \in [a,b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

Theorem 1.17.

A bounded function on an interval $[a, b]$ is Riemann-integrable if and only if it is Riemann-integrable on $[a, c]$ and on $[c, b]$, for all $c \in [a, b]$. Moreover if f is Riemann-integrable on $[a, b]$, then

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx. \tag{1.4}$$

(This identity is called the Chasles identity)

Proof .

Assume that f is Riemann-integrable on $[a, b]$, so $\forall\ \varepsilon > 0$, there exists a partition σ of $[a, b]$ such that $U(f, \sigma) - L(f, \sigma) \leq \varepsilon$. Let $\sigma' = \sigma \cup \{c\}$; then $U(f, \sigma') - L(f, \sigma') \leq U(f, \sigma) - L(f, \sigma) \leq \varepsilon$. Consider $\sigma' = \sigma_1 \cup \sigma_2$, with σ_1 a partition of $[a, c]$ formed from the points of σ' in $[a, c]$ and σ_2 a partition of $[c, b]$ formed from the points of σ' in $[c, b]$. It follows that $U(f, \sigma_1) - L(f, \sigma_1) \leq \varepsilon$ and $U(f, \sigma_2) - L(f, \sigma_2) \leq \varepsilon$. So f is separately Riemann-integrable over $[a, c]$ and $[c, b]$.

If f is separately Riemann-integrable over $[a, c]$ and $[c, b]$, so $\forall\ \varepsilon > 0$, there is a partition σ_1 of $[a, c]$ and a partition σ_2 of $[c, b]$ such that $U(f, \sigma_1) - L(f, \sigma_1) \leq \varepsilon$ and $U(f, \sigma_2) - L(f, \sigma_2) \leq \varepsilon$. The set $\sigma = \sigma_1 \cup \sigma_2$ is a partition of $[a, b]$ and $U(f, \sigma) - L(f, \sigma) \leq 2\varepsilon$, which proves that f is Riemann-integrable on $[a, b]$.

Consider for a Riemann-integrable function f on $[a, b]$ the numbers: $I = \int_a^b f(x)\ dx$, $I_1 = \int_a^c f(x)\ dx$ and $I_2 = \int_c^b f(x)\ dx$.

$\forall\ \varepsilon > 0$, there exists α >0 such that for any partitions σ of $[a, b]$, σ_1 of $[a, c]$ and σ_2 of $[c, b]$, with ($\|\sigma\| < \alpha$, $\|\sigma_1\| < \alpha$ and $\|\sigma_2\| < \alpha$ we have: $|U(f, \sigma) - I| \leq \varepsilon$, $|U(f, \sigma_1) - I_1| \leq \varepsilon$ and $|U(f, \sigma_2) - I_2| \leq \varepsilon$. We consider the partition $\sigma' = \sigma_1 \cup \sigma_2$, $\|\sigma'\| < \alpha$, $|U(f, \sigma') - I| \leq \varepsilon$; similarly $|U(f, \sigma') - I_1 - I_2| \leq |U(f, S_1) - I_1| + |U(f, S_2) - I_2| \leq 2\varepsilon$. So $I = I_1 + I_2$.

$\square$

Remark 3 :

By convention if $b < a$, we set $\int_b^a f(x)\ dx = -\int_a^b f(x)\ dx$.

Exercise 1.1 :

Compute the following integrals:

1. $F(x) = \int_0^\pi |x - t| \sin t dt$ for $x \in \mathbb{R}$.

2. $F(x) = \int_0^\pi |x - t| \sin t dt$ for $x \in \mathbb{R}$.

Solution

1. I $x \leq 0$, $F(x) = \displaystyle\int_0^{\pi} (t-x)\sin t dt = \pi - 2x$.

   If $0 \leq x \leq \pi$, then $F(x) = \displaystyle\int_0^{x} (x-t)\sin t dt + \int_x^{\pi} (t-x)\sin t dt = \pi - 2\sin x$.

   If $x \geq \pi$, then $F(x) = \displaystyle\int_0^{\pi} (x-t)\sin t dt = 2x - \pi$.

2. If $x \leq 0$, then $F(x) = \displaystyle\int_0^{\pi} (t-x)\sin t dt = \pi - 2x$.

   If $0 \leq x \leq \pi$, then $F(x) = \displaystyle\int_0^{x} (x-t)\sin t dt + \int_x^{\pi} (t-x)\sin t dt = \pi - 2\sin x$.

   If $x \geq \pi$, then $F(x) = \displaystyle\int_0^{\pi} (x-t)\sin t dt = 2x - \pi$.

Definition 1.18.
A function f defined on an interval $[a, b]$ is said to be piecewise continuous if there is a partition $\sigma = (x_0 = a, \ldots, x_n = b)$ of $[a, b]$ such that f is continuous on each open interval $]x_i, x_{i+1}[$ and f admits a right limit of x_i for all $0 \leq i \leq n-1$ and a left limit of x_{i+1} for all $1 \leq i \leq n$.

Exercise 1.1 :
Show that any piecewise continues function on an interval $[a, b]$ is Riemann integrable.

Theorem 1.19.
The space of Riemann-integrable functions on $[a, b]$ is a vector space on $\mathbb{R}$.

Theorem 1.20.
If f is Riemann-integrable on an interval $[a, b]$, then $|f|$ is too.

Proof .
Let $[c, d] \subset [a, b]$.
• If f is non negative on $[c, d]$, then $\sup\limits_{[c,d]} |f| = \sup\limits_{[c,d]} f$ and $\inf\limits_{[c,d]} |f| = \inf\limits_{[c,d]} f$.
• If f is non positive on $[c, d]$, then $\sup\limits_{[c,d]} |f| = -\inf\limits_{[c,d]} f$ and $\inf\limits_{[c,d]} |f| = -\sup\limits_{[c,d]} f$.
• If f has no constant sign on $[c, d]$, then $\sup\limits_{[c,d]} f \geq 0$ and $\inf_{[c,d]} f \leq 0$.
It follows that $\sup\limits_{[c,d]} |f| = \max(\sup\limits_{[c,d]} f, -\inf\limits_{[c,d]} f)$. We deduce that in all cases $\sup\limits_{[c,d]} |f| - \inf\limits_{[c,d]} |f| \leq \sup\limits_{[c,d]} f - \inf\limits_{[c,d]} f$, which gives that $U(|f|, \sigma) - L(|f|, \sigma) \leq U(f, \sigma) - L(f, \sigma)$, for any partition σ of $[a, b]$. It results that $|f|$ is Riemann-integrable. $\square$

Proposition 1.21.
If two functions f and g are Riemann-integrable on a interval $[a, b]$, then $\sup(f, g)$ and $\inf(f, g)$ are Riemann-integrable.

Proof .
$\sup(f,g) = \frac{1}{2}(f+g+|f-g|)$ and $\inf(f,g) = \frac{1}{2}(f+g-|f-g|)$. □

Theorem 1.22.
The product of two Riemann-integrable functions is a Riemann-integrable function.

Proof .
It suffices to prove the result for two non negative functions. Let f and g be two non negative Riemann-integrable functions on $[a,b]$. Let M be an upper bound of f and g over $[a,b]$. For any a partition σ of $[a,b]$, $U(fg,\sigma) - L(fg,\sigma) \leq M(U(f,\sigma) - L(f,\sigma)) + M(U(g,\sigma) - L(g,\sigma))$. It follows that $f.g$ is Riemann-integrable. □

Theorem 1.23.
Let f be a non negative Riemann-integrable function on $[a,b]$. Then for all $\alpha > 0$, the function $f^\alpha(x)$ is Riemann-integrable.

Proof .
Let $\varepsilon > 0$, there is a partition $\sigma = (x_0 = a, x_1, \dots, x_n = b)$ such that:

$$\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i - m_i) < \varepsilon,$$

with

$$M_i = \sup_{x\in]x_i,x_{i+1}[} f(x) \quad \text{et } m_i = \inf_{x\in]x_i,x_{i+1}[} f(x).$$

Note that $\forall\ t \in [0,1]$; $1 - t^\alpha \leq (1-t)\sup(1,\alpha)$, which gives that

$$M_i^\alpha - m_i^\alpha \leq (M_i - m_i)M_i^{\alpha-1}\sup(\alpha,1).$$

If $\alpha > 1$: $M_i^{\alpha-1} \leq M^{\alpha-1}$, with $M = \sup\limits_{x\in[a,b]} f(x)$. In this case, we have:

$$\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i^\alpha - m_i^\alpha) < \alpha\varepsilon M^{\alpha-1},$$

which gives the result in this case.
If $\alpha < 1$ and if $M_i \leq \varepsilon$ we have: $M_i^\alpha - m_i^\alpha \leq \varepsilon^\alpha$ and if $M_i > \varepsilon$ we have: $M_i^{\alpha-1} < \varepsilon^{\alpha-1}$ which yields

$$\begin{aligned}
\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i^\alpha - m_i^\alpha) &\leq \sum_{i=0}^{n-1}(x_{i+1} - x_i)\varepsilon^\alpha + \sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i - m_i)\varepsilon^{\alpha-1} \\
&= (b-a)\varepsilon^\alpha + \varepsilon^{\alpha-1}\sum_{i=0}^{n-1}(x_{i+1} - x_i)(M_i - m_i) \leq \varepsilon^\alpha(b-a+1).
\end{aligned}$$

In general, we have the following theorem:

Theorem 1.24.
Let $f\colon [a,b] \longrightarrow [c,d]$ be a Riemann integrable function and $\varphi\colon [c,d] \longrightarrow \mathbb{R}$ a continuous function. Then $\varphi \circ f$ is Riemann integrable.

Proof .
Let $\varepsilon > 0$, we will construct a partition $\sigma = (x_0 = a, x_1, \dots, x_n = b)$ of $[a,b]$ such that: $U(\varphi \circ f, \sigma) - L(\varphi \circ f, \sigma) < \varepsilon$.
The function φ is uniformly continuous on $[c,d]$ and bounded, then there is $M > 0$ such that $|\varphi(x)| \leq M$, $\forall x \in [c,d]$ and if $\varepsilon' = \dfrac{\varepsilon}{2M + (b-a)}$, there is $0 < \alpha < \varepsilon'$ such that for $|x-y| < \alpha$, $|\varphi(x) - \varphi(y)| \leq \varepsilon'$, for all $x, y \in [c,d]$.
Since f is Riemann-integrable on $[a,b]$, there exist a partition $\sigma = (x_0 = a, x_1, \dots, x_n = b)$ of $[a,b]$ such that:

$$U(f,\sigma) - L(f,\sigma) < \alpha^2. \tag{1.5}$$

Let $M_j = \sup\{f(x);\ x \in [x_j, x_{j+1}]\}$, $m_j = \inf\{f(x);\ x \in [x_j, x_{j+1}]\}$, $\tilde{M}_j = \sup\{\varphi \circ f(x);\ x \in [x_j, x_{j+1}]\}$, $\tilde{m}_j = \inf\{\varphi \circ f(x);\ x \in [x_j, x_{j+1}]\}$.
We denote $J_1 = \{0 \leq j \leq n-1;\ M_j - m_j < \alpha$ et $J_2 = \{0 \leq j \leq n-1;\ M_j - m_j \geq \alpha$.
If $j \in J_1$, then by the uniform continuity of $\varphi \circ f$, we have $|\varphi \circ f(x) - \varphi \circ f(y)| < \varepsilon'$ for all $x, y \in [x_j, x_{j+1}]$, which yields $\tilde{M}_j - \tilde{m}_j \leq \varepsilon'$, then

$$\sum_{j \in J_1} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq \varepsilon'(b-a). \tag{1.6}$$

By (6.1),

$$\alpha^2 > \sum_{j \in J_2} (M_j - m_j)(x_{j+1} - x_j) \geq \alpha \sum_{j \in J_2} (x_{j+1} - x_j).$$

Then $\displaystyle\sum_{j \in J_2} (x_{j+1} - x_j) < \alpha < \varepsilon'$ and since $\tilde{M}_j - \tilde{m}_j \leq 2M$, we have:

$$\sum_{j \in J_2} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq 2M \sum_{j \in J_2} (x_{j+1} - x_j) < 2M\varepsilon'. \tag{1.7}$$

It results by (6.2) and (6.5) that

$$U(\varphi \circ f, \sigma) - L(\varphi \circ f, \sigma) = \sum_{j=0}^{n-1} (\tilde{M}_j - \tilde{m}_j)(x_{j+1} - x_j) \leq \varepsilon'((b-a) + 2M) = \varepsilon.$$

$\square$

Remark 4 :

1. The integral of a non negative Riemann-integrable function is a non negative real number.

2. If f is Riemann-integrable on $[a, b]$, then

$$|\int_a^b f(x)\ dx| \leq \int_a^b |f(x)|\ dx \leq (b-a) \sup_{x\in[a,b]} |f(x)|.$$

Corollary 1.25.
If f is Riemann-integrable on $[a, b]$, then the function $F(x) = \displaystyle\int_a^x f(t)\ dt$ is continuous on $[a, b]$.

Proof .
$F(x) - F(y) = \displaystyle\int_y^x f(t)\ dt$. Since f is bounded on $[a, b]$, there exist $M > 0$ such that $|F(x) - F(y)| \leq M|x - y|$. □

Corollary 1.26.
Let f be a Riemann-integrable function on $[a, b]$. If $m = \inf_{x\in[a,b]} f(x)$ and $M = \sup_{x\in[a,b]} f(x)$, there exist $\lambda \in [m, M]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\ dx = \lambda.$$

Proof .
We have: $m(b-a) \leq \displaystyle\int_a^b f(x)\ dx \leq M(b-a)$, then $\displaystyle\frac{1}{b-a}\int_a^b f(x)\ dx \in [m, M]$.

Corollary 1.27. [First Mean Value Formula]
Let f and g be two Riemann-integrable functions on an interval $[a, b]$. Assume that f is continuous and g has a constant sign on $[a, b]$. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)\ dx.$$

Proof .
If $\displaystyle\int_a^b g(x)\ dx = 0$, then $\displaystyle\int_a^b f(x)g(x)\ dx = 0$.
If $\displaystyle\int_a^b g(x)\ dx \neq 0$, we set $g_1 = \displaystyle\frac{1}{\int_a^b g(x)\ dx} g$, then $\displaystyle\int_a^b g_1(x)\ dx = 1$ and if $m = \displaystyle\inf_{x\in[a,b]} f(x)$ and $M = \displaystyle\sup_{x\in[a,b]} f(x)$, there is $\lambda \in [m, M]$ such that $\displaystyle\int_a^b f(x)g_1(x)dx = \lambda.$

$\square$

Theorem 1.28. [The Fundamental Theorem of Calculus]
Let $f\colon [a,b] \longrightarrow [c,d]$ be continuous function, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is differentiable and $F'(x) = f(x)$.

Proof .
For $x \in [a,b]$ and $h \in \mathbb{R}^*$ such that $x+h \in [a,b]$.

$$\begin{aligned} \frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\Big(\int_a^{x+h} f(t)\ dt - \int_a^x f(t)\ dt\Big) \\ &= \frac{1}{h}\int_x^{x+h} f(t)\ dt = f(c) \end{aligned}$$

where $c \in [x, x+h]$ or $c \in [x+h, x]$. Since f is continuous, $\lim\limits_{h\to 0} f(c) = f(x)$. Then $F'(x) = f(x)$. $\square$

Corollary 1.29.
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a differentiable function and f' is Riemann integrable, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Theorem 1.30. [The Cauchy-Schwarz Inequality]
Let f and g be two Riemann-integrable functions on an interval $[a,b]$, then

$$\Big(\int_a^b f(x)g(x)\ dx\Big)^2 \leq \int_a^b f^2(x)\ dx \int_a^b g^2(x)\ dx.$$

Proof .
Let λ be a real number.

$$P(\lambda) = \int_a^b (f(x)+\lambda g(x))^2\ dx = \lambda^2 \int_a^b g^2(x)\ dx + 2\lambda \int_a^b f(x)g(x)\ dx + \int_a^b f(x)^2\ dx$$

If $\int_a^b g^2(x)\ dx > 0$, $P(\lambda)$ is a non negative polynomial. It follows that its discriminant is non positive, which gives the desired inequality.
If $\int_a^b g^2(x)\ dx = 0$, $P(\lambda) \geq 0$, then $\int_a^b (fg)(x)\ dx = 0$ and the inequality holds.
$\square$

Corollary 1.31. [Minkowsky Inequality]
Let f and g be two Riemann-integrable functions on an interval $[a,b]$, then $\left(\int_a^b (f(x)+g(x))^2\,dx\right)^{\frac{1}{2}} \leq \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} + \left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$.

Proof .
$\int_a^b (f(x)+g(x))^2\,dx = \int_a^b f^2(x)\,dx + \int_a^b g^2(x)\,dx + 2\int_a^b f(x)g(x)\,dx$. By the Cauchy-Schwarz inequality we have

$$\left(\int_a^b (f(x)+g(x))^2\,dx\right)^{\frac{1}{2}} \leq \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} + \left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}.$$

Remark 5 :
If f is a non negative Riemann-integrable function and $\int_a^b f(x)\,dx = 0$, then $\int_a^b f(x)g(x)\,dx = 0$ for all Riemann-integrable function g. In particular $\int_a^b f^{\alpha}(x)\,dx = 0$, $\forall \alpha > 0$.

Theorem 1.32. [Hölder Inequality for Integrals]
Let f and g be two non negative Riemann-integrale functions on an interval $[a,b]$. Then for all conjugate positive numbers p, q, $(\frac{1}{p}+\frac{1}{q}=1)$ we have:

$$\int_a^b f(x)g(x)\,dx \leq \Big(\int_a^b f^p(x)\,dx\Big)^{\frac{1}{p}} \Big(\int_a^b g^q(x)\,dx\Big)^{\frac{1}{q}}.$$

Proof .
If $\int_a^b f^p(x)\,dx = 0$ or $\int_a^b g^q(x)\,dx = 0$, the result is trivial.
If $\int_a^b f^p(x)\,dx \neq 0$ and $\int_a^b g^q(x)\,dx \neq 0$, we set $f_1(x) = \dfrac{f(x)}{\left(\int_a^b f^p(t)\,dt\right)^{1/p}}$ and $g_1(x) = \dfrac{g(x)}{\left(\int_a^b g^q(t)\,dt\right)^{1/q}}$, we get $\int_a^b f_1^p(x)\,dx = \int_a^b g_1^q(x)\,dx = 1$. From the convexity of the function $t \longmapsto t^p$ on $]0,+\infty[$, for $p>1$, we get $f_1^{\frac{1}{p}} g_1^{\frac{1}{q}} \leq \dfrac{1}{p} f_1 + \dfrac{1}{q} g_1$. We deduce the desired result.

$$\int_a^b f(x)g(x)\,dx \leq \left(\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)\,dx\right)^{\frac{1}{q}}.$$

$\square$

Theorem 1.33. [Second Mean Value Formula]
Let f be a decreasing non negative continuous function on the interval $[a,b]$ and let g be a Riemann-integrable function on $[a,b]$. Then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(a)\int_a^c g(x)\ dx.$$

Proof .
Consider the function $G(x) = \displaystyle\int_a^x g(t)\ dt$. G is continuous on $[a,b]$. Let $m = \inf_{x\in[a,b]} G(x)$ and $M = \sup_{x\in[a,b]} G(x)$. To prove the theorem it suffices to prove that $mf(a) \leq \displaystyle\int_a^b f(x)g(x)\ dx \leq Mf(a)$. Let $\sigma_n = (x_0 = a, \ldots, x_n)$ be the uniform partition of $[a,b]$ i.e. $x_{i+1} - x_i = \dfrac{b-a}{n}$, $x_j = a + j\frac{b-a}{n}$. We set $\lambda_i = \dfrac{G(x_{i+1}) - G(x_i)}{x_{i+1} - x_i}$.

$$\lim_{n\to+\infty} \sum_{i=0}^{n-1} (x_{i+1} - x_i)(fg)(x_i) = \int_a^b f(x)g(x)\ dx.$$

$$\begin{aligned}
|\sum_{i=0}^{n-1}(x_{i+1}-x_i)f(x_i)(g(x_i)-\lambda_i)| &\leq f(a)\sum_{i=0}^{n-1}(x_{i+1}-x_i)(M_i - m_i)\\
&= f(a)(U(g,\sigma_n) - L(g,\sigma_n) \underset{n\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

with $M_i = \sup\limits_{t\in]x_i,x_{i+1}[} g(t)$ and $m_i = \inf\limits_{t\in]x_i,x_{i+1}[} g(t)$. It results that

$$\lim_{n\to+\infty} \sum_{i=0}^{n-1} f(x_i)(G(x_{i+1}) - G(x_i)) = \int_a^b f(x)g(x)\ dx.$$

$$\begin{aligned}
\sum_{i=0}^{n-1} f(x_i)(G(x_{i+1}) - G(x_i)) &= \sum_{i=0}^{n-1} f(x_i)G(x_{i+1}) - \sum_{i=0}^{n-1} f(x_i)G(x_i)\\
&= \sum_{i=0}^{n-1}(f(x_{i-1}) - f(x_i))G(x_i) + f(x_{n-1})G(b).
\end{aligned}$$

Since f is decreasing and non negative, we deduce

$$
\begin{aligned}
m\big[f(x_{n-1}) + \sum_{i=0}^{n-1}(f(x_{i-1}) - f(x_i))\big] &\leq \sum_{i=0}^{n-1} f(x_i)(G(x_{i+1}) - G(x_i)) \\
&\leq M\big[f(x_{n-1}) + \sum_{i=0}^{n-1}(f(x_{i-1}) - f(x_i))\big].
\end{aligned}
$$

Then

$$
mf(a) \leq \int_a^b f(x)g(x)\ dx \leq Mf(a).
$$

Corollary 1.34.
Let f be a monotone continuous function on an interval $[a, b]$ and let g be a Riemann-integrable function, then there exist $c \in [a, b]$ such that

$$
\int_a^b f(x)g(x)\ dx = f(a)\int_a^c g(x)\ dx + f(b)\int_c^b g(x)\ dx.
$$

Proof .
We can assume that f is increasing. We use the previous theorem to the functions $h(x) = f(b) - f(x)$ and g. □

Theorem 1.35.
Let f be a Riemann-integrable function on the interval $[a, b]$.

1. If $\lim\limits_{x \to t, (x>t)} f(x) = s$ exists, then the function $F(x) = \displaystyle\int_a^x f(t)\ dt$ is differentiable at the right of t and $F'(t+) = s$.

2. If $\lim\limits_{x \to t, (x<t)} f(x) = s$ exists, then the function F is differentiable at the left of t and $F'(t-) = s$.

Proof .

1. For $\varepsilon > 0$, there exists $\alpha > 0$ such that $|f(x) - s| \leq \varepsilon$, for all $x \in]t, t + \alpha[$. If $u \in [t, t + \alpha]$, then $|\displaystyle\int_t^u (f(x) - s)\ dx| \leq \varepsilon(u - t)$ and $|F(u) - F(t) - s(u - t)| \leq \varepsilon(u - t)$. Hence $\frac{F(u)-F(t)}{u-t} - s| \leq \varepsilon$.

2. With the same arguments we get the result.

□

Theorem 1.36.
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a continuous function and $u\colon I \longrightarrow [a,b]$ a differentiable function. Then the function $F(x) = \displaystyle\int_a^{u(x)} f(t)\, dt$ is differentiable on I and $F'(x) = u'(x)f(u(x))$, for all $x \in I$.

Proof .
Let G be an antiderivative of f such that $G(a) =$. Then $G \circ u = F$ and

$$F'(x) = (G \circ u)'(x) = G'(u(x)).u'(x) = f(u(x))u'(x).$$

$\square$

Theorem 1.37. [Integral by Substitution]
If g is continuously differentiable (C^1) on $[a,b]$, and if f is continuous on $g([a,b]$. Then

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_a^b f \circ g(t) g'(t)\, dt.$$

Proof .
Let $F(t) = \displaystyle\int_{g(a)}^{g(t)} f(x)\, dx$, $G(t) = \displaystyle\int_a^t f \circ g(x) g'(x)\, dx$, $G'(t) = g'(t) f \circ g(t)$, $F(a) = G(a) = 0$ and $F'(t) = g'(t) f \circ g(t)$. Then $F' = G'$ on the interval $[a,b]$ and $F = G$ on the interval $[a,b]$. $\square$

Example 1.2 :

1. If the function f is even, then $\displaystyle\int_{-a}^{a} f(t)\, dt = 2\int_0^a f(t)\, dt$ and if f is odd, then $\displaystyle\int_{-a}^{a} f(t)\, dt = 0$

2. If the function f is $T-$periodic, with $T > 0$ on $\mathbb{R}$. Then $\displaystyle\int_a^{a+T} f(t)\, dt = \int_0^T f(t)\, dt$, for all $a \in \mathbb{R}$.
   $\displaystyle\int_a^{a+T} f(t)\, dt = \int_a^0 f(t)\, dt + \int_0^T f(t)\, dt + \int_T^{a+T} f(t)\, dt$. Then $\displaystyle\int_T^{a+T} f(t)\, dt = \int_0^a f(t)\, dt$, (substitution $t = T + x$).

Theorem 1.38. [Integration by Parts]
Let f and g be two continuously differentiable functions (C^1) on an interval I, then

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int f'(x)g(x)\, dx.$$

Moreover if $[a,b] \subset I$, then

$$\int_a^b f(x)g'(x)\ dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\ dx.$$

Example 1.3 :

$$\int_0^1 x\tan^{-1} x dx = \frac{x^2}{2}\tan^{-1} x\Big]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2} dx = \frac{\pi}{4} - \frac{1}{2}.$$

Theorem 1.39.
Let f and g be two functions of class C^n on an interval I, then

$$\int f(x)g^{(n)}(x)dx = \sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-1-p)}(x) + (-1)^n \int g(x)f^{(n)}(x)dx.$$

Proof .

$$\begin{aligned}
\Big(\sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-1-p)}(x)\Big)' &= \sum_{p=0}^{n-1}(-1)^p f^{(p+1)}(x)g^{(n-1-p)}(x) \\
&\quad + \sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-p)}(x) \\
&= \sum_{p=0}^{n-1}(-1)^p f^{(p)}(x)g^{(n-p)}(x) \\
&\quad - \sum_{p=1}^{n}(-1)^p f^{(p)}(x)g^{(n-p)}(x) \\
&= f(x)g^{(n)}(x) - (-1)^n g(x)f^{(n)}(x).
\end{aligned}$$

□

Theorem 1.40. [Taylor Formula with integral Reminder]
Let f be function of class $\mathcal{C}^{n+1}$ defined on an interval I in $\mathbb{R}$. For a and x in I, we have:

$$f(x) = f(a) + \sum_{k=1}^{n}\frac{(x-a)^k}{k!}f^{(k)}(a) + \int_a^x \frac{(x-t)^n}{(n)!}f^{(n+1)}(t)dt.$$

Proof .
We apply the theorem (1.39) to the function f and the function $g(t) = \dfrac{(x-t)^{n-1}}{(n-1)!}$.
□

1.3 The Lebesgue Theorem

Definition 1.41.
A subset $E \subset \mathbb{R}$ is said to be a null set (or a set of zero measure or a negligible set or zero set) if for any $\varepsilon > 0$ there is a countable number of open intervals $(]a_n, b_n[)_n$ such that $\sum_{n=1}^{+\infty}(b_n - a_n[< \varepsilon$ and $E \subset \cup_{n=1}^{+\infty}]a_n, b_n)$.

Theorem 1.42. [Lebesgue's Theorem on Riemann Integrable Functions]
A bounded function $f\colon [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable if and only if the set of discontinuity points of f is a null set.

Proof .
Let $D = \{x \in [a,b] : f \text{ is discontinuous at } x\}$. We have

$$D = \{x \in [a,b];\ w_x(f) > 0\} = \bigcup_{n=1}^{+\infty}\{x \in [a,b];\ w_x(f) \geq \frac{1}{n}\}.$$

Let $D_n = \{x \in [a,b];\ w_x(f) \geq \frac{1}{n}\}$. Note that D is a null set if and only if each D_n is a null set.
Now assume that f be Riemann integrable on $[a,b]$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary. Since f is Riemann integrable, there exists a partition $\sigma = (x_0, x_1, \ldots, x_n)$ of $[a,b]$ such that

$$U(f,\sigma) - L(f,\sigma) = \sum_{k=1}^{n}(M_k - m_k)(x_k - x_{k-1}) < \varepsilon,$$

Let $J_k = \{j :]x_{j-1}, x_j[\cap D_k \neq \emptyset\}$.
If $J_k = \emptyset$, then $D_k \subset \{x_0, x_1, \cdots, x_n\}$, hence D_k is finite and then it is a null set. Otherwise, for each $j \in J_k$, there exists $t \in D_k \cap]x_{j-1}, x_j[$ and hence $\frac{1}{k} \leq w_t(f) \leq M_j - m_j$. Thus we have

$$\sum_{j\in J_k}\frac{1}{k}(x_j - x_{j-1}) \leq \sum_{j\in J_k}(M_j - m_j)(x_j - x_{j-1}) < \varepsilon$$

and hence $\sum_{j\in J_k}(x_j - x_{j-1}) < k\varepsilon$. Then $D_k \setminus \sigma \subset \bigcup_{j\in J_k}]x_{j-1}, x_j[$, where $\sum_{j\in J_k}(x_j - x_{j-1}) < k\varepsilon$. Since ε is arbitrary, $D_k \setminus \sigma$ is a null set. Thus $D_k \subset (D_k \setminus \sigma) \cup \sigma$ is a null set.
Conversely if D is a null set, to show f is Riemann integrable, we take an arbitrary $\varepsilon > 0$. Since D is a null set, there is a countable family of open intervals $(I_j =]a_j, b_j[)_j$ such that $\sum_{j=1}^{+\infty}(b_j - a_j) < \varepsilon$ and $D \subset \cup_{j=1}^{+\infty} I_j$. For all

$x \in [a,b] \setminus D$, $w_x(f) = 0$ and hence by definition there exists an open interval J_x containing x such that $\sup\{|f(y) - f(z)|;\ y, z \in J_x \cap [a,b]\} < \varepsilon$.
The set $\mathcal{F} = \{I_j;\ j \in \mathbb{N}\} \cup \{J_x;\ x \in [a,b] \setminus D\}$ is an open cover of the compact set $[a,b]$. So $\mathcal{F}$ has a finite subcover $\mathcal{F}' = \{I_j;\ j = 1, \ldots, m\} \cup \{J_{x_j};\ j = 1, \ldots, p\}$.
Let $\sigma = \{t_0, t_1, \ldots, t_n\}$ with $a = t_0 < t_1 < \cdots < t_n = b$ be the partition of $[a,b]$ determined by those endpoints of $(I_j)_{1 \le j \le m}$ and $(J_{x_j})_{1 \le j \le p}$ which are inside $[a,b]$. Also let $M_j = \sup_{t \in [t_{j-1},\ t_j]} f(t)$, $m_j = \inf_{t \in [t_{j-1},\ t_j]} f(t)$ and $\delta_j = t_j - t_{j-1}$, $j = 1, 2, \ldots, n$ and $|f(x)| \le M$.
Then for each $j \in \{1, 2, \ldots, n\}$ the interval $]t_{j-1}, t_j[$ is contained in some I_k, $1 \le k \le m$ or some J_{x_k}, $1 \le k \le p$ and let $\mathcal{J} = \{j;\ ]t_{j-1}, t_j) \subset I_k$ for some $k = 1 \le m\}$.
Note that if $j \notin \mathcal{J}$ then $]t_{j-1}, t_j[\subseteq J_{x_k}$ for some $k = 1, 2, \ldots, p$ and $M_k - m_k \le \sup\{|f(t) - f(s)|;\ t, s \in J_{x_k} \cap [a,b]\} < \varepsilon$. Then

$$
\begin{aligned}
U(f,\sigma) - L(f,\sigma) &= \sum_{j=1}^{n} (M_j - m_j)\delta_j \\
&= \sum_{j \in \mathcal{J}} (M_j - m_j)\delta_j + \sum_{j \notin \mathcal{J}} (M_j - m_j)(t_j - t_{j-1}) \\
&\le \sum_{j \in \mathcal{J}} 2M(t_j - t_{j-1}) + \sum_{j \notin \mathcal{J}} \varepsilon(t_j - t_{j-1}) \\
&\le \sum_{j \in \Lambda} 2M(b_j - a_j) + (b-a)\varepsilon \\
&\le \sum_{j \in \mathbb{N}} 2M(b_j - a_j) + (b-a)\varepsilon \\
&< 2M\varepsilon + (b-a)\varepsilon = (2M + b - a)\varepsilon.
\end{aligned}
$$

can be made arbitrary small. Hence f is Riemann integrable on $[a,b]$.

□

1.4 Exercises

1-1-1 Let f be a function of class $\mathcal{C}^1$ on $[a,b] \subset \mathbb{R}$. For $n \in \mathbb{N}$ and $x_i = a + i\,\frac{b-a}{n}$ for $i \in [0,n]$, compute :

$$\lim_{n \to +\infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i) f'(x_i).$$

1-1-2 Evaluate the following limits:

(a) $\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{1}{n+k}$,

(b) $\lim_{n \longrightarrow +\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2}$,

(c) $\lim_{n \longrightarrow +\infty} \frac{1}{n} \sum_{k=1}^{n} \sin(\frac{kx}{n})$, $x \in \mathbb{R}$,

(d) $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2}$,

(e) $\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$,

(f) $\lim_{n \longrightarrow +\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}}$,

(g) $\lim_{n \longrightarrow +\infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2 \sin(\frac{k\pi}{n})$,

(h) $\lim_{n \longrightarrow +\infty} \frac{1}{n^4} \prod_{k=1}^{2n} (n^2 + k^2)^{\frac{1}{n}}$,

(i) $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{n} \cos\left(\frac{k\pi}{n}\right)$,

(j) $\lim_{n \to +\infty} \sum_{k=1}^{2^n} \frac{k^3}{2^{4n}}$.

(k) $\lim_{n \to +\infty} \sum_{j=1}^{(k-1)n} \frac{1}{n+j}$

(l) $\lim_{n \to +\infty} \frac{1}{n^2} \sum_{j=1}^{n-1} \sqrt{j(n-j)}$,

(m) $\lim_{n \to +\infty} \prod_{j=1}^{n} \left(1 + \frac{j}{n}\right)^{\frac{1}{n}}$.

1-1-3 For any $n, k \in \mathbb{N}$, define $S_n(k) = \sum_{j=n+1}^{kn} \frac{1}{j}$.

(a) i. Prove that $\frac{1}{j+1} \leq \int_{j}^{j+1} \frac{dx}{x} \leq \frac{1}{j}$.

ii. Deduce that $\int_{n+1}^{kn+1} \frac{dx}{x} \leq S_n(k) \leq \int_{n}^{kn} \frac{dx}{x}$.

(b) Compute $\lim_{n \to +\infty} S_n(k)$.

(c) The sequence $(T_n)_n$ defined by $T_n = \sum_{j=1}^{n} \frac{1}{j}$ is it convergent?

1-1-4 Let f be a non negative function, Riemann-Integrable on $[a, b] \subset \mathbb{R}$, such that: $\int_a^b f(x)\, dx = 0$.

Prove that $\frac{1}{1+f}$ is Riemann-Integrable, and compute $\int_a^b \frac{1}{1+f(x)} dx$.

(Hint: Compute $\int_a^b \frac{dx}{1+f(x)} - \int_a^b 1 dx$).

1-1-5 Compute the following integrals:

1) $\displaystyle\int_0^x \sin^3 t\cos t dt,$

2) $\displaystyle\int_1^x t^2\sqrt{1+t^3}dt,$

3) $\displaystyle\int_0^x \frac{tdt}{1+\sqrt{t}},$

4) $\displaystyle\int_0^x \frac{t^3+t^2+t+1}{1+t+t^2}dt.$

5) $\displaystyle\int_0^\pi \frac{dx}{1+\sin x},$

6) $\displaystyle\int_0^\pi \frac{dx}{1+\sin^2 x},$

7) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin^3 x\cos x}{1+\cos^2(2x)}dx,$

8) $\displaystyle\int_0^\pi \frac{dx}{3+\cos(2x)},$

9) $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx,$

10) $\displaystyle\int_1^2 \frac{e^x dx}{(3+e^x)\sqrt{e^x-1}},$

11) $\displaystyle\int \frac{dx}{\cosh(3x)-\cosh x},$

12) $\displaystyle\int \frac{dx}{\sinh^3 x+\cosh^3 x-1},$

13) $\displaystyle\int \frac{dx}{5\cosh x+3\sinh x+4},$

14) $\displaystyle\int (1-x^2)\sqrt{1-x^2}dx,$

15) $\displaystyle\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}},$

16) $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{(1+x)^2},$

1-1-6 (a) Compute $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n(x)dx$, $J_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n(x)dx$ and $K_n = \displaystyle\int_{-1}^{+1} (x^2-1)^n dx$.

(b) i. Find a relation between $L_n = \displaystyle\int_0^x \frac{dt}{(1+t^2)^n}$ and L_{n-1}, for $n \geq 2$.

ii. Compute the integral $I = \displaystyle\int_0^x \frac{1+t}{(t^2+1)^3}dt$, $(x > 0)$.

1-1-7 State the condition on the different real numbers a, b, c, d such that the primitives of $x \longmapsto \dfrac{(x-a)(x-b)}{(x-c)^2(x-d)^2}$ are rational functions.

1-1-8 (a) Compute $K = \displaystyle\int_0^{\frac{\pi}{4}} e^{-2x}\cos(2x)dx$.

(b) Let $I = \displaystyle\int_0^{\frac{\pi}{4}} e^{-2x}\cos^2(x)dx$ and $J = \displaystyle\int_0^{\frac{\pi}{4}} e^{-2x}\sin^2(x)dx$. Compute $I+J$, $I-J$ and deduce the values of I and J.

1-1-9 Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a Riemann integrable function such that: $\forall x \in \mathbb{R}, f(x) = f(1-x)$. Prove that $\displaystyle\int_0^1 xf(x)dx = \frac{1}{2}\int_0^1 f(x)dx$.

1-1-10 Compute $F(x) = \displaystyle\int_0^\pi |x-t|\sin t\ dt$ for any x of $\mathbb{R}$. The function F is it continuous?

1-1-11 Let f be a real locally integrable function defined on $\mathbb{R}$ and fulfills

$$f(x) = \int_0^{ax} f(t)dt, \quad a \in \mathbb{R}.$$

(a) Prove that f is C^∞ and for all $n \in \mathbb{N}$, $f^{(n)}(x) = a^{\frac{n(n+1)}{2}} f(a^n x)$.

(b) Assume that $|a| < 1$. Using the Taylor formula prove that $f = 0$.

1-1-12 Define the function F by:

$$F(x) = \int_0^{\cos^2 x} \cos^{-1} \sqrt{t}dt + \int_0^{\sin^2 x} \sin^{-1} \sqrt{t}dt.$$

(a) Prove that F is differentiable on $\mathbb{R}$ and compute $F'(x)$ for $x \in [0, \frac{\pi}{2}]$.

(b) i. Prove that F is even, π-periodic on $\mathbb{R}$ and deduce that it is constant on $\mathbb{R}$.

ii. Prove that $\cos^{-1} u + \sin^{-1} u = \frac{\pi}{2}$, $\forall u \in [-1, 1]$

iii. Deduce that $F(x) = \frac{\pi}{4}$, $\forall x \in \mathbb{R}$.

1-1-13 Let a be a real positive number and f, g two continuous functions on $[0, a]$. Assume that $f(x) = f(a-x)$ and $g(x) + g(a-x) = k$; $\forall\, x \in [0, a]$.

(a) Prove that $\displaystyle\int_0^a f(t)g(t)dt = \frac{k}{2} \int_0^a f(t)dt$.

(b) Compute the integral $\displaystyle\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$.

1-1-14 Let a be a non negative real number. Give all the non negative continuous functions $f\colon [0,1] \longrightarrow \mathbb{R}$ such that

$$\int_0^1 f(x)dx = 1, \quad \int_0^1 x f(x)dx = a \quad \text{and} \quad \int_0^1 x^2 f(x)dx = a^2.$$

(Hint: Compute $\displaystyle\int_0^1 (x-a)^2 f(x)dx$).

1-1-15 Compute $\displaystyle\lim_{x \to 0} \frac{1}{x} \int_0^x (1 + \sin 2t)^{\frac{1}{t}} dt$.

1-1-16 Define $I_n = \displaystyle\int_0^1 \ln^n(1+x)dx$, for $n \in \mathbb{N}$.

(a) Compute $\lim\limits_{n\longrightarrow+\infty} I_n$.

(b) Express I_n in term of n and I_{n+1}.

(c) Compute I_n in term of n.

1-1-17 For $n \geq 1$, we set:

$$U_n = \int_0^1 x^n.\frac{\sin 2x}{x^2-2}dx, \quad V_n = n.\int_0^1 x^n \frac{\sin 2x}{x^2-2}dx.$$

(a) Compute $\lim\limits_{n\longrightarrow+\infty} U_n$.

(b) Let $a \in]0,1[$, compute $\lim\limits_{n\longrightarrow+\infty} n.\int_0^a x^n f(x)dx$, with $f(x) = \dfrac{\sin(2x)}{x^2-2}$.

(c) Deduce $\lim\limits_{n\longrightarrow+\infty} V_n$. (Hint: remark that $V_n = n.\int_0^1 x^n(f(x) - f(1))dx + \dfrac{nf(1)}{n+1}$.)

1-1-18 (a) Compute $\lim\limits_{n\longrightarrow+\infty} \int_0^{\frac{\pi}{2}-\varepsilon} (-\sin x)^n dx$ and $\lim\limits_{n\longrightarrow+\infty} \int_0^{\frac{\pi}{2}} (-\sin x)^n dx$.

(b) i. Prove that $\forall x \in]0, \frac{\pi}{2}]$, $\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1$.

ii. Deduce the limit $\lim\limits_{r\longrightarrow+\infty} \int_0^{\frac{\pi}{2}} (\sin rx).e^{-r\sin x}dx$.

(c) i. Prove that $\forall x \geq 0$, $\cos x \geq 1 - \frac{x^2}{2}$.

ii. Deduce $\lim\limits_{\substack{x\longrightarrow 0\\ x>0}} h(x)$, where $h(x) = \int_x^{2x} \dfrac{\cos t}{t}dt$.

1-1-19 Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Define the map F by: $F(x) = \int_0^x tf(t)dt$.

(a) i. Prove that F is $\mathcal{C}^1$ on $\mathbb{R}$.

ii. Prove that $F''(0)$ exists.

(b) What we can say about F if f is even?

(c) Determine F, in the following cases:

$f_1(x) = \tan^{-1} x, \quad f_2(x) = x\tan^{-1} x,$

$f_3(x) = \dfrac{\tan^{-1} x}{(1+x^2)^2}, \quad f_4(x) = \dfrac{1}{\cosh^2 x}.$

1-1-20 Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^2$. Prove that:

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)) + \frac{1}{2}\int_a^b (x-a)(x-b)f^{''}(x)dx.$$

1-1-21 For $c \in]0,1[$ and $n \in \mathbb{N}$, we set: $I_n = \displaystyle\int_0^c x^n \ln(1+x^2)dx$,

$J_n = \displaystyle\int_0^1 x^n \ln(1+x^2)dx.$

Prove that $\displaystyle\lim_{n\longrightarrow+\infty} I_n = \lim_{n\longrightarrow+\infty} J_n = \lim_{n\longrightarrow+\infty} \frac{I_n}{J_n} = 0.$

1-1-22 Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

Prove that the map $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\varphi(x) = \int_a^b f(x-t)(1+t^2+\sin t)dt$$

is differentiable and compute $\varphi'(x)$.

1-1-23
(a) Prove that $\displaystyle\lim_{n\longrightarrow+\infty} n\int_0^{\frac{\pi}{2}} \cos x \sin^n x dx = 1.$

Let g be a Riemann integrable function on $[0, \frac{\pi}{2}]$ and $a \in [0, \frac{\pi}{2}[$.

(b) Compute:

i. $\displaystyle\lim_{n\longrightarrow+\infty} n\sin^n a.$

ii. $\displaystyle\lim_{n\longrightarrow+\infty} n\int_0^a \cos x \sin^n x g(x)dx.$

(c) Deduce that if $\displaystyle\lim_{x\longrightarrow(\frac{\pi}{2})^-} g(x)$ exists, then

$$\lim_{n\longrightarrow+\infty} n\int_0^{\frac{\pi}{2}} \cos x(\sin x)^n g(x)dx = \lim_{x\longrightarrow(\frac{\pi}{2})^-} g(x).$$

1-1-24
(a) Let $g\colon [a,b] \longrightarrow \mathbb{R}$ be a non negative continuous function. Prove that

$$\int_a^b g(x)dx = 0 \iff g \equiv 0.$$

(b) Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^1$ such that $f(a) = 0$.

i. Prove that

$$|f(x)|^2 \leq (x-a)\int_a^x |f'(t)|^2 dt \leq (x-a)\int_a^b |f'(t)|^2 dt.$$

ii. Deduce that

$$\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{2}\int_a^b |f'(x)|^2 dx.$$

iii. Prove that the equality in the previous question holds if and only if f is constant on $[a,b]$.

2 Improper Integrals

2.1 Presentation of the Improper Integral

Definition 2.1.

1. Let f be a piecewise continuous function on the interval $[a,b[$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$.

   We say that the integral of f on the interval $[a,b[$ is convergent if the function $F(x) = \displaystyle\int_a^x f(t)dt$ defined on $[a,b[$ has a finite limit when x tends to b ($x < b$). This limit is called the improper integral of f on $[a,b[$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

2. Let f a piecewise continuous function on the interval $]a,b]$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R}$.

   We say that the integral of f on the interval $]a,b]$ is convergent if the function $G(x) = \displaystyle\int_x^b f(t)dt$ defined on $]a,b]$ has a finite limit when x tends to a ($x > a$). This limit is called the improper integral of f on $]a,b]$ and will be denoted by: $\displaystyle\int_a^b f(x)dx$.

3. Let f be a piecewise continuous function on the interval $]a,b[$, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$.

   We say that the integral of f on the interval $]a,b[$ is convergent if the integral of f is convergent on $]a,c]$ and on $[c,b[$ for any c in $]a,b[$.

4. Let f be a piecewise continuous function on an interval I. The function is called integrable on I (or the integral is absolutely convergent) if the integral of $|f|$ on the interval I is convergent.

Example 2.1 :

1. $\displaystyle\int_0^{+\infty} \frac{dx}{1+x}$ is divergent, $\displaystyle\int_0^{+\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$, $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}} = 2$.
2. Let $\alpha \in \mathbb{R}$ and $a \in \mathbb{R}_+^*$. The integral $\displaystyle\int_a^{+\infty} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha > 1$ and the integral $\displaystyle\int_0^{a} \frac{dx}{x^\alpha}$ is convergent if and only if $\alpha < 1$.
3. For $\beta \in \mathbb{R}$ and $a \in]1, +\infty[$, we set

$$F_\beta(x) = \int_a^x \frac{dt}{t(\ln t)^\beta},$$

for $x \geq a$. In taking the change of variable $u = \ln t$, we get:
$F_1(x) = \ln(\ln x) - \ln(\ln a)$ and for $\beta \neq 1$;
$F_\beta(x) = \displaystyle\int_{\ln a}^{\ln x} \frac{du}{u^\beta} = \frac{1}{1-\beta}[\frac{1}{(\ln x)^{\beta-1}} - \frac{1}{(\ln a)^{\beta-1}}]$. Thus the integral $\displaystyle\int_a^{+\infty} \frac{dx}{x(\ln x)^\beta}$ is convergent if and only if $\beta > 1$.

Definition 2.2.
Let f be a locally Riemann integrable function on an interval I. The intgeral of f on I is called absolutely convergent if the integral of $|f|$ on I is convergent.

Proposition 2.3.
Let f be a locally Riemann integrable function on the interval $[a, b[$.

1. If the integral $\displaystyle\int_a^b f(x)dx$ is absolutely convergent, then $\displaystyle\int_a^b f(x)dx$ is convergent.
2. If there exists a non negative piecewise continuous function g on $[a, b[$, such that $\displaystyle\int_a^b g(x)dx$ converges and $|f(x)| \leq g(x)$, then $\displaystyle\int_a^b f(x)dx$ is absolutely convergent.

Remark 6 :

If $\int_a^b f(x)dx$ is convergent, then $\int_a^b f(x)dx$ is not in general absolutely convergent.
Consider the function $\frac{\sin x}{x}$ on the interval $[1, +\infty[$.
By integration by parts, $\int_1^s \frac{\sin x}{x}dx = \cos 1 - \frac{\cos s}{s} - \int_1^s \frac{\cos x}{x^2}dx$; this shows that the integral of the function $\frac{\sin x}{x}$ is convergent on $[1, +\infty[$. (we can also use the second mean value formula theorem 1.33). Moreover

$$\begin{aligned}
\int_\pi^{n\pi} \frac{|\sin x|}{x}dx &= \sum_{k=1}^{n-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin x|}{x}dx \\
&\geq \sum_{k=1}^{n-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin x|dx \\
&= \sum_{k=1}^{n-1} \frac{2}{(k+1)\pi}
\end{aligned}$$

As the sequence $(v_n)_n$ defined by $v_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}$ is divergent, then the integral of f is not absolutely convergent.
Another proof: we remark that $|\sin x| \geq \sin^2 x = \frac{1-\cos 2x}{2}$. As the integral $\int_1^{+\infty} \frac{\cos(2x)}{2x}dx$ is convergent, the integral $\int_1^{+\infty} \frac{|\sin x|}{x}dx$ is divergent.

2.2 Convergence Tests of Improper Integrals

Theorem 2.4. [The Cauchy Test]
Let f be a piecewise continuous function on $[a, b[$, $b \in \mathbb{R} \cup \{+\infty\}$.
$\int_a^b f(x)dx$ converges if and only if

$$\forall \varepsilon > 0, \exists\ c \ \text{ tel que } \forall x, y \in]c, b[;\ \left|\int_x^y f(t)dt\right| \leq \varepsilon.$$

(We can suppose only f locally Riemann integrable function).

Corollary 2.5.
Let $f \colon [a, b[\longrightarrow \mathbb{R}$ a bounded function and $a, b \in \mathbb{R}$. If f is piecewise continuous on $[a, b[$, then the integral of f on $[a, b[$ is convergent.

Example 2.2 :

The integral of the function $f(t) = \dfrac{\sin t}{t}$ is convergent on $]0, 1]$.

Also the function $g(t) = \sin \dfrac{1}{t}$ on $]0, 1]$.

Theorem 2.6.

Let f be a non negative locally Riemann integrable function on $[a, b[$. The integral $\displaystyle\int_a^x f(t)dt$ converges if and only if there exists $M > 0$ such that $\forall x \in [a, b[;\ \displaystyle\int_a^x f(t)dt \leq M$.

Corollary 2.7.

Let f and g be two non negative locally Riemann integrable functions on $[a, b[$. Assume that $f(t) \leq g(t);\ \forall t \in [a, b[$. Then

If $\displaystyle\int_a^b g(x)dx$ converges; the integral $\displaystyle\int_a^b f(x)dx$ converges.

If $\displaystyle\int_a^b f(x)dx$ diverges, the integral $\displaystyle\int_a^b g(x)dx$ diverges.

Corollary 2.8.

Let f be a non negative locally Riemann integrable function on the interval $[a, b[$ and let $\mathcal{E} = \{(x_n)_n \in [a, b[;\ \lim_{n\to+\infty} x_n = b\}$. For any $x \in [a, b[$, we define $F(x) = \displaystyle\int_a^x f(t)dt$. Then following properties are equivalent

1. The integral of f on $[a, b[$ is convergent.
2. $\{F(x);\ x \in [a, b[\}$ is bounded.
3. For any sequence $(x_n)_n \in \mathcal{E}$, the sequence $(F(x_n)_n$ is convergent.
4. There exists a sequence $(x_n)_n \in \mathcal{E}$ such that the sequence $(F(x_n)_n$ is convergent.

Example 2.3 :

1. $f(t) = e^{-t^2}, t \in [0, +\infty[$, we have $f(t) \leq e^{-t}$ and $\displaystyle\int_0^{+\infty} e^{-x}dx = 1$, thus $\displaystyle\int_0^{+\infty} e^{-x^2}dx$ is convergent.
2. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{\sin\ x}$ diverges because $\dfrac{1}{\sin x} \geq \dfrac{1}{x}\ \forall\ x\ \in]0, \dfrac{\pi}{2}]$.

Proposition 2.9.

Let I be an interval and $f\colon I \longrightarrow \mathbb{R}^+$ a non negative locally Riemann integrable function. The integral of f on I converges if and only if there exists an increasing sequence of intervals $([a_n, b_n])_n$ which covers I and a real $M \geq 0$ such that $\displaystyle\int_{a_n}^{b_n} f(x)dx \leq M$, for any $n \in \mathbb{N}$. In this case

$$\int_I f(x)dx = \sup_{n\in\mathbb{N}} \int_{a_n}^{b_n} f(x)dx.$$

Theorem 2.10.
Let $f\colon [a,b[\longrightarrow \mathbb{R}$ and $g\colon [a,b[\longrightarrow \mathbb{R}^+$ be two locally Riemann integrable functions. Assume that there exists $\ell \in \mathbb{R}\setminus\{0\}$ such that $f \approx \ell g$ (when t tends to b^-). Then $\displaystyle\int_a^b f(x)dx$ converges if and only if $\displaystyle\int_a^b g(x)dx$ converges.

Proof .
If $f \approx \ell g$ (when t tends to b^-), then there exists a function h such that $f(t) = \ell h(t)g(t)$ and $\lim\limits_{t\to b^-} h(t) = 1$. Thus $f(t)-\ell g(t) = (h(t)-1)\ell g(t)$ and, thus there exists c such that $\forall t \in]c,b[,\ |f(t)-\ell g(t)| \leq g(t)$, let $|f(t)| \leq (1+|\ell|)g(t)$.
If the integral $\displaystyle\int_a^b g(x)dx$ converges, then the integral $\displaystyle\int_a^b f(x)dx$ converges absolutely.
If the integral $\displaystyle\int_a^b f(x)dx$ converges, as $\ell \neq 0$, there exists c such that $\forall t \in$]c,b[; $|f(t)-\ell g(t)| \leq \dfrac{|\ell|}{2}g(t)$. If $x < y \in]c,b[$, we have: $\left|\displaystyle\int_x^y f(t)-\ell g(t)dt\right| \leq \dfrac{|\ell|}{2}\displaystyle\int_x^y g(t)dt$, thus $\dfrac{|\ell|}{2}\displaystyle\int_x^y g(t)dt \leq \left|\displaystyle\int_x^y f(t)dt\right| \underset{x,y\to b}{\longrightarrow} 0$.

Remark 7 :
If g change of sign the previous result is not true. It suffices to take the function $f(t) = \dfrac{|\sin t|}{t} + \dfrac{\sin t}{\sqrt{t}}$ and $g(t) = \dfrac{\sin t}{\sqrt{t}}$, for $t \in [1,+\infty[$. The integral of the function g is convergent on $[1,+\infty[$, it suffices to use the Cauchy test and the second Mean Value Formula. The integral of the function f is divergent.

Theorem 2.11.
Let $f\colon [1,+\infty[\longrightarrow \mathbb{R}^+$ be a piecewise continuous function.

1. If there exists $\alpha > 1$ such that $\lim\limits_{x\to+\infty} x^\alpha f(x) = 0$, then the integral of f is convergent on $[1,+\infty[$.
2. If there exists $\alpha < 1$ such that $\lim\limits_{x\to+\infty} x^\alpha f(x) = +\infty$, then the integral of f is not convergent on $[1,+\infty[$.

Theorem 2.12.
Let $a, b \in \mathbb{R}$ and $f\colon]a,b] \longrightarrow \mathbb{R}+$ be a locally Riemann integrable function.

1. If there exists $\alpha < 1$ such that $\lim\limits_{x \to a^+} (x-a)^\alpha f(x) = 0$, then the integral of f is convergent on $]a, b]$.
2. If there exists $\alpha > 1$ such that $\lim\limits_{x \to +\infty} (x-a)^\alpha f(x) = +\infty$, then the integral of f is not convergent on $]a, b]$.

Theorem 2.13. [Abel's Theorem]
Let $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$, and f and g be two continuous functions on the interval $[a, b[$. Assume that:

i) there exists $M \geq 0$ such that $\left|\int_x^y f(t)dt\right| \leq M$ for any x, y in $[a, b[$.

ii) g is monotonic on $[a, b[$ and $\lim\limits_{t \to b} g(t) = 0$.

Then $\displaystyle\int_a^b f(x)g(x)dx$ converges.

Proof .
We can assume that g is decreasing. By second mean value formula, theorem 1.33, for any $x < y$ in $[a, b[$,

$$\begin{aligned} \left|\int_x^y f(t)g(t)dt\right| &= g(x)\left|\int_x^c f(t)dt\right| \\ &\leq Mg(x) \underset{x \to b^-}{\longrightarrow} 0. \end{aligned}$$

□

Example 2.4 :

1. Let f be a non negative continuous function, decreasing and $\lim\limits_{x \to +\infty} f(x) = 0$, then the integral $\displaystyle\int_0^{+\infty} e^{\mathrm{i}\lambda x} f(x)dx$ converges for $\lambda \neq 0$.
2. Let $f\colon [a, +\infty[\longrightarrow [0, +\infty[$ be a decreasing continuous function. Define for all $n \in \mathbb{N}$; $x_n = \displaystyle\sum_{k=0}^{n} f(a+k)$ and $y_n = x_n - \displaystyle\int_a^{a+k+1} f(x)dx$. Then

   i) the sequence $(y_n)_n$ is convergent,

   the integral $\displaystyle\int_a^{+\infty} f(x)dx$ converges if and only if the sequence $(x_n)_n$ converges.

   Indeed:

$$\begin{aligned} f(a+n+1) &= \int_{a+n}^{a+n+1} f(a+n+1)dx \\ &\leq \int_{a+n}^{a+n+1} f(x)dx \leq \int_{a+n}^{a+n+1} f(a+n)dx = f(a+n) \end{aligned}$$

$y_n = \sum_{k=0}^{n}(f(a+k) - \int_{a+k}^{a+k+1} f(x)dx)$, thus the sequence $(y_n)_n$ is non negative and increasing. Moreover $y_n \leq \sum_{k=0}^{n}(f(a+k) - f(a+k+1)) \leq f(a)$, thus the sequence $(y_n)_n$ is convergent.

The sequence $(x_n)_n$ is increasing and $\int_{a}^{a+n+1)} f(x)dx \leq x_n$ and $x_{n+1} \leq f(a) + \int_{a}^{a+n+1} f(x)dx$, thus the sequence $(x_n)_n$ converges if and only if the integral $\int_{a}^{+\infty} f(x)dx$ converges.

As application the sequence $z_n = (\sum_{k=1}^{n} \frac{1}{k}) - \ln n$ is convergent. Its limit is called the Euler constant.

2.3 Exercises

1-2-1 Prove that the integral $I_n = \displaystyle\int_1^{+\infty} \frac{dx}{x^{n+1}\sqrt{x-1}}$ is convergent. Compute by induction I_n for $n \in \mathbb{N}$.

1-2-2 Let $x \in [-1, 1[$, prove that the integral $f(x) = \displaystyle\int_1^{+\infty} \frac{dt}{\sqrt{t(t-1)(t-x)}}$ converges and that it diverges for $x = 1$.
Prove that the function f is continuous on the interval $[-1, 1[$.

1-2-3 the integral $\displaystyle\int_0^1 x^\alpha g(x)dx$ converges.

(a) Prove that $\displaystyle\lim_{x\to 0^+} x^{\alpha+1} g(x) = 0$. (We will be able to use the integral $\displaystyle\int_x^{2x} t^\alpha g(t)dt$).

(b) Deduce that if h is a continuous monotonic function on $[1, +\infty[$ and the integral $\displaystyle\int_1^{+\infty} x^\alpha h(x)dx$ converges, then $\displaystyle\lim_{x\to+\infty} x^{\alpha+1}h(x) = 0$.

1-2-4 Let f be a continuous function on $[1, +\infty[$ such that $\displaystyle\int_1^{+\infty} f(x)dx$ converges.
What can we say about $\lim_{x\to+\infty} f(x)$?

1-2-5 Let $0 < \alpha < 1$, β such that $1 - \alpha < \beta \leq 1$ and $\lambda > 0$.

(a) Prove that the following integrals are convergent

$$I(\lambda) = \int_1^{+\infty} \frac{dx}{x^\alpha(1+\lambda x)}; \quad J(\lambda) = \int_1^{+\infty} \frac{dx}{(1+x^\alpha)(1+\lambda x)}$$

(b) Prove that $I(\lambda) - \lambda^{\alpha-1} \displaystyle\int_0^{+\infty} \frac{dx}{x^\alpha(1+x)}$ has a limit when λ tends to 0.

(c) Prove that the integral $K(\lambda) = \displaystyle\int_1^{+\infty} \frac{dx}{x^\beta(1+x^\alpha)(1+\lambda x)}$ is convergent and that $\displaystyle\lim_{\lambda\to 0^+} K(\lambda) = \int_1^{+\infty} \frac{dx}{x^\beta(1+x^\alpha)}$.

1-2-6 Study the nature of the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}+\sin x}dx$.

1-2-7 Let f be a function of class $\mathcal{C}^2$ on $]0,1]$ such that the integral $\displaystyle\int_0^{\pi} xf(\sin x)dx$ converges.

(a) Prove that $\displaystyle\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

(b) Prove that $\displaystyle\int_0^{\pi} x\ln(\sin x)dx = -\frac{\pi^2}{2}\ln 2$, namely that $\displaystyle\int_0^{\pi}\ln(\sin x)dx = -\pi\ln 2$.

1-2-8 Compute, when they are convergent, the following integrals:

$$\int_a^{2a}\frac{a^2+2x^3}{\sqrt{x^4-a^4}}dx,\quad \int_0^1\frac{x\ln x}{(1-x^2)^{3/2}}dx,\quad \int_0^1\frac{\ln x}{(1-x)^{\frac{3}{2}}}dx,\quad \int_0^{\frac{\pi}{2}}\sin 2x\ln(\tan x)dx,$$

$$\int_0^{\frac{\pi}{2}}\cos x\ln(\tan x)dx,\quad \int_1^{+\infty}\frac{x^4+1}{x^3(x+1)(1+x^2)}dx,\quad \int_1^{+\infty}\frac{dx}{x\ ^4\sqrt{1+x^2}},\quad \int_0^{+\infty}x^ne^{-x}dx$$

1-2-9 Give the nature of the following integrals.

$$\int_0^1\frac{\ln x}{1-x^2}dt,\quad \int_0^1\frac{\ln^2 x}{1+x^2}dt,\quad \int_0^{+\infty}\frac{dx}{\sqrt{x}(1+|\ln x|)},\quad \int_0^{+\infty}\frac{dx}{\sqrt{x}(1+x^2)},$$
$$\int_0^{+\infty}\frac{x\sin x}{(1+x^2)}dx,\quad \int_0^{+\infty}\frac{\cos x}{(1+x^{\alpha})}dx,\quad \int_0^{+\infty}\frac{\cos(\alpha x)}{1+e^x}dx,\quad \int_0^{+\infty}\frac{\sin x}{\sqrt{x+\cos x}}dx,$$
$$\int_0^1\frac{dx}{\sqrt{1-x^2}},\quad \int_0^1\frac{\ln x}{(1-x)^{\alpha}}dx,\quad \int_0^{+\infty}\frac{x^{\alpha-1}}{1+x}dx,\quad \int_0^{+\infty}x^{\alpha}e^{-x}dx,$$
$$\int_0^{+\infty}\frac{x\sin x}{1+x^2}dx,\quad \int_0^{+\infty}\sin(x^2)dx,\quad \int_0^{+\infty}\frac{2\tan^{-1}x-\pi}{\sqrt{x}}dx,\quad \int_0^{+\infty}\left(\frac{\tan^{-1}x}{x}-\frac{\pi}{2(1+x)}\right)dx,$$
$$\int_0^{+\infty}\sin(x)\sin(\frac{1}{x})dx,,\quad \int_0^{+\infty}\sin\left(x+\frac{1}{x}\right)\frac{dx}{\sqrt{x}},\quad \int_0^{\frac{\pi}{2}}\tan x dx,\quad \int_0^{\frac{\pi}{2}}\sqrt{\tan x}dx,$$
$$\int_1^{+\infty}\frac{e^{\sin x}}{x}dx,\quad \int_1^{+\infty}\frac{1}{x}\left(e^{\frac{1}{x}}-\cos\frac{1}{x}\right)dx,\quad \int_0^1\frac{dx}{\cos^{-1}x}.$$

1-2-10 (a) Let $s\in\mathbb{R}$.

i. Prove that the integral $\displaystyle\int_1^{+\infty} t^{s-1}\cos x dx$ is absolutely convergent for $s<0$.

ii. Deduce that the integral $\int_1^{+\infty} t^s \sin t dt$ is defined for $s < 0$.

(b) Use the previous result to study if the following integral $\int_1^{+\infty} \sqrt{t}\sin(t^2)dt$.

1-2-11 Let $f\colon [0,+\infty[\longrightarrow \mathbb{R}$ be a continuous function. Assume that the integral $\int_0^{+\infty} f(x)dx$ is convergent and that $\lim_{x\longmapsto+\infty} f(x) = \ell$.

(a) Prove that $\ell = 0$.

(b) i. Say if the following proposition is true: $\int_0^{+\infty} g(x)dx$ exists, then $\lim_{x\longmapsto+\infty} g(x) = 0$.

ii. What can we say if g is non negative?

1-2-12 (a) Using the Abel Theorem, prove the convergence of the following integral: $I = \int_0^{+\infty} \frac{\sin x}{x}dx$.

(b) Using the Cauchy criterion, prove that the integral $\int_1^{+\infty} \frac{|\sin t|}{t}dt$ is not convergent.

1-2-13 (a) Prove that $J = \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ is convergent.

(b) Compute this integral using the change of variable $x = a\cos^2 t + b\sin^2 t$.

1-2-14 (a) Set $\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt$ called the Gamma function.

i. Prove that Γ is well defined for $x > 0$.

ii. Compute $\Gamma(x+1)$ in term of $\Gamma(x)$.

(b) i. Prove that $\Gamma(\frac{x}{p}+\frac{y}{q}) \leq \Gamma(x)^{1/p}\Gamma(y)^{1/q}$ for $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

ii. Deduce that $\ln(\Gamma)$ is a convex function.

1-2-15 (a) Discuss according to the values of $\alpha, \beta \in \mathbb{R}^+$ the nature of the integral $\int_0^{+\infty} \frac{\ln(1+t^\alpha)}{t^\beta}dt$.

(b) Discuss according to the values of $\alpha \in \mathbb{R}$ the nature of the integral $\int_0^{+\infty} \frac{\sin^2 t}{t^\alpha}dt$.

(c) i. Study the convergence of the integral $I(\alpha) = \displaystyle\int_0^{\frac{\pi}{2}} \frac{dt}{1+\cos\alpha\cos t}$.

ii. Compute $I(\alpha)$ when it exists.

1-2-16 (a) Verify that $\forall u \in [\frac{1}{2}, 1]$, $-\ln u \leq \ln 2$.

(b) Prove that the integral $\displaystyle\int_0^1 \frac{\ln u}{\sqrt{1-u^2}} du$ is convergent.

(c) Deduce that the integrals $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin\theta) d\theta$ and $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\cos\theta) d\theta$ are convergent and have the same value.

(d) Prove that the integral $\displaystyle\int_0^{\pi} \ln(\sin\theta) d\theta$ is convergent and

$$\int_0^{\pi} \ln(\sin\theta) d\theta = 2\int_0^{\frac{\pi}{2}} \ln(\sin\theta) d\theta.$$

(e) Prove that $\displaystyle\int_0^1 \frac{\ln u}{\sqrt{1-u^2}} du = -\frac{\pi}{2}\ln 2$.

1-2-17 (a) Study the convergence of the integral $\displaystyle\int_0^1 \frac{\ln t}{\sqrt{1-t}} dt$ and compute its value.

(b) i. Justify the convergence of the integral $\displaystyle\int_0^{+\infty} \frac{\sin^2 t}{t^2} dt$.

ii. Find the following limits

$$\lim_{\varepsilon\to 0} \frac{1-\cos\varepsilon}{\varepsilon} \quad \text{and} \quad \lim_{\varepsilon\to+\infty} \frac{1-\cos\varepsilon}{\varepsilon}$$

iii. Using an integration by parts in the integral $\displaystyle\int_a^b \frac{\sin t}{t} dt$, prove that the integral $\displaystyle\int_0^{+\infty} \frac{\sin t}{t} dt$ is convergent and we have:

$$\int_0^{+\infty} \frac{\sin t}{t} dt = \int_0^{+\infty} \frac{\sin^2 t}{t^2} dt.$$

1-2-18 Prove that the integral $\displaystyle\int_0^{+\infty} \frac{\ln t}{1+t^2} dt$ is convergent and compute its value.

3 Problems on Chapter-1

3-1 (a) Consider the integrals: $I_n = \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x} dx$, $J_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x} dx$, $K_n = \int_0^{\frac{\pi}{4}} \frac{\cos(4n+1)x}{\cos x} dx$.

a) Prove that I_n does not depends of n and compute its value.

b) Express $I_n - J_n$ in term of K_n and deduce $\lim\limits_{n \longrightarrow +\infty} J_n$.

(b) a) State an induction relation between J_n and J_{n-1}.

b) Give the expression of J_n in term of n.

c) Deduce $\lim\limits_{n \longrightarrow +\infty} (1 - \frac{1}{3} + \frac{1}{5} + \ldots + \frac{(-1)^{n-1}}{2n-1}) = \lim\limits_{n \longrightarrow +\infty} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1}$.

3-2 For any function f defined on $[0, +\infty[$ and Riemann integrable on any closed and bounded interval of $[0, +\infty[$, we define $F(x) = \frac{1}{x} \int_0^x f(t)dt$, for $x > 0$.

(a) a) Prove that if $\lim\limits_{x \longrightarrow +\infty} f(x) = \ell$, then $\lim\limits_{x \longrightarrow +\infty} F(x) = \ell$.

b) Compute $F(x)$, for $f(x) = \cos x$ and study the reciprocal of a).

c) Let $(u_n)_n$ be a sequence such that $\lim\limits_{n \longrightarrow +\infty} u_n = \ell$ and let f defined by $f(x) = u_n$ for $x \in [n, n+1[$.

i) Prove that $\lim\limits_{x \longrightarrow +\infty} f(x) = \ell$.

ii) Compute $\int_0^n f(t)dt$ and deduce that

$$\lim_{n \longrightarrow +\infty} \frac{u_0 + u_1 + \ldots + u_n}{n} = \ell.$$

In which follows f is a continuous function of $[0, 1]$ and $F(x) = \frac{1}{x} \int_0^x f(t)dt$.

(b) Prove that F can be extended to a continuous function on $[0, 1]$. In which follows we still denote F this extension.

(c) Prove that F is differentiable on $]0, 1]$ and compute F' in term of F and f.

(d) Assume that $F = \lambda f$.

a) Prove that f is differentiable on $]0, 1]$ and give a simple relation between f and f'.

b) How we can choose λ to have f a polynomial.

(e) Assume that there is $t \in]0,1[$ such that $F(t) = \sup_{x\in[0,1]} f(x)$. Prove f is constant on $[0,t]$.

(f) Using the Taylor formula prove that if f is differentiable at 0, then F is differentiable at 0 and compute $F'(0)$.

3-3 Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function which fulfills the following properties:

i) $\forall\ (x,y) \in \mathbb{R}^2,\ f(x+y) + f(x-y) = 2f(x)f(y)$.

ii) The restriction of f on any interval $[a,b]$ is Riemann integrable and the map $F(x) = \displaystyle\int_0^x f(t)dt$ is not the zero function.

(a) Prove that F is continuous on $\mathbb{R}$.

(b) Let $x_0 \in \mathbb{R}$ be such that $F(x_0) \neq 0$. For all $y \in \mathbb{R}$, define $G(y) = F(x_0+y) - F(y-x_0)$.
Prove that $G(y) = \displaystyle\int_y^{y+x_0} f(t)dt - \int_y^{y-x_0} f(t)dt = 2F(x_0)f(y)$. (We will be able to make the change of variable $u = t-y$ and $v = y-t$ respectively).

(c) Deduce that: f $\mathcal{C}^\infty$ on $\mathbb{R}$.

(d) For x in $\mathbb{R}$, we set $H(y) = f(x+y) + f(x-y) = 2f(x)f(y)$.
a) Compute $H'(0)$ and deduce that $f'(0) = 0$.
b) Compute $H''(0)$ in two manner and deduce that f fulfills the following differential equation $f''(x) = f''(0)f(x)$, $\forall\ x \in \mathbb{R}$.

(e) a) In using i) show that $f(0) = 0$ or $f(0) = 1$.
b) Prove that if $f(0) = 0$, then f is identically zero.
c) Deduce that $f(0) = 1$.

3-4 For all $n \in \mathbb{N}$, define the function f_n by

$$f_n(x) = \int_0^x \frac{dt}{\cosh^n t}.$$

(a) Prove that f_n is $\mathcal{C}^\infty$ and odd. Compute f_n' and f_n''.

(b) Compute $f_1(x)$, $f_2(x)$, $\lambda_1 = \lim\limits_{x\to+\infty} f_1(x)$ and $\lambda_2 = \lim\limits_{x\to+\infty} f_2(x)$.

(c) Prove that f_n is bounded on $[0,+\infty[$.

(d) Prove that $\lim\limits_{x\to+\infty} f_n(x) = \lambda_n$ exists in $\mathbb{R}$.

(e) State the induction formula

$$(n-1)f_n(x) = (n-2)f_{n-2}(x) + \frac{\sinh x}{\cosh^{n-1} x}, \quad \forall\ n \geq 3.$$

(f) Deduce an induction formula which express λ_n in term of λ_{n-2} and compute λ_n.

3-5 Define the Wallis integral of rank $n \in \mathbb{N}$ by:

$$W_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$$

(a) Prove that $W_{n+2} = \dfrac{n+1}{n+2} W_n$.

(b) Deduce that $W_{2n} = \dfrac{(2n)!}{2^{2n}(n!)^2} \dfrac{\pi}{2}$ and $W_{2n+1} = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$.

(c) Prove that the sequence $(nW_nW_{n-1})_n$ is constant and deduce that $nW_nW_{n-1} = \frac{\pi}{2}$.

(d) Prove that $W_{n+1} \leq W_n \leq W_{n-1}$ and deduce that

$$\lim_{n\to+\infty} \frac{W_n}{W_{n-1}} = 1 \quad \text{and} \quad \lim_{n\to+\infty} \sqrt{n} W_n = \sqrt{\frac{\pi}{2}}$$

(e) Let $B_n = \displaystyle\int_0^{\sqrt{n}} (1 - \frac{t^2}{n})^n dt$.

a) Prove that $B_n = \sqrt{n} \displaystyle\int_0^1 (1-u^2)^n du$.

b) Prove that $B_n = \sqrt{n} \displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n+1} v dv$.

(f) For $x > 0$, define $A_n(x) = \displaystyle\int_0^x (1 + \frac{t^2}{n})^{-n} dt$, for $n \in \mathbb{N}$.

a) Prove that

$$A_n(x) = \int_0^{\frac{x}{\sqrt{n}}} (1+u^2)^{-n} du.$$

and

$$A_n(x) = \sqrt{n} \int_0^{\tan^{-1} \frac{x}{n}} \cos^{2n-2} v dv.$$

b) Prove that $\displaystyle\lim_{n\to+\infty} \lim_{x\to+\infty} A_n(x) = \lim_{n\to+\infty} \sqrt{n} W_{2n-2} = \frac{\sqrt{\pi}}{2}$.

(g) a) Prove that $\forall y \in \mathbb{R}$, $e^y \geq 1 + y$ and deduce that $(1 + \dfrac{x^2}{n})^n \leq e^{x^2}$.

$\forall x \in [0, \sqrt{n}]$, $0 \leq (1 - \dfrac{x^2}{n})^n \leq e^{-x^2}$.

b) Prove that $\lim_{x\to+\infty} A_n(x) \geq \lim_{n\to+\infty} \int_0^{\sqrt{n}} e^{-x^2}dx \geq \lim_{n\to+\infty} B_n$.
Deduce that
$$\frac{\sqrt{\pi}}{2} = \lim_{n\to+\infty} \int_0^{\sqrt{n}} e^{-x^2}dx.$$

3-6 (a) a) Using the Taylor's formula prove that $\frac{8}{3} < e$.
b) Compute $\int_0^1 \ln(1+t)dt$ and show that $\int_0^1 \ln(1+t)dt < \ln(\int_0^1 (1+t)dt)$.
c) State the inequality $\ln(1+v) \leq v$, for $v > -1$.

(b) a) Let $u\colon [0,1] \longrightarrow \mathbb{R}$ be a continuous positive function.
Using the question c) prove that if $\int_0^1 u(t)dt = 1$, one has: $\int_0^1 \ln u(t)dt \leq 0$.
How we can choose u to have the equality?
b) Let $f\colon [0,1] \longrightarrow \mathbb{R}$ be a continuous positive function. Prove that
$$\int_0^1 \ln f(t)dt \leq \ln(\int_0^1 f(t)dt). \tag{1.8}$$

(c) Give a proof of (7.10) using the concavity of the function ln and the Riemann sums.

3-7 We set
$$\psi(t) = \sum_{n=1}^{+\infty} \frac{1}{(n+t)^2}$$

(a) Prove that ψ is of class C^∞ on $[0,+\infty[$ and compute $\lim_{t\to+\infty} \psi(t)$.

(b) a) Prove that for any $t \geq 0$ and $n \geq 1$
$$\frac{1}{(n+t)(n+t+1)} \leq \frac{1}{(n+t)^2} \leq \frac{1}{(n+t)(n+t-1)}.$$
b) Deduce that $\frac{1}{t+1} \leq \psi(t) \leq \frac{1}{t}$, for any $t > 0$, and that $\psi(t) \approx \frac{1}{t}$; $(t \to +\infty)$.
c) Prove that
$$\psi(t) - \frac{1}{t+1} = \sum_{n=1}^{+\infty} \frac{1}{(n+t)^2(n+t+1)}.$$

(c) a) Remark that $\dfrac{1}{(n+t+1)^3} \leq \displaystyle\int_n^{n+1} \dfrac{dx}{(t+x)^3} \leq \dfrac{1}{(n+t)^3}$ and prove that for any $t > 0$

$$\psi(t) - \frac{1}{t+1} = \frac{\eta(t)}{t^2},$$

with η a bounded function on $]0, +\infty[$.

b) Deduce that the integral $\displaystyle\int_0^{+\infty} (\psi(t) - \frac{1}{1+t})dt$ is convergent.

(d) Justify the following equality

$$\int_0^{+\infty} (\psi(t) - \frac{1}{t+1})dt = \sum_{n=1}^{+\infty} (\frac{1}{n} - \ln(\frac{n+1}{n})).$$

2 Infinite Series

1 Tests of Convergence of Infinite Series

Definition 1.1.

1. Let $(u_n)_n$ be a sequence of real numbers (eventually complex numbers). Consider the sequence $(S_n)_n$ defined by: $S_n = \sum_{k=1}^{n} u_k$.
   If the sequence $(S_n)_n$ is convergent, we say that the series $\sum_{n\geq 1} u_n$ is convergent.
   The limit of the sequence $(S_n)_n$ if it exists is denoted by $\sum_{n=1}^{+\infty} u_n$.
2. The series $\sum_{n\geq 1} u_n$ is called divergent if the sequence $(S_n)_n$ is divergent.

Remark 8 :

1. If the series $\sum_{n\geq 1} u_n$ converges, then $\lim_{n\longrightarrow+\infty} u_n = 0$. ($u_n = S_n - S_{n-1}$.)
2. The condition $\lim_{n\longrightarrow+\infty} u_n = 0$ is not, however, sufficient to ensure the convergence of the series $\sum_{n\geq 1} u_n$. For instance, the series $\sum_{n\geq 1} \sqrt{n+1} - \sqrt{n}$ is divergent because $S_n = \sqrt{n+1} - 1$, for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} u_n = 0$.

Theorem 1.2. [Cauchy Criterion]
Let $(u_n)_n$ be a sequence of real numbers. The series $\sum_{n\geq 1} u_n$ converges if and only if,

$$\forall\, \varepsilon > 0,\ \exists\, N_\varepsilon \in \mathbb{N};\ |\sum_{n=p}^{q} u_n| \leq \varepsilon, \quad \forall\, q \geq p \geq N_\varepsilon. \tag{2.1}$$

Definition 1.3.
A series $\displaystyle\sum_{n\geq 1} u_n$ is called absolutely convergent if the series $\displaystyle\sum_{n\geq 1} |u_n|$ is convergent.

Remark 9 :
Every absolutely convergent series is convergent but the converse is false, it suffices to take the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$. Indeed, if $S_n = \displaystyle\sum_{p=1}^{n} \frac{(-1)^{p+1}}{p}$, then $S_{2n+1} - S_{2n} = \frac{-1}{2n+1} \xrightarrow[p\to+\infty]{} 0$. To prove that the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is convergent, it suffices to prove that the sequence $(S_{2n})_n$ is convergent.
We have: $S_{2n+2} - S_{2n} = \dfrac{1}{2n+2} - \dfrac{1}{2n+1} \leq 0$ and $S_{2n+1} - S_{2n-1} = \frac{1}{2n} - \frac{1}{2n+1} \geq 0$, then the sequences $(S_{2n})_n$ and $(S_{2n+1})_n$ are adjacent, which shows that the sequence $(S_n)_n$ is convergent.
We remark also that $\displaystyle\sum_{k=n+1}^{2n} \frac{1}{k} \geq \frac{n}{2n} = \frac{1}{2}$, then the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent.
There are several standard tests for convergence of a series of non negative terms. These tests are based primarily on the fact that an increasing sequence is convergent if, and only, if, it is bounded above. It follows that a series $\sum_{n\geq 1} u_n$ with non negative terms is convergent if, and only, if, the sequence $(S_n)_n$ defined by: $S_n = \sum_{k=1}^{n} u_k$ is bounded.

1.1 Comparison Test

Theorem 1.4. [Comparison Test]
Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative real numbers. Assume that there exists an integer $k \in \mathbb{N}$ such that $u_n \leq v_n$, for every $n \geq k$. Then if the series $\displaystyle\sum_{n\geq 1} v_n$ is convergent, the series $\displaystyle\sum_{n\geq 1} u_n$ is also convergent.

Proof .
Let $S_n = \displaystyle\sum_{j=k}^{n} u_j$ and $T_n = \displaystyle\sum_{j=k}^{n} v_j$. We have $S_n \leq T_n$. The series $\displaystyle\sum_{n\geq 1} v_n$ is convergent if and only if the sequence $(T_n)_n$ is bounded above, which gives the result.
The result can also be deduced by the Cauchy Criterion (1.2). $\square$

Corollary 1.5.
Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that there exists $a > 0$ and $b > 0$ such that $au_n \leq v_n \leq bu_n$ for every $n \geq k$, then the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge or diverge together.

Corollary 1.6.
Let $(u_n)_n$ and $(v_n)_n$ be two sequences with non negative numbers. Assume that $\lim_{n\to+\infty} \frac{u_n}{v_n} = \ell$.

1. If $\ell > 0$, the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ converge or diverge together..
2. If $\ell = 0$, the convergence of the series $\sum_{n\geq 1} v_n$ involves the convergence of the series $\sum_{n\geq 1} u_n$.
3. If $\ell = +\infty$, the convergence of the series $\sum_{n\geq 1} u_n$ involves the convergence of the series $\sum_{n\geq 1} v_n$.

Theorem 1.7.
Let $(u_n)_n$ and $(v_n)_n$ be two sequences of positive numbers. If there exists $m \in \mathbb{N}$ such that, $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$, whenever $n \geq m$, then the convergence of the series $\sum_{n\geq 1} v_n$ involves the convergence of the series $\sum_{n\geq 1} u_n$.

Proof .
Let $N \in \mathbb{N}$ be large enough such that $\forall n \geq N$, $\frac{u_{n+1}}{u_n} \leq \frac{v_{n+1}}{v_n}$. Thus $\frac{u_{n+1}}{v_{n+1}} \leq \frac{u_n}{v_n}$ for $n \geq N$. The sequence $\left(\frac{u_n}{v_n}\right)_{n\geq N}$ is decreasing and $\frac{u_n}{v_n} \leq \frac{u_N}{v_N} = M \in \mathbb{R}_+^*$, $\forall n \geq N$. Then $u_n \leq Mv_n$ for all $n \geq N$, which yields the result. $\square$

1.2 Integral Test

Theorem 1.8. [Integral Test]
Let f be a decreasing continuous function on $[1, +\infty[$. We define $u_n = f(n)$, for all $n \in \mathbb{N}$. Then:

$$\int_1^{+\infty} f(x)dx \text{ is convergent } \iff \sum_{n\geq 1} u_n \text{ is convergent.}$$

Proof .
Let $S_n = \sum_{k=0}^{n} u_k$ and $v_n = \int_1^n f(t)dt$. We have: $f(n+1) \leq \int_n^{n+1} f(t)dt \leq f(n)$, thus

$$\sum_{k=1}^{n} f(k+1) \leq \int_1^{n+1} f(t)dt \leq \sum_{k=1}^{n} f(k).$$

If the sequence $(S_n)_n$ is convergent, then it is bounded above. Hence the sequence $(v_n)_n$ is also bounded above, and since it is increasing it is convergent. Conversely if the sequence $(v_n)_n$ is convergent, the sequence $(S_n)_n$ is bounded above and then it is convergent. □

Corollary 1.9. [Convergence of Riemann series]
The series $\sum_{n\geq 1} \frac{1}{n^\alpha}$ is convergent if and only if, $\alpha > 1$.

Proposition 1.10. [Application: Comparison with Riemann series]
Let $(u_n)_n$ be a sequence with non negative real numbers. Assume that there exist $0 < a < b$ such that $0 < a \leq n^\alpha u_n \leq b < +\infty$ for every n large enough, then the series $\sum_{n\geq 1} u_n$ is convergent if and only if, $\alpha > 1$.

This proposition results from Theorem (1.4)

Exercise 1.2 :
Show that the Bertrand series $\sum_{n\geq 2} \frac{1}{n^\alpha \ln^\beta n}$ is convergent if and only if $\alpha > 1$ or $\alpha = 1$ and $\beta > 1$.

Solution
If $\alpha \leq 0$, $\lim_{n\to+\infty} \frac{n}{n^\alpha (\ln n)^\beta} = +\infty$, then the series is divergent.

If $0 < \alpha < 1$, we take $\alpha < \gamma < 1$ and consider the sequence $v_n = \frac{1}{n^\gamma}$. $\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = +\infty$, then the series $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ is divergent.

If $\alpha > 1$, we take $1 < \gamma < \alpha$ and consider the sequence $v_n = \frac{1}{n^\gamma}$, $\lim_{n\to+\infty} \frac{n^\gamma}{n^\alpha (\ln n)^\beta} = 0$, then the series $\sum_{n\geq 2} \frac{1}{n^\alpha (\ln n)^\beta}$ is convergent.

If $\alpha = 1$, we consider the sequence $u_n = \frac{1}{n \ln^\beta n}$ and $f(x) = \frac{1}{x \ln^\beta x}$, for $x \geq 2$. The function f is decreasing for x large. Then the series $\sum_{n\geq 2} \frac{1}{n(\ln n)^\beta}$ is convergent if and only if $\int_2^\infty \frac{dx}{x \ln^\beta x}$.

The integral

$$\int_2^{\infty} \frac{dx}{x \ln^{\beta} x} \overset{t=\ln x}{=} \int_{\ln 2}^{\infty} \frac{dt}{t^{\beta}}$$

is convergent if and only if $\beta > 1$. $\square$

1.3 Root Test or the Cauchy Test

Theorem 1.11. [Root Test or the Cauchy Test]
Let $(u_n)_n$ be a sequence of real numbers and $\ell = \overline{\lim}_{n\to+\infty} \sqrt[n]{|u_n|}$.

1. If $\ell < 1$, the series $\sum\limits_{n\geq 1} u_n$ is absolutely convergent.
2. If $\ell > 1$, the general term of the series does not tends to 0 and the series $\sum_{n\geq 1} u_n$ is divergent.
3. If $\ell = 1$, we can not conclude about the convergence of the series.

Proof .

1. Let α be such that $\ell < \alpha < 1$, there exists $N \in \mathbb{N}$ such that $\sqrt[n]{|u_n|} < \alpha$, for every $n \geq N$. Then $u_n \leq \alpha^n$. Since the series $\sum_{n\geq 1} \alpha^n$ is convergent, the series $\sum_{n\geq 1} u_n$ is convergent.
2. Let $1 < \beta < \ell$, there exists an increasing sequence of integers $(n_k)_k$ such that $\lim\limits_{k\to+\infty} |u_{n_k}|^{1/n_k} = \ell > \beta$. Hence there exists $k_0 \in \mathbb{N}$ such $|u_{n_k}| \geq \beta^{n_k}$, for all $k \geq k_0$. It follows that $\lim\limits_{k\longrightarrow+\infty} |u_{n_k}| = +\infty$ and the series $\sum\limits_{n\geq 1} u_n$ is divergent.
3. We know that the series $\sum\limits_{n\geq 1} \frac{1}{n}$ is divergent and $\sum\limits_{n\geq 1} \frac{1}{n^2}$ is convergent, but in the two cases $\lim\limits_{n\to+\infty} n^{-\frac{1}{n}} = \lim\limits_{n\to+\infty} n^{-\frac{2}{n}} = 1$.

$\square$

1.4 The Ratio Test or the D'Alembert's Test

Proposition 1.12.
Let $(u_n)_n$ be a sequence of real numbers. Assume that $\lim\limits_{n\longrightarrow+\infty} |\frac{u_{n+1}}{u_n}| = \ell$. Then

1. If $\ell < 1$, the series $\sum_{n\geq 1} u_n$ is absolutely convergent.

2. If $\ell > 1$ the series $\sum_{n\geq 1} u_n$ is divergent.

3. If $\ell = 1$, we can not conclude about the convergence of the series.

We prove that is this case $\lim_{n\longrightarrow+\infty} \sqrt[n]{|u_n|} = \ell$.

Proof .

1. Let α be a real number such that $\ell < \alpha < 1$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\dfrac{|u_{n+1}|}{|u_n|} < \alpha$, then $u_n \leq \alpha^n \dfrac{|u_N|}{\alpha^N}$. Since the series $\sum_{n\geq 1} \alpha^n$ is convergent, the series $\sum_{n\geq 0} u_n$ is absolutely convergent.

2. Let $1 < \beta < \ell$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, $\dfrac{|u_{n+1}|}{|u_n|} \geq \beta$, then $u_n \geq \beta^n \dfrac{|u_N|}{\alpha^N}$. Since the series $\sum_{n\geq 1} \beta^n$ is divergent, the series $\sum_{n\geq 0} u_n$ is not convergent.

3. We know that $\sum_{n\geq 1} \frac{1}{n}$ diverges and $\sum_{n\geq 1} \dfrac{1}{n^2}$ converges, but in the two cases $\lim_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = 1$.

Assume $\lim_{n\longrightarrow+\infty} |\dfrac{u_{n+1}}{u_n}| = \ell$ and $0 < \ell < +\infty$.

For $0 < \alpha < \ell < \beta < +\infty$, there exists $N \in \mathbb{N}$ such that $\forall n \geq N,\ \alpha < \frac{|u_{n+1}|}{|u_n|} < \beta$. It follows that

$$\alpha^n \frac{|u_N|}{\beta^{N-1}} \leq \alpha|u_n| \leq |u_{n+1}| \leq \beta|u_n| \leq \beta^{n-N+1}|u_N| = \beta^n \frac{|u_N|}{\beta^{N-1}},\ \forall n \geq N.$$

We deduce that

$$\alpha = \lim_{n\to+\infty} \alpha^{1-N/n} \sqrt[n]{|u_N|} \leq \lim_{n\to+\infty} \sqrt[n]{|u_n|} \leq \lim_{n\to+\infty} \beta^{1-N/n} \sqrt[n]{|u_N|} = \beta.$$

Thus $\alpha \leq \lim_{n\to+\infty} \sqrt[n]{|u_n|} \leq \beta$ for every $0 < \alpha < \ell < \beta < +\infty$, this which yields that $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = \ell$.

If $\ell = +\infty$ and $0 < \alpha$. The above proof yields that $\alpha \leq \lim_{n\to+\infty} \sqrt[n]{|u_n|}$, then $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = +\infty$.
If $\ell = 0$ and $0 < \beta$. The above proof yields that $\lim_{n\to+\infty} \sqrt[n]{|u_n|} \leq \beta$, then $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = 0$.

$\square$

Examples 1 :

1. Let $z \in \mathbb{C}$, the series $\sum_{n\geq 0} \frac{z^n}{n!}$ is absolutely convergent on $\mathbb{C}$, because for every $z \in \mathbb{C}$; $|\frac{u_{n+1}}{u_n}| = \frac{|z|}{n+1} \xrightarrow[n\longrightarrow+\infty]{} 0$. We denote e^z the sum of this series. $e^z = \displaystyle\sum_{n=0}^{+\infty} \frac{z^n}{n!}$.
2. For $|z| < 1$, the series $\sum_{n\geq 1} \frac{z^n}{n}$ is absolutely convergent.

1.5 The Abel's Criterion

Theorem 1.13. [Abel's Criterion]
Let $(u_n)_n$ be a sequence of real numbers and let $(v_n)_n$ be a sequence of non negative real numbers such that:

1. the sequence $(v_n)_n$ is decreasing and converges to 0.
2. the sequence $\left(S_n = \displaystyle\sum_{k=1}^{n} u_k\right)_n$ is bounded.

Then the series $\displaystyle\sum_{n\geq 1} u_n v_n$ is convergent.

Proof .
We use the Cauchy criterion (1.2) for the existence of the limit of sequences. Let $q > p \geq 1$,

$$\begin{aligned}\sum_{k=p+1}^{q} u_k v_k &= \sum_{k=p+1}^{q} (S_k - S_{k-1}) v_k = \sum_{k=p+1}^{q} S_k v_k - \sum_{k=p}^{q-1} S_k v_{k+1} \\ &= \sum_{k=p+1}^{q-1} (v_k - v_{k+1}) \ + S_q v_q - S_p v_{p+1}\end{aligned}$$

Since $|S_k| \leq M$, then $|\displaystyle\sum_{k=p+1}^{q} u_k v_k| \leq 2M v_{k+1} \xrightarrow[k\to+\infty]{} 0$.

Remark 10 :
The result holds also if we suppose that the sequence $(S_n)_n$ is bounded and the sequence $(b_n)_n$ converges to 0 and the series $\sum_{n=0}^{+\infty}(b_n - b_{n+1})$ is convergent.

Examples 2 :

1. Let $b_n = \dfrac{(-1)^{[\sqrt{n}]}}{n}$, for $n \geq 1$ and $a_n = e^{in\theta}$ for $0 < \theta < 2\pi$.
   $|\sum_{n=p}^{q} a_n| \leq \dfrac{1}{\sin\theta/2}$ and we can prove that $\sum_{n=2}^{+\infty} |b_n - b_{n-1}| \leq \sum_{n=2}^{+\infty} \dfrac{2}{(n-1)^2}$.
   It results that the series $\sum_{n\geq 1} \dfrac{(-1)^{[\sqrt{n}]} e^{in\theta}}{n}$ converges for all $0 < \theta < 2\pi$.

2. Let $s_n = \sum_{k=1}^{n} \dfrac{1}{k} - \ln n$, $n \geq 1$. We set $u_1 = S_1 = 1$ and for all $n \geq 2$; $u_n = S_n - S_{n-1} = \dfrac{1}{n} + \ln\dfrac{n-1}{n} = \dfrac{1}{n} + \ln(1 - \dfrac{1}{n}) = \dfrac{1}{n} + (-\dfrac{1}{n} - \dfrac{1}{2n^2} + o(\dfrac{1}{n^2}))$, then $u_n = \frac{-1}{2n^2} + o(\frac{1}{n^2})$, thus $(s_n)_n$ converges. We set $\gamma = \lim\limits_{n\to+\infty} s_n$, γ is called the "**Euler constant**.

1.6 Exercises

2-1-1 Consider a sequence $(u_n)_{n\geq 1}$ of real numbers such that the series $\sum_{n\geq 1} n\, u_n$ is convergent. Prove that the series $\sum_{n\geq 1} u_n$ is convergent.

2-1-2 Let $(u_n)_{n\geq 1}$ be a decreasing sequence such that the series $\sum_{n\geq 1} u_n$ is convergent.

(a) Prove that $\lim_{n\to+\infty} nu_n = 0$.

(b) Prove that $\sum_{n\geq 1} n(u_n - u_{n+1})$ converges and $\sum_{n=1}^{+\infty} n(u_n - u_{n+1}) \sum_{n=1}^{+\infty} u_n$.

(c) Compute for $0 \leq r < 1$ the following sums:
$\sum_{n=1}^{+\infty} nr^n$ and $\sum_{n=1}^{+\infty} n^2 r^n$.

2-1-3 (a) Prove that the series $\sum_{n\geq 0} \frac{(-1)^n}{n+1}$ is convergent.

(b) Show that $\left| \sum_{k=0}^{n} \frac{(-1)^k}{k+1} - \int_0^1 \frac{dt}{1+t} \right| \leq \frac{1}{n+2}$.

(c) Deduce that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \ln 2$.

2-1-4 Find the following sums:

1) $\sum_{n=2}^{+\infty} \frac{1}{n^2-1}$,

2) $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)(n+2)}$,

3) $\sum_{n=1}^{+\infty} \frac{n^2}{n!}$,

4) $\sum_{n=0}^{+\infty} \frac{2n^3+1}{n!}$,

5) $\sum_{n=2}^{+\infty} \ln\left(1 - \frac{1}{n^2}\right)$,

6) $\sum_{n=1}^{+\infty} \ln\left(\cos \frac{1}{2^n}\right)$,

7) $\sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x (\ln t)^n dt$.

2-1-5 Study the convergence of the following series:

1) $\sum_{n\geq 1} \frac{2^n n!}{n^n}$,

2) $\sum_{n\geq 1} \frac{3^n n!}{n^n}$,

3) $\sum_{n\geq 1} \frac{n!}{n^n}$,

4) $\sum_{n\geq 2} \frac{(-1)^n \ln n}{n}$,

5) $\sum_{n\geq 2} \ln\left(1 + \frac{(-1)^n}{n^\alpha}\right)$,

6) $\sum_{n\geq 1} \left(\frac{n}{n+1}\right)^{n^2}$,

7) $\sum_{n\geq 0} \frac{\cos n}{\sqrt{n} + \cos n}$,

8) $\sum_{n\geq 0} \frac{1}{C_{2n}^n}$,

9) $\sum_{n\geq 1} \frac{(2n)!}{n^n (n-1)!}$,

10) $\sum_{n\geq 1} n \sin(\frac{1}{n})$,

11) $\sum_{n\geq 1} e - \left(1 + \frac{1}{n}\right)^n$,

12) $\sum_{n\geq 1} \cosh^\alpha n - \sinh^\alpha n$,

13) $\sum_{n\geq 1} \cos^{-1}\left(\frac{n^3+1}{n^3+2}\right)$,

14) $\sum_{n\geq 1} \ln \frac{(n^3+1)^2}{(n^2+1)^3}$,

15) $\sum_{n\geq 1} (\frac{1}{2})^{\sqrt{n}}$,

16) $\sum_{n\geq 1} \sqrt{1 + \frac{(-1)^n}{\sqrt{n}}} - 1$,

17) $\sum_{n\geq 1} \frac{(\ln n)^n}{n^{\ln n}}$,

18) $\sum_{n\geq 1} \frac{1}{(\ln n)^{\ln n}}$,

19) $\sum_{n\geq 1} \sin\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)$,

20) $\sum_{n\geq 1} \frac{(-1)^n}{n^\alpha + (-1)^n}$,

21) $\sum_{n\geq 3} \frac{1}{n \ln n \big(\ln(\ln n)\big)^\alpha}$,

22) $\sum_{n\geq 1} \left(\cos\frac{1}{\sqrt{n}}\right)^n - \frac{1}{\sqrt{e}}$,

23) $\sum_{n\geq 1} \ln\frac{1}{\sqrt{n}} - \ln\left(\sin\frac{1}{\sqrt{n}}\right)$,

2-1-6 Let a, b and c three real numbers. Consider the sequence $(u_n)_n$ defined by:

$$u_n = a \ln n + b \ln(n+1) + c \ln(n-1), \; n \geq 2.$$

(a) Express in term of a, b and c the necessary condition of the convergent of the series $\sum_{n\geq 2} u_n$.

(b) If this condition is satisfied, prove that the series $\sum_{n\geq 2} u_n$ is absolutely convergent.

(c) Chooses $a=-2$, $b=c=1$, prove that the series $\sum_{n\geq 2} u_n$ is convergent and compute its sum.

2-1-7 Consider $f(n)=\dfrac{n!}{n^n e^{-n}\sqrt{n}}$ and $S_n=\ln f(n)$, for $(n\geq 1)$.

(a) Prove that the series $\sum_{n\geq 2} u_n$ is convergent, where $u_n=S_n-S_{n-1}$.

(b) Deduce the convergence of the sequence $(S_n)_n$.

(c) Set $\ell=\lim_{n\to+\infty} S_n$. Determine in term of ℓ an equivalent of $n!$ when $n\to+\infty$.

2-1-8 Define the sequence of real numbers $(u_n)_n$ by:
u_0 arbitrary and $u_{n+1}=1-e^{-u_n}$, $\forall n\geq 0$.

(a) Study the convergent of the sequence $(u_n)_n$.

(b) Assume $u_0>0$, compute $\lim_{n\to+\infty}\dfrac{u_{n+1}-u_n}{u_n^2}$ and study the convergence of the series $\sum_{n\geq 0} u_n^2$.

2-1-9 Verify that the series $\sum_{n\geq 1}\dfrac{(-1)^n}{\sqrt{n}}+\dfrac{1}{n}$ is alternate and divergent. Conclude.

2-1-10 (a) Consider the function $f(x)=|\sin(2\pi x)|$, for $x\geq 1$.

Prove that $\displaystyle\int_1^{+\infty} f(t)dt$ diverges and the series $\sum_{n\geq 1} f(n)$ converges.

(b) Consider the function

$$g(x)=\begin{cases} n^2x+1-n^3 & \text{for} \quad x\in\left[n-\frac{1}{n^2},n\right] \quad (n\geq 2)\\ -n^2x+1+n^3 & \text{for} \quad x\in\left[n,n+\frac{1}{n^2}\right] \quad (n\geq 2)\\ 0 & \text{for} \quad x \text{ does not in any of these intervals}\end{cases}.$$

Prove that $\displaystyle\int_0^{+\infty} g(t)dt$ converges and the series $\sum_{n\geq 1} g(n)$ diverges.

Conclude.

2-1-11 Let f be a function of class C^1 such that the integral $\displaystyle\int_0^{+\infty} f(t)dt$ is convergent and the integral $\displaystyle\int_0^{+\infty} f'(t)dt$ is absolutely convergent.

(a) Prove that the series $\displaystyle\sum_{n\geq 0} f(n)$ converges. (Hint: We can use Taylor formula with integral remainder).

(b) Study the convergence of the following series $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin(\pi\sqrt{n})}{n}$.

2-1-12 (a) Prove that for any $\theta \in \left]0, \frac{\pi}{2}\right[$:

$$\sin(2m+1)\theta = (\sin^{2m+1}\theta)P_m(\cot^2\theta),$$

where P_m the polynomial defined by: $P_m(x) = \displaystyle\sum_{k=0}^{m}(-1)^k C_{2m+1}^{2k+1} x^{m-k}$.

(One will be able to use the Moivre Formula).

(b) Deduce the roots of the polynomial P_m and the following relation

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

(c) Prove that: $\quad \forall t \in \left]0, \frac{\pi}{2}\right[, \quad \cot^2 t \leq \frac{1}{t^2} \leq \cot^2 t + 1.$

(d) Apply this result for $t = \dfrac{k\pi}{2m+1}$, deduce that $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

2-1-13 Let $\displaystyle\sum_{n\geq 0} u_n$ and $\displaystyle\sum_{n\geq 0} v_n$ be two convergent series with non negative terms.

(a) Prove that the series $\displaystyle\sum_{n\geq 0} u_n^2$ and $\displaystyle\sum_{n\geq 0} \sqrt{u_n v_n}$ are convergent.

Let $\displaystyle\sum_{n\geq 0} w_n$ be a series with non negative terms and such that $\displaystyle\lim_{n\longrightarrow+\infty}(nw_n) = \ell$.

(b) Prove that if the series $\displaystyle\sum_{n\geq 0} w_n$ is convergent, then $\ell = 0$.

2-1-14 Let u_0 be a number real of $]0,1[$ and define the sequence $(u_n)_n$ by: $u_{n+1} = u_n - u_n^2$.

(a) Prove that the sequence $(u_n)_n$ is a decreasing sequence.

(b) Prove that $\forall n \in \mathbb{N},\ u_n \in]0,1[$.

(c) Deduce that the sequence $(u_n)_n$ is convergent and compute its limit.

(d) Prove that the series $\sum_{n\geq 0} u_n^2$ converges and give its sum.

(e) Prove that the series $\sum_{n\geq 0} \ln(\frac{u_{n+1}}{u_n})$ and $\sum_{n\geq 0} u_n$ are divergent.

(f) Define for $n \in \mathbb{N}$, $v_n = \dfrac{1}{u_n} - \dfrac{1}{u_{n-1}}$.

  i. Prove that $\lim_{+\infty} v_n = 1$.

  ii. Deduce that $u_n \approx \frac{1}{n}$.

  iii. Study the convergence of the series $\sum_{n\geq 1} \sin(u_n^2)$ and $\sum_{n\geq 1} \dfrac{u_n}{\sqrt{n}}$.

2-1-15 Let $(u_n)_n$ be a sequence of real numbers. Assume that $|u_n| < 1$, for any $n \in \mathbb{N}$.

(a) Prove that the series $\sum_{n\geq 0} \ln(1+u_n)$ is absolutely convergent if and only if the series $\sum_{n\geq 0} u_n$ is absolutely convergent.

(b) What can we say about the convergence?

(c) Assume that the series $\sum_{n\geq 0} u_n$ is absolutely convergent.

(a) Prove that the series $\sum_{n\geq 0} u_n^2$, $\sum_{n\geq 0} \dfrac{u_n}{1+u_n}$ are absolutely convergent.

(b) What can we say about the convergence?

2-1-16 Let $(u_n)_n$ be a sequence of non negative numbers. Define $v_n = \dfrac{u_n}{1+u_n}$.

Prove that the series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ converge or diverge together.

2-1-17 Let $(u_n)_n$, $(v_n)_n$ and $(w_n)_n$ be three reals sequences such that the series $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} w_n$ converge, and $u_n \leq v_n \leq w_n$ for any n.

Prove that the series $\sum_{n\geq 0} v_n$ is convergent.

2-1-18 Consider the sequence $(u_n)_n$, with $u_n = \displaystyle\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(x^2)dx$.

(a) Prove that the series $\displaystyle\sum_{n\geq 1} u_n$ is an alternate series.

(b) Prove that $\forall n \in \mathbb{N}, \quad |u_n| = \displaystyle\int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{2\sqrt{t}}dt.$

Deduce that the series $\displaystyle\sum_{n\geq 1} u_n$ is convergent.

Prove that it is conditionally convergent.

2-1-19 Study the convergence and the absolutely convergence of the following series $\displaystyle\sum_{n\geq 2} u_n$, where $u_n = \dfrac{(-1)^n}{n^{\frac{3}{4}} + \cos n}$.

2-1-20 Let $(u_n)_{n\geq 0}$ be a sequence defined by : $u_0 > 0, \forall\, n \in \mathbb{N},\ u_{n+1} = u_n + u_n^2$.

(a) Prove that $\displaystyle\lim_{n\to+\infty} u_n = +\infty$.

(b) Set $v_n = 2^{-n}\ln u_n$.

Prove that the sequence $(v_n)_n$ is convergent. (Study the series $\displaystyle\sum_{n\geq 0} v_{n+1} - v_n$)

(c) Deduce that there exists $\alpha > 0$ such that $u_n \approx \alpha^{2^n}$.

2-1-21 Let $f(x) = \dfrac{1}{1 + \cosh x \sin^2 x}$ defined on $[0, +\infty[$.

(a) Prove that $\displaystyle\int_0^{+\infty} f(x)dx$ is convergent if and only if the series $\displaystyle\sum_{n\geq 0} u_n$ is convergent, with $u_n = \displaystyle\int_{n\pi}^{(n+1)\pi} f(x)dx$.

(b) Prove that for any $n \in \mathbb{N},\ 0 \leq u_n \leq \displaystyle\int_{n\pi}^{(n+1)\pi} \frac{1}{1 + \frac{e^{n\pi}}{2}\sin^2 x}dx.$

(c) Deduce that $\forall n \in \mathbb{N}, u_n \leq \dfrac{\pi^2}{\sqrt{2}}e^{\frac{-n\pi}{2}}$ and that the integral $\displaystyle\int_0^{+\infty} f(x)dx$ is convergent.

2-1-22 Let $(a_n)_n$ be a sequence of non negative numbers such that the series $\sum_{n\geq 0} a_n$ is convergent. Define the sequences $(R_n)_n$ and $(b_n)_n$ by: $R_n = \sum_{k=n+1}^{+\infty} a_k$ and $b_n = \frac{a_n}{R_{n-1}^{\alpha}}$, with $\alpha \in]0,1[$ fixed.

(a) Prove that for any $n \in \mathbb{N}^*$, $b_n \leq \frac{R_{n-1}^{1-\alpha} - R_n^{1-\alpha}}{1-\alpha}$. (We will be able to use the integral $\int_{R_n}^{R_{n-1}} \frac{dt}{t^{\alpha}}$).

Deduce that the series $\sum_{n\geq 1} b_n$ is convergent.

(b) Set for any $n \in \mathbb{N}^*$, $c_n = \frac{a_n}{R_n}$, $d_n = \frac{a_n}{R_{n-1}}$ and $e_n = \ln(\frac{R_{n-1}}{R_n})$.

Prove that the series $\sum_{n\geq 1} c_n$ and $\sum_{n\geq 1} d_n$ are divergent. (Prove that the series $\sum_{n\geq 1} e_n$ diverges and $0 \leq e_n \leq c_n$ and $c_n = \frac{d_n}{1-d_n}$.)

(c) If (u_n) is a given non negative sequence such that the series $\sum_{n\geq 0} u_n$ converges, is there exists a sequence $(v_n)_n$ such that the series $\sum_{n\geq 0} u_n v_n$ converges and $\lim_{n\to+\infty} v_n = +\infty$?

2 Series Product

Definition 2.1.
Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers. For $n \in \mathbb{N}$, we set

$$c_n = \sum_{k=1}^{n} u_k v_{n-k}. \tag{2.2}$$

The series $\sum_{n\geq 1} c_n$ is called the series product of the two given series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$.

In this definition we are not interested in whether the product of the series exists, because it depends on some conditions. Indeed we have the following example:

Consider $\sum_{n\geq 1} c_n$ the series product of the series $\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n+1}}$ with itself. The series $\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n+1}}$ is convergent but the series $\sum_{n\geq 1} c_n$ is divergent. Indeed:

$$c_n=\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{k+1}}\frac{(-1)^{n-k}}{\sqrt{n-k+1}}=(-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{k+1}\sqrt{n-k+1}}.$$

Then $|c_n|\geq 1$ and the series $\sum_{n\geq 1} c_n$ is divergent.

The following theorem affirms the existence of the series product under some conditions.

Theorem 2.2.

Let $(u_n)_n$ and $(v_n)_n$ be two sequences of real numbers.

1. Assume that the series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} v_n$ are absolutely convergent. Then the series $\sum_{n\geq 1} c_n$ is absolutely convergent and we have
$$\sum_{n=1}^{+\infty} c_n=(\sum_{n=1}^{+\infty} u_n)(\sum_{n=1}^{+\infty} v_n). \tag{2.3}$$

2. Assume that the series $\sum_{n\geq 1} u_n$ is absolutely convergent and the series $\sum_{n\geq 1} v_n$ is convergent. Then the series $\sum_{n\geq 1} c_n$ is convergent and we have:
$$\sum_{n=1}^{+\infty} c_n=(\sum_{n=1}^{+\infty} u_n)(\sum_{n=1}^{+\infty} v_n). \tag{2.4}$$

Proof .

It suffices to proves 2). We set

$$A_n=\sum_{k=1}^{n} u_k,\qquad B_n=\sum_{k=1}^{n} v_k,\qquad C_n=\sum_{k=1}^{n} c_k,$$

$$A=\sum_{n=1}^{+\infty} u_n,\qquad \alpha=\sum_{n=1}^{+\infty}|u_n|\quad\text{and}\quad B=\sum_{n=1}^{+\infty} v_n.$$

Then

$$C_n=\sum_{j=1}^{n} c_j=\sum_{j=1}^{n} u_jB_{n-j}=\sum_{j=1}^{n} u_j(B_{n-j}-B)+BA_n.$$

Since $\lim_{n\to+\infty} B.A_n = A.B$, then to show that $\lim_{n\longrightarrow+\infty} C_n = A.B$, it suffices to show that the sequence $(\Delta_n)_n$ converges to 0, where $\Delta_n = \sum_{j=1}^{n} a_j(B_{n-j} - B)$.

Let $\varepsilon > 0$: $\exists N \in \mathbb{N}$ such that $|B_n - B| < \frac{\varepsilon}{2\alpha}$ and $\displaystyle\sum_{j=N}^{+\infty} |a_j| \leq \frac{\varepsilon}{2M}$, $\forall\ n \geq N$.

Thus for every $n \geq 2N$,

$$|\Delta_n| \leq \sum_{j=1}^{N} |a_j||B_{n-j} - B| + \sum_{j=N+1}^{n} |a_j||B_{n-j} - B| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

It results that $\lim_{n\longrightarrow+\infty} |\Delta_n| = 0$. $\square$

2.1 Exercises

2-2-1

3 Integrals Depending on Parameters

We recall in this chapter, that a piecewise continuous function f is called integrable on I if the integral $\displaystyle\int_I |f(x)|dx$ is convergent.

1 Convergence Theorem

Theorem 1.1. [Monotone Convergence Theorem]
Let $\Big(f_n\colon I \longrightarrow \mathbb{R}\Big)_n$ be a sequence of integrable piecewise continuous functions on I. Assume that

i) the sequence $(f_n)_n$ is increasing, (i.e. $f_n \leq f_{n+1}$)

ii) the sequence $(f_n)_n$ is pointwise convergent to a integrable piecewise continuous function f on I.

Then f is integrable on I if and only if the sequence $\Big(\int_I f_n(x)dx\Big)_n$ is bounded above. Moreover with these assumptions

$$\int_I f(x)dx = \sup_{n\in\mathbb{N}} \int_I f_n(x)dx = \lim_{n\to+\infty} \int_I f_n(x)dx.$$

Remark 11 :
Let $\Big(f_n\colon I \longrightarrow \mathbb{R}\Big)_n$ be a sequence of integrable piecewise continuous functions on I. We assume that

i) the sequence $(f_n)_n$ is decreasing, (i.e. $f_n \geq f_{n+1}$)

ii) the sequence $(f_n)_n$ is pointwise convergent to a integrable piecewise continuous function f on I. Then f is integrable on I if and only if the sequence $\Big(\int_I f_n(x)dx\Big)_n$ is lower bounded. Moreover with these assumptions

$$\int_I f(x)dx = \inf_{n\in\mathbb{N}} \int_I f_n(x)dx = \lim_{n\to+\infty} \int_I f_n(x)dx.$$

Theorem 1.2. [Dominated Convergence Theorem]
Let $\Big(f_n\colon I \longrightarrow \mathbb{R}\Big)_n$ be a sequence of integrable piecewise continuous functions on I. We assume that

i) the sequence $(f_n)_n$ is increasing, (i.e. $f_n \leq f_{n+1}$),

ii) the sequence $(f_n)_n$ is pointwise convergent to a integrable piecewise continuous function f on I,

iii) there exists an integrable function $\varphi\colon I \longrightarrow \mathbb{R}^+$ such that $|f_n| \leq \varphi$, for any $n \in \mathbb{N}$. (This assumption is called the domination assumption).

Then for any $n \in \mathbb{N}$, f_n is integrable on I and f is integrable on I. Moreover

$$\lim_{n\to+\infty} \int_I f_n(x)dx = \int_I f(x)dx.$$

1.1 Continuity

Theorem 1.3.
Let Ω be a subset of $\mathbb{R}^m$ and $f\colon \Omega \times I \longrightarrow \mathbb{C}$ a continuous function on $\Omega \times I$ and fulfills the domination assumption, (i.e. there exists an integrable function $\varphi\colon I \longrightarrow \mathbb{R}^+$ such that $|f(x,t)| \leq \varphi(t)$, for all $x \in \Omega$.) Then the function $x \longmapsto F(x) = \displaystyle\int_I f(x,t)dt$ is continuous on Ω.

Theorem 1.4.
Let Ω a subset of $\mathbb{R}^m$ and $f\colon \Omega \times I \longrightarrow \mathbb{C}$ a continuous function on $\Omega \times I$ and fulfills the local domination assumption, (i.e. for any compact $K \subset \Omega$, there exists an integrable function $\varphi\colon I \longrightarrow \mathbb{R}^+$ such that $|f(x,t)| \leq \varphi(t)$, for all $x \in K$.) Then the function $x \longmapsto F(x) = \displaystyle\int_I f(x,t)dt$ is continuous on Ω.

1.2 Differentiability

Theorem 1.5.
Let J be an interval and $f\colon J \times I \longrightarrow \mathbb{R}$ a continuous function on $J \times I$. We assume that

i) For any $x \in J$, the function $t \longmapsto f(x,t)$ is integrable on I

ii) $\dfrac{\partial f}{\partial x}$ exists, continuous on $J \times I$ and fulfills the domination assumption, (i.e. there exists an integrable function $\varphi\colon I \longrightarrow \mathbb{R}^+$ such that $\left|\dfrac{\partial f}{\partial x}\right| \leq \varphi(t)$, for all $x \in J$.)

Then the function $x \longmapsto F(x) = \displaystyle\int_I f(x,t)dt$ is of class $\mathcal{C}^1$ on J.

2 Generalized Integral Depending on Parameter

2.1 Convergence Theorem of Generalized Integral

Let $f(t,x)$ be a function defined on $]a,b[\times]\alpha,\beta[$; with $-\infty \leq a < b \leq +\infty$ and $-\infty \leq \alpha < \beta \leq +\infty$. We intend to study the continuity and the differentiability of the function

$$F(x) = \int_a^b f(t,x)dt.$$

To study this problem it suffices to study the case $a \in \mathbb{R}$. In which follows we consider the case $a \in \mathbb{R}$. To study the function F, we consider a sequence $(u_n)_n$ of $[a,b[$ which converges to b and we study the sequence

$$F_n(x) = \int_a^{u_n} f(t,x)dt$$

and we apply for each function F_n the previous results and deduce the regularity of the function $F = \lim\limits_{n\to+\infty} F_n$.

Definition 2.1.
Let X be a subset of $\mathbb{R}$ and f a function defined on $[a,b[\times X$ such that the integral $\displaystyle\int_a^b f(t,x)dt$ converges for any $x \in X$.
We say that the integral $\displaystyle\int_a^b f(t,x)dt$ converges unoformly on X if, $\forall\varepsilon > 0$, $\exists c$ independent of x such that $|\int_s^b f(t,x)dt| \leq \varepsilon$; for any $c \leq s < b$.

We remark that if the integral $\displaystyle\int_a^b f(t,x)dt$ converges unoformly on X, then for any sequence $(u_n)_n$ of $[a,b[$ convergent to b, the sequence $F_n(x) = \displaystyle\int_a^{u_n} f(t,x)dt$ converges unoformly on X.

Theorem 2.2. [The Cauchy Criterion]
Let X be a subset of $\mathbb{R}$ and f a function defined on $[a,b[\times X$ such that the integral $\displaystyle\int_a^b f(t,x)dt$ converges for any $x \in X$.
The integral $\displaystyle\int_a^b f(t,x)dt$ converges uniformly on X if and only if $\forall\varepsilon > 0$, $\exists c$ independent of x such that $|\int_u^v f(t,x)dt| \leq \varepsilon$, for any $c \leq u \leq v < b$.

Theorem 2.3.
Let X be a subset of $\mathbb{R}$ and f a function defined on $[a,b[\times X$. We assume that there exists an integrable function defined on on $[a,b[$ such that $|f(t,x)| \leq \varphi(t)$, for any $x \in X$. Then

i) The integral $\displaystyle\int_a^b f(t,x)dt$ converges absolutely for any $x \in X$.

ii) The integral $\displaystyle\int_a^b f(t,x)dt$ converges unoformly on X.

Example 2.1 :

1. Consider the integral $\displaystyle\int_0^{+\infty} e^{-t^2} e^{\text{i}tx} dt$, for $x \in \mathbb{R}$. As $|e^{-tx} e^{\text{i}tx}| \leq e^{-t^2}$ which is integrable, thus $\displaystyle\int_0^{+\infty} e^{-t^2} e^{\text{i}tx} dt$ converges unoformly on $\mathbb{R}$.

2. Vonsider the integral $\displaystyle\int_0^{+\infty} e^{-tx} \frac{\sin t}{t} dt$. This integrable converges unoformly on any interval $[a, +\infty[$; for any $a > 0$.

Theorem 2.4. [Abel Rule for the Uniform Convergence]
Let X be subset of $\mathbb{R}$ and f, g two functions defined one $[a, +\infty[\times X$ such that

i) There exists a real M independent of x such that $|\displaystyle\int_a^u f(t,x)dt| \leq M$, for any $u \in [a, +\infty$.

ii) The function $t \longmapsto g(t,x)$ is decreasing for any $x \in X$ and there exists a non negative decreasing function φ on $[a, +\infty[$ such that $|f(t,x)| \leq \varphi(t)$ and $\displaystyle\lim_{t \to +\infty} \varphi(t) = 0$. Then the integral $\displaystyle\int_a^{+\infty} f(t,x)dt$ converges uniformly on X.

Therefore the integral $\displaystyle\int_0^{+\infty} e^{-tx} \frac{\sin t}{t} dt$ converges uniformly on $]0, +\infty[$. It suffices to take $f(t) = \sin t$ and $g(t,x) = \frac{e^{-tx}}{t} \leq \frac{1}{t}$.

2.2 Continuity

Theorem 2.5.

Let f be a continuous function on $[a, b[\times]\alpha, \beta[$ such that the integral $\displaystyle\int_a^b f(t,x)dt$ converges uniformly on any compact $[\zeta, \xi] \subset]\alpha, \beta[$. Then the function

$$F(x) = \int_a^b f(t,x)dt$$

is continuous on $]\alpha, \beta[$.

2.3 Differentiability

Theorem 2.6.
Let f be a continuous function on $[a,b[\times]\alpha,\beta[$ such that $\dfrac{\partial f}{\partial x}$ exists and is continuous on $[a,b]\times]\alpha,\beta[$, for any $x\in]\alpha,\beta[$, the integral $\displaystyle\int_a^b f(t,x)dt$ converges and the integral $\displaystyle\int_a^b \frac{\partial f}{\partial x}(t,x)dt$ converges uniformly on any compact $[\zeta,\xi]\subset]\alpha,\beta[$. Then the function

$$F(x)=\int_a^b f(t,x)dt$$

is differentiable on $]\alpha,\beta[$ and

$$F'(x)=\int_a^b \frac{\partial f}{\partial x}(t,x)dt$$

.

Example 2.2 :

1. Let $F_n(x)=\displaystyle\int_0^1 t^x\ln^n t dt$, for $x\in]-1,0]$. F_n is well defined. Moreover the functions $f_n(t,x)=t^x\ln^n t$ and $\dfrac{\partial f_n}{x}(t,x)=f_{n+1}(t,x)$ are continuous on $]0,1]\times]-1,0]$ and for $x\in[a,0]$, with $-1<a<0$, one has $t^x|\ln^n t|\le t^a|\ln^n t|$. Thus the integral $\displaystyle\int_0^1 t^x\ln^n t dt$ converges uniformly on $[a,0]$ and F_n is continuous and of class C^∞ on $]-1,0]$. $F_n(x)=\dfrac{(-1)^n n!}{(x+1)^{n+1}}$.

2. Consider the function G defined for $x\ge 0$ by:

$$G(x)=\int_0^{+\infty}\frac{e^{-xt^2}}{1+t^2}dt.$$

The function $g(t,x)=\frac{e^{-xt^2}}{1+t^2}$ is continuous on $[0,+\infty[\times[0,+\infty[$. $g(t,x)\le\dfrac{1}{1+t^2}$, thus G is continuous on $[0,+\infty[$.
$\dfrac{\partial g}{\partial x}(t,x)=-t^2e^{\frac{-xt^2}{1+t^2}}$ which is continuous on $[0,+\infty[\times[0,+\infty[$ and $\displaystyle\int_0^{+\infty}\frac{\partial g}{\partial x}(t,x)dt$ converges uniformly on any interval $[a,+\infty[$, for any $a>0$, because

$|\frac{\partial g}{\partial x}(t,x)| \leq e^{-at^2}$ which is integrable, for $x \geq a$. Therefore the function G is differentiable on $]0,+\infty[$ and

$$G'(x) = \int_0^{+\infty} \frac{\partial g}{\partial x}(t,x)dt.$$

2.4 Exercises

3-2-1 Let E be the vector space of continuous functions on $[0,1]$, and let K be the function of two variables defined by:

$$K(x,y)=\begin{cases}(x-1)y & \text{si } y\leq x\\ x(y-1) & \text{si } x\leq y\end{cases}$$

To any function f of E we associate the function

$$\tilde{f}(x)=\int_0^1 K(x,y)f(y)dy.$$

(a) Prove that for any $f\in E$, $\tilde{f}$ is of class $\mathcal{C}^2$, $\tilde{f}(1)=\tilde{f}(0)=0$ and $\tilde{f}''=f$.

(b) Prove that for any f,g of E :

$$\int_0^1 \tilde{g}(x)f(x)dx=\int_0^1 \tilde{f}(x)g(x)dx.$$

3-2-2 (a) Study the convergence of the following integral with respect to the parameter $x\in\mathbb{R}$.

$$\int_1^{+\infty}\frac{t^{-(x+1)}}{\sqrt{t^2-1}}dt.$$

Let I be the set of x for which the integral is convergent.

(b) For $x\in I$, define

$$F(x)=\int_1^{+\infty}\frac{t^{-(x+1)}}{\sqrt{t^2-1}}dt.$$

Prove that F is of class C^∞ on I.

3-2-3 We claim to compute the following integral

$$F(x)=\int_0^{+\infty}\frac{1-\cos(tx)}{t^2}.e^{-t}dt;\quad x>0$$

(a) Verify the existence of this integral.

(b) Prove that $F''(x)=\dfrac{1}{1+x^2}$

(c) Deduce the expression of F.

3-2-4 For $x > 0$ define the functions $F(x) = \int_0^{+\infty} \frac{\sin t}{t+x} dt$ and $G(x) = \int_0^{+\infty} \frac{e^{-tx}}{1+t^2} dt$.

(a) Prove that F and G fulfills the same differential equation $y'' + y = \frac{1}{x}$.

(b) Prove that $F = G$.

(c) Deduce the value of the Dirichlet integral $\int_0^{+\infty} \frac{\sin t}{t} dt$.

3-2-5 Let f be a continuous function and bounded on $\mathbb{R}_+$. We define for $x > 0$ the function $F(x) = \int_0^{+\infty} f(t)e^{-xt} dt$ and $G(x) = \int_0^{+\infty} tf(t)e^{-xt} dt$.

(a) Verify that F and G are well defined for $x > 0$.

(b) Determine the limit of F at $+\infty$.

(c) Prove that F is differentiable and compute $F'(x)$.

3-2-6 Let $\psi(t) = \frac{1}{\pi(1+t^2)}$ and f a continuous function on $\mathbb{R}$ such that $\int_{-\infty}^{+\infty} |f(t)| dt < +\infty$.

Define

$$\varphi(x) = \int_{-\infty}^{+\infty} f(x-t)\psi(t) dt.$$

(a) Prove that φ is continuous on $\mathbb{R}$.

(b) Prove that φ is of class C^∞ on $\mathbb{R}$.

(c) Prove that

$$\int_{-\infty}^{+\infty} \varphi(x) dx = \int_{-\infty}^{+\infty} f(t) dt. \int_{-\infty}^{+\infty} \psi(t) dt.$$

(d) Let $\tilde{\varphi}(x) = \int_{-\infty}^{+\infty} \frac{\cos(x-t)}{\pi(1+t^2)} dt$.

a) Prove that $\tilde{\varphi}$ is of class C^∞ and fulfills a differential equation of second order.

b) Compute $\tilde{\varphi}(0)$ and deduce the expression of $\tilde{\varphi}$.

3-2-7 (a) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$.

a) Compute I_{2p} and I_{2p+1}, for any $p \in \mathbb{N}$.

b) Prove that for any $n \in \mathbb{N}$, $I_n I_{n+1} \leq I_n^2 \leq I_n I_{n-1}$ and deduce the Wallis formula:

$$I_n \sim_{+\infty} \sqrt{\frac{\pi}{2n}}.$$

(b) a) Prove that $f : x \mapsto \int_0^{\frac{\pi}{2}} \sin^x t dt$ is $\mathcal{C}^\infty$ on $]-1, +\infty[$.

b) Give a simple equivalent of f at $+\infty$.

c) Give an asymptotic rxpansion of three terms of f at -1.

3-2-8 Let $F(x) = \displaystyle\int_0^{+\infty} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$.

(a) Prove that F is of class C^1 on $]0, +\infty[$.

(b) Find a relation between $F(x)$ and $F(\frac{1}{x})$.

(c) Determine the limit of $F(x)$ when $x \longrightarrow +\infty$.

(d) Remark that $F(x) > \displaystyle\int_0^1 \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$ and determine $\lim\limits_{x \to 0} F(x)$.

(e) a) Prove that $F(x) = 2 \displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$.

b) Prove that $F(x) \sim_0 2 \displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{x^2+t^2}}$.

c) Deduce a simple equivalent of F in a neighborhood of 0 and $+\infty$.

3-2-9 Define $f(x) = \displaystyle\int_0^1 \frac{t^x(1-t)}{\ln t} dt$.

(a) Determine the domain of definition of f.

(b) Prove that f is differentiable on $]-1, +\infty[$ and determine $f'(x)$ for any $x > -1$.

(c) Give $\lim\limits_{x \longrightarrow +\infty} f(x)$ and deduce the value of $f(x)$ for any $x > -1$.

3-2-10 Let f be a continuous function on $[0, +\infty[$ and

$$\mathcal{D} = \{(u, t) \in \mathbb{R}^2;\ 0 < u < x,\ 0 < t < u\}.$$

Define the function

$$g(x) = \iint_{\mathcal{D}} \frac{f(t)}{\sqrt{(x-u)(u-t)}} dudt.$$

(a) Prove that g is well defined.

(b) Compute

$$\int_t^x \frac{du}{\sqrt{(x-u)(u-t)}.}$$

(We will be able make the change of variables $u = t\cos^2\varphi + x\sin^2\varphi$.)

(c) Prove that $g(x) = \pi \displaystyle\int_0^x f(t)dt$ and deduce the expression of f in term of g.

3-2-11 Define the function F by:

$$F(x) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2\sin^2 t}} dt.$$

(a) Prove that the domain of definition of F is $]-1,1[$.

(b) Prove that F is of class $\mathcal{C}^2$ on $]-1,1[$, and give the expression in integral form of F' and F''.

(c) a) Use the change of variables $u = x\sin t$ to prove that

$$F(x) \geq \int_0^x \frac{du}{1-u^2}, \quad 0 < x < 1.$$

b) Deduce $\lim_{x\to 1^-} F(x)$.

4 Sequences and Series of Functions

1 Sequences of Functions

Definition 1.1.
Let $(f_n)_n$ be a sequence of functions defined on a subset A of $\mathbb{R}$.

1. The sequence $(f_n)_n$ is called pointwise convergent on A if for every $x \in A$, the sequence $(f_n(x))_n$ is convergent.
2. The sequence $(f_n)_n$ is called uniformly convergent to f on A if

$$\lim_{n\to+\infty} \sup_{x\in A} \|f_n(x) - f(x)\| = 0.$$

Remark 12 :

1. The sequence $(f_n)_n$ converges to f on A if and only if

$$\forall x \in A,\ \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \ |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N.$$

2. The sequence $(f_n)_n$ converges uniformly to f on A if and only if

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \ |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq N \text{ and } \forall x \in A.$$

Examples 3 :

1. Let $(f_n)_n$ the sequence of functions defined on $I = [0,1]$ by: $f_n(x) = x^n$, for all $x \in I$ and $n \in \mathbb{N}$. The sequence $(f_n)_n$ converges to the function f defined by:

$$f(x) = \begin{cases} 0 & \text{if} \quad 0 \leq x < 1 \\ 1 & \text{if} \quad x = 1 \end{cases}$$

$\sup\limits_{x \in [0,1]} |f_n(x) - f(x)| = \sup\limits_{x \in [0,1[} x^n = 1$, then the sequence $(f_n)_n$ is not uniformly convergent on $[0,1]$ and also on $[0,1[$. Moreover, the sequence $(f_n)_n$ converges uniformly on any interval $[0,a]$, $\forall a \in [0,1[$. Indeed, $\lim\limits_{n \to +\infty} \left(\sup\limits_{x \in [0,a]} x^n \right) = \lim\limits_{n \to +\infty} a^n = 0$.

2. Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}$ by: $f_n(x) = \dfrac{\sin(nx)}{n}$. The sequence $(f_n)_n$ converges uniformly to 0 on $\mathbb{R}$. ($|f_n(x)| \leq \frac{1}{n}$).

3. Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}^+ = [0,+\infty[$ by: $f_n(x) = \dfrac{x}{n+x}$. The sequence $(f_n)_n$ converges to 0 on $\mathbb{R}^+$ and not uniformly since $\sup\limits_{x \in \mathbb{R}^+} f_n(x) = 1$. Moreover the sequence $(f_n)_n$ converges uniformly on any closed interval $[a,b] \subset \mathbb{R}^+$.

4. Let $f_n(x) = xe^{-nx}$ for $x \in \mathbb{R}^+$. We have $\sup_{x \in \mathbb{R}^+} f_n(x) = \frac{1}{n}$. Then the sequence $(f_n)_n$ converges uniformly to 0 on $\mathbb{R}^+$.

5. Let $f_n(x) = \dfrac{x\sqrt{n}}{1+nx^2}$ for $x \in \mathbb{R}$. The sequence $(f_n)_n$ converges to 0, but $\sup_{x \in \mathbb{R}} f_n(x) = \frac{1}{2e}$. Then the sequence $(f_n)_n$ is not uniformly convergent on $\mathbb{R}$. Moreover for all $a > 0$, the sequence $(f_n)_n$ converges uniformly on $[a,+\infty[$. Indeed for n large enough $\sup_{x \in [a,+\infty[} f_n(x) = f_n(a)$.

1.1 Cauchy Criterion for the Convergence

Theorem 1.2. (Cauchy Criterion for the uniform convergence)
Let $(f_n)_n$ be a sequence of functions defined on an open subset Ω of $\mathbb{R}$. The sequence $(f_n)_n$ converges uniformly on a $A \subset \Omega$ if and only if

$$\lim_{p,q \to +\infty} \sup_{x \in A} |f_p(x) - f_q(x)| = 0.$$

This is still equivalent to:

$$\forall\, \varepsilon > 0, \exists N,\ \sup_{x \in A} |f_{n+p}(x) - f_n(x)| \leq \varepsilon, \quad \forall\, n \geq N,\ \forall\, p \in \mathbb{N}.$$

Remark 13 :
If the sequence $(f_n)_n$ converges uniformly to f on $A \subset \Omega$, then for any sequence $(x_n)_n \in A$, the sequence $\big(u_n = |f_n(x_n) - f(x_n)|\big)_n$ converges to 0. This is because $u_n \leq \sup\limits_{x \in A} |f_n(x) - f(x)|$.

1.2 Continuity and Uniform Convergence

Theorem 1.3.
Let $(f_n)_n$ be a sequence of functions defined on an open subset $\Omega \subset \mathbb{R}$ which converges uniformly to f on a subset $I \subset \Omega$. Let $a \in I$ and assume that $\lim\limits_{x \to a} f_n(x) = \ell_n$ exists for any n, then the sequence $(\ell_n)_n$ converges and $\lim\limits_{x \to a} f(x) = \lim\limits_{n \to +\infty} \ell_n$. Otherwise

$$\lim_{\substack{x \to a, \\ x \in I}} \big(\lim_{n \to +\infty} f_n(x) \big) = \lim_{n \to +\infty} \big(\lim_{\substack{x \to a, \\ x \in I}} f_n(x) \big). \tag{4.1}$$

Proof .
To prove that the sequence $(\ell_n)_n$ is convergent, we prove that it is a Cauchy sequence.
For $\varepsilon > 0$, there exists $\exists N$ such that $|f_n(x) - f_m(x)| \leq \varepsilon$, $\forall n, m \geq N$ and $\forall x \in I$. The inequality is still true if x tends to a. Then $\forall \varepsilon > 0$, $|\ell_n - \ell_m| \leq \varepsilon$, $\forall n, m \geq N$. The sequence $(\ell_n)_n$ is a Cauchy sequence in $\mathbb{R}$. Let $\ell = \lim\limits_{n \to +\infty} \ell_n$.
For $n_0 \geq N$, we have:

$$|f(x) - \ell| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - \ell_{n_0}| + |\ell_{n_0} - \ell|.$$

Since the sequence $(f_n)_n$ converges uniformly to f, $|f(x) - f_{n_0}(x)| < \varepsilon$, $\forall x \in I$. (We take $m = n_0$ and we tends n to $+\infty$). Since $\lim_{x \to a} f_{n_0}(x) = \ell_{n_0}$, there exists $\eta > 0$ such that $\forall x \in I$, with $0 < |x - a| < \eta$ we have: $|f_{n_0}(x) - \ell_{n_0}| < \varepsilon \Rightarrow |\ell_{n_0} - \ell| < \varepsilon$. We have: $\forall x \in I$ such that $|x - a| < \eta$, $|f(x) - \ell| < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$, which proves the result.

$\square$

Example 1.1 :
Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}^+$ by: $f_n(x) = \displaystyle\int_0^n \frac{\sin t}{t} e^{-xt} dt$.
$f_n(x) - f_m(x) = \displaystyle\int_n^m \frac{\sin t}{t} e^{-xt} dt$, $\quad (m > n)$. The function $t \longrightarrow \frac{e^{-xt}}{t}$ is decreas-

ing on $[n, m]$, by the second mean formula, [1] $|f_n(x) - f_m(x)| \leq \dfrac{e^{-xn}}{n}.2 \leq 2/n$, then $\sup\limits_{x\in\mathbb{R}+} |f_n(x) - f_m(x)| \leq 2/n$, which proves that the sequence $(f_n)_n$ converge uniformly on $\mathbb{R}^+$.

Moreover $\lim\limits_{x\to 0^+} f_n(x) = \displaystyle\int_0^n \frac{\sin t}{t} dt$, because $\left| f_n(x) - \displaystyle\int_0^n \frac{\sin t}{t} dt \right| \leq xn \xrightarrow[x\to 0^+]{} 0$. Then

$$\lim_{x\to 0^+} \int_0^{+\infty} \frac{\sin t}{t} e^{-xt} dt = \int_0^{+\infty} \frac{\sin t}{t} dt.$$

Theorem 1.5.

Let $(f_n)_n$ be a sequence of functions defined on an open subset $I \subset \mathbb{R}$. Assume that:

1. The sequence $(f_n)_n$ converges uniformly to f on any closed interval $[a, b] \subset I$,
2. For any $n \in \mathbb{N}$, the function f_n is continuous at $c \in I$.

Then f is continuous at c.

Proof .

We consider a sequence $(x_n)_n \in \Omega$ which converges to c. By Theorem 1.3 $\lim\limits_{x\to c} f(x) = \lim\limits_{n\to+\infty} f_n(c) = f(c)$. □

Examples 4 :

1. Let $(f_n)_n$ be the sequences of functions defined on $\mathbb{R}^+$ by: $f_n(x) = \displaystyle\int_0^n \frac{\sin t}{t+x} dt$. The function f_n are continuous on $\mathbb{R}^+$.

   $|f_n(x) - f_n(0)| \leq \left| \displaystyle\int_0^n \frac{\sin t}{t} (\frac{x}{t+x}) dt \right| \leq x(\ln(n+x) - \ln x)$, then $\lim\limits_{x\to 0} |f_n(x) - f_n(0)| = 0$. It results that f_n is continuous at 0.

   For $x_0 > 0$, $|f_n(x) - f_n(x_0)| \leq M_n(x_0)|x - x_0|$, $\forall x > \frac{x_0}{2}$, with $M_n(x_0) = \displaystyle\int_0^n \frac{dt}{(t+x_0)(t+\frac{x_0}{2})}$.

---

[1]

Theorem 1.4. [Second Mean Formula]

Let f be non negative decreasing continuous function on the interval $[a, b]$ and let g be a Riemann integrable function on $[a, b]$. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)\ dx = f(a) \int_a^c g(x)\ dx.$$

In use the second mean formula, we get: $|f_n(x) - f_m(x)| \leq \frac{2}{n+x}$, for all $n < m$ and $x > 0$. Then $\sup\limits_{x \in \mathbb{R}^+} |f_n(x) - f_m(x)| \leq \dfrac{2}{n}$ and the sequence $(f_n)_n$ converges uniformly on $\mathbb{R}^+$. It results that the function f defined by $f(x) = \displaystyle\int_0^{+\infty} \frac{\sin t}{t+x} dt$ is continuous on $\mathbb{R}^+$.

2. For $x > 0$, we set $f_n(x) = \dfrac{1}{x} \displaystyle\int_0^n \frac{\sin t}{t+x} dt$. The functions f_n are continuous on $\mathbb{R}_+^*$. The sequence $(f_n)_n$ convergences uniformly on $[h, +\infty[$, $\forall h > 0$. It results that the function g defined by $g(x) = \dfrac{1}{x} \displaystyle\int_0^{+\infty} \frac{\sin t}{t+x} dt$ is continuous on $\mathbb{R}_+^*$.

Theorem 1.6.
Let $(f_n)_n$ be a sequence of continuous functions on an open set $\Omega \subset \mathbb{R}$ and converges uniformly on compact subsets of I to a function f. Then f is continuous on I.

1.3 Integrability and Uniform Convergence

Let $(f_n)_n$ be a sequence of Riemann integrable functions on an interval $[a, b]$. Assume that the sequence $(f_n)_n$ converges to the function f. Various problems arise, however

1. the function f is it Riemann integrable?

2. if f is Riemann integrable on $[a, b]$, can we have
$$\lim_{n \to +\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt?$$

The answer to the question a) is negative, it suffices to take the function f defined on $[a, b]$ by:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a, b] \\ 0 & \quad\text{if not} \end{cases}$$

This function is not Riemann integrable and it is a limit of Riemann integrable functions. ($\mathbb{Q}$ is countable).
The answer to the question b) is also negative. We can take $f_n(x) = nx(1-x^2)^n$ defined on $[0, 1]$. The sequence $(f_n)_n$ converges to 0 and $\lim\limits_{n \to +\infty} \displaystyle\int_0^1 f_n(x)dx = \dfrac{1}{2}$.
We still have the following theorem:

Theorem 1.7.
Let $(f_n)_n$ be a sequence of Riemann-integrable functions on an interval $[a, b]$. If the sequence $(f_n)_n$ converges uniformly to a function f on $[a, b]$, then f is Riemann-integrable on $[a, b]$ and we have:

$$\lim_{n \to +\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt.$$

Moreover the sequence $(F_n)_n$ defined by: $F_n(x) = \displaystyle\int_a^x f_n(t)dt$ converges uniformly to the function F defined by: $F(x) = \displaystyle\int_a^x f(t)dt$ on $[a, b]$.

Proof .
As the sequence $(f_n)_n$ is uniformly convergent to f on $[a, b]$, the function f is bounded. Indeed, for $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $\sup\limits_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon$, $\forall n \geq N_\varepsilon$. Then $\sup\limits_{x\in[a,b]} |f(x)| \leq \sup\limits_{x\in[a,b]} |f_{N_\varepsilon}(x)| + \varepsilon < +\infty$.
Let $\sigma = \{x_1, \ldots, x_p\}$ be a partition of $[a, b]$ and let $n \geq N_\varepsilon$. As $\forall x \in [a, b]$ $f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon$, we have:
$M_k^n - \varepsilon \leq M_k \leq M_k^n + \varepsilon$ and $m_k^n - \varepsilon \leq m_k \leq m_k^n + \varepsilon$, with $M_k = \sup\limits_{x\in[x_k, x_{k+1}]} f(x)$, $M_k^n = \sup\limits_{x\in[x_k, x_{k+1}]} f_n(x)$, $m_k = \inf\limits_{x\in[x_k, x_{k+1}]} f(x)$ and $m_k^n = \inf\limits_{x\in[x_k, x_{k+1}]} f_n(x)$.
It results that:

$$U(f_n, \sigma) - \varepsilon(b - a) \leq U(f, \sigma) \leq U(f_n, \sigma) + \varepsilon(b - a)$$

$$L(f_n, \sigma) - \varepsilon(b - a) \leq L(f, \sigma) \leq L(f_n, \sigma) + \varepsilon(b - a).$$

Then

$$L(f_n) - \varepsilon(b - a) \leq L(f) \leq U(f) \leq U(f_n) + \varepsilon(b - a). \tag{4.2}$$

Since the functions f_n are Riemann integrable, we have $U(f_n) = L(f_n)$ for all $n \in \mathbb{N}$, and $0 \leq U(f) - L(f) \leq 2\varepsilon(b - a)$, for all $\varepsilon > 0$. It results that f is Riemann integrable on $[a, b]$ and for all $n \in \mathbb{N}$: $\left| \displaystyle\int_a^b f(t)dt - \int_a^b f_n(t)dt \right| < \varepsilon(b - a)$. Moreover we also have

$$\forall x \in [a, b], |F_n(x) - F(x)| \leq (b - a) \sup_{t\in[a,b]} |f_n(t) - f(t)|.$$

$\square$

Corollary 1.8.
Let $\left(f_n\colon [a,b] \longrightarrow \mathbb{R}\right)_n$ be a sequence of piecewise continuous functions on $[a,b]$ and uniformly convergent to f on $[a,b]$, then f is Riemann-integrable on $[a,b]$ and we have:

$$\int_a^b f(t)dt = \lim_{n\to+\infty} \int_a^b f_n(t)dt.$$

1.4 Differentiability

Theorem 1.9.
Let $(f_n)_n$ be a sequence of continuously differentiable functions (of class $\mathcal{C}^1$) on an interval $[a,b] \subset \mathbb{R}$. Assume that:

1. the sequence $(f_n)_n$ is pointwise convergent to f on $[a,b]$.
2. the sequence $(f'_n)_n$ is uniformly convergent on $[a,b]$.

Then f continuously differentiable on $[a,b]$ and: $\forall x \in [a,b]$, $f'(x) = \lim_{n\to+\infty} f'_n(x)$ and $(f_n)_n$ converges uniformly to f on $[a,b]$. In particular f is of class $\mathcal{C}^1$ on $[a,b]$.

Proof .
We have $\displaystyle\int_a^x f'_n(t)dt = f_n(x) - f_n(a)$. Let g be the limit of the sequence $(f'_n)_n$. We have $\displaystyle\int_a^x g(t)dt = f(x) - f(a)$. Moreover g is continuous, then f is differentiable and $f'(x) = g(x), \forall x \in [a,b]$. $\square$

Exercise 1.3 :
Let $(f_n)_n$ be a sequence of differentiable functions on an interval $[a,b]$. Assume that the sequence $(f'_n)_n$ is uniformly convergent on $[a,b]$ and there exists $x_0 \in [a,b]$ such that the sequence $(f_n(x_0))_n$ is convergent. Prove that the sequence $(f_n)_n$ is uniformly convergent on $[a,b]$ to a differentiable function f and $f'(x) = \lim\limits_{n\to+\infty} f'_n(x)$.
(Hint: use the mean value theorem to the function $f_n - f_m$, for n and m large enough.)

1.5 Exercises

4-1-1 Define the sequence of functions $(f_n)_n$ on $\mathbb{R}$ by: $f_n(x) = n^2x(1-x)^n$.

(a) Determine the domain of pointwise convergence of the sequence $(f_n)_n$.

(b) Compute $\displaystyle\int_0^1 f_n(x)dx$ and deduce that the sequence (f_n) is not uniformly convergent on the interval $[0, 2[$.

(c) Compute the limit of $f_n(\frac{1}{n})$, when $n \longrightarrow +\infty$, and deduce an other time the previous result.

4-1-2 Study the pointwise and the uniform convergence of the following sequences of functions $(f_n)_n$ defined by:

(a) $f_n(x) = \dfrac{nx}{1+n^2x^2}$ on $\mathbb{R}$,

(b) $f_n(x) = \begin{cases} 2n^2x & \text{if} \quad x \in [0, \frac{1}{2n}] \\ 0 & \text{if} \quad x \in [\frac{1}{n}, 1] \\ 2n - 2n^2x & \text{if} \quad x \in [\frac{1}{2n}, \frac{1}{n}] \end{cases}$ on $[0, 1]$,

(c) $f_n(x) = \begin{cases} x^2 \sin(\frac{1}{nx}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ on $\mathbb{R}$,

(d) $f_n(x) = \begin{cases} \frac{\sin(x)}{x} e^{-nx} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$ on $\mathbb{R}_+$,

(e) $f_n(x) = n^\alpha x(1 - nx - |1 - nx|)$ on $\mathbb{R}_+$, $\alpha \in \mathbb{R}$,

(f) $f_n(x) = \begin{cases} n^\alpha x(1 - nx) & \text{if} \quad 0 \leq x < \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$, $\alpha \in \mathbb{R}$

(g) $f_n(x) = \begin{cases} nx - \frac{1}{n} & \text{if} \quad x \in [0, \frac{1}{n}[\\ 1 - x & \text{if} \quad x \in [\frac{1}{n}, 1] \end{cases}$ on $[0, 1]$.

(h) $f_n(x) = \begin{cases} \dfrac{\sin nx}{n\sqrt{x}} & \text{if } x > 0 \\ \\ 0 & \text{if } x = 0 \end{cases}$

(i) $f_n(x) = \dfrac{x^n}{1+x^n}$ on each of the following intervals, with $0 < a < 1$.
$[0, 1-a]$, $[1-a, 1+a]$, $[1+a, +\infty[$.

(j) $f_n(x) = \begin{cases} \dfrac{\sin^2 nx}{nx} & \text{if} \quad x \notin \pi\mathbb{Z} \\ 0 & \text{if} \quad x \in \pi\mathbb{Z} \end{cases}$,

4-1-3 (a) Consider the function $\varphi_n\colon]0,n[\longrightarrow \mathbb{R}$ defined for $n \geq 2$ by:

$$\varphi_n(x) = e^{-x} - \left(1 - \frac{x}{n}\right)^n.$$

i. Prove that φ_n' has a unique zero on the interval $]0,n[$.

ii. Study the variations of φ_n on $[0,n]$.

(b) Study the pointwise and uniform convergence of the sequence of functions $(f_n)_{n\geq 1}$ defined on $[0,+\infty[$ by:

$$f_n(x) = \begin{cases} \left(1 - \frac{x}{n}\right)^n & \text{if} \quad 0 \leq x \leq n \\ 0 & \text{if} \quad x > n \end{cases}$$

4-1-4 Study the pointwise and the uniform convergence of the following sequences of functions $(f_n)_n$ defined by:

(a) $f_n(x) = (\cos^n x)\sin x$ for $x \in [0, \frac{\pi}{2}]$.

(b) $g_n(x) = (1 + \dfrac{x}{n})^n$, if $x \geq -n$ and $g_n(x) = 0$ if $x < -n$.
Consider the case of the uniform convergence on $]-\infty, a]$, for $a \in \mathbb{R}$.

4-1-5 Let $(f_n)_n$ be the sequence of functions defined by on $\mathbb{R}\backslash\{-2\}$ by: $f_n(x) = \dfrac{(x+1)^n - 1}{(x+1)^n + 1}$.

Study the pointwise and the uniform convergence of the sequence $(f_n)_n$ on $\mathbb{R}\backslash\{-2\}$ and on any closed interval which does not contain neither -2 and 0.

4-1-6 Let $u_n(x) = n^2 x e^{-nx}$, $x \in [0,1]$.

(a) Find the pointwise limit of the sequence of functions $(u_n)_n$

(b) Find $\displaystyle\lim_{n\longrightarrow+\infty} \int_0^1 u_n(x)dx$.

(c) The convergence of the sequence $(u_n)_n$ on $[0,1]$ is it uniform?

4-1-7 Let $(f_n)_n$ be the sequence of functions defined on $[0,+\infty[$ by: $f_n(x) = \dfrac{nx}{1+nx}$.

(a) Determine the pointwise limit f of the sequence $(f_n)_n$.

(b) The convergence of $(f_n)_n$ to f is it uniform on $[0,1]$? on $[1,+\infty[$? and on $[0,+\infty[$?

(c) Let F_n be the function defined on $[0, +\infty[$ by: $F_n(x) = \displaystyle\int_0^x f_n(t)dt$.

i. Determine the pointwise limit F of the sequence $(F_n)_n$.

ii. The convergence of $(F_n)_n$ to F on $[0, 1]$ is it uniform?

4-1-8 Let $(f_n)_n$ be the sequence of functions defined by: $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$, for $x \in \mathbb{R}$.

(a) Prove that the sequence $(f_n)_n$ converges uniformly on $\mathbb{R}$.

(b) Prove that the functions f_n are differentiable on $\mathbb{R}$ and the limit of the sequence $(f_n)_n$ is not differentiable.

4-1-9 Define a sequence of functions $(f_n)_n$ on $\mathbb{R}_+^*$ by:

$$f_n(x) = n|\ln x|^n.$$

(a) Determine the domain D of the pointwise convergence of the sequence $(f_n)_n$.

(b) Study the uniform convergence of the sequence $(f_n)_n$ to f on D and on the compacts of D.

4-1-10 Define the sequence of functions $(f_n)_n$ on $\mathbb{R}_+$ by: $f_n(x) = \dfrac{ne^{-x}(x^3 + x)}{1 + nx}$.

(a) Determine the limit f of the sequence $(f_n)_n$ and deduce that the sequence $(f_n)_n$ is not uniformly convergent on $\mathbb{R}_+$.

(b) Prove that the sequence $(f_n)_n$ converges uniformly on any closed and bounded interval of $]0, +\infty[$ to f.

(c) Prove that the sequence $(|f_n - f|)_n$ is bounded on $[0, 1]$.

(d) Deduce that $\displaystyle\lim_{n\to+\infty} \int_0^1 f_n(t)dt = \int_0^1 f(t)dt$.

4-1-11 Define the sequence $(f_n)_n$ of functions defined on $\mathbb{R}_+$ by: $f_n(x) = e^{-nx^n}$.

(a) Determine the domain D of pointwise convergence of the sequence $(f_n)_n$.

(b) Prove that the sequence $(f_n)_n$ converges uniformly on $[1, +\infty[$.

(c) Prove that the sequence $(f_n)_n$ is not uniformly convergent on $[0, 1[$.

(d) Study the uniform convergence of the sequence $(f_n)_n$ on the compact subsets of $[0,1[$?

Let $g_n = f_n'$.

(e) Determine the domain of pointwise convergence of the sequence $(g_n)_n$.

(f) Study the convergence of the sequence $\left(g_n(\frac{n-1}{n^2})^{\frac{1}{n}}\right)_n$.

(g) Study the uniform convergence of the sequence $(g_n)_n$ on the following intervals, $[0,+\infty[$, $[0,1[$ and $[1,+\infty[$.

4-1-12 Define the sequence of functions $(f_n)_n$ on $\mathbb{R}_+^*$ by:

$$f_n(x) = (-1)^n x^{n^\beta} \ln(\frac{x^2+x+n}{n+x}).$$

(a) Prove that $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^{2+n^\beta}}{n}$.

(b) Determine, eventually according to the values of β the domain D_β of the pointwise convergence of the sequence $(f_n)_n$.

(c) Study the uniform convergence on D_β, and on the compacts of D_β.

4-1-13 Let f be a continuous function on $\mathbb{R}$. Assume that there exists a sequence $(P_n)_n$ of polynomials which converges uniformly on $\mathbb{R}$ to f.

(a) Prove that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0,\ P_n - P_{n_0}$ is bounded on $\mathbb{R}$.

(b) Deduce that f is a polynomial function.

4-1-14 Study the pointwise and uniform convergence of the following sequence of functions $(f_n)_n$.

(a) $f_n(x) = \begin{cases} x^{2n}\ln x & \text{if } x \in]0,1] \\ 0 & \text{if } x = 0 \end{cases}$

(b) $f_n(x) = \begin{cases} nx^n\ln x & \text{if } x \in]0,1] \\ 0 & \text{if } x = 0 \end{cases}$

(c) $f_n(x) = \begin{cases} \dfrac{\sin^2 nx}{n\sin x} & \text{if } x \in]0,1] \\ 0 & \text{if } x = 0 \end{cases}$

(d) $f_n(x) = 4^n\left(x^{2^{n+1}} - x^{2^n}\right)$.

(e) $f_n(x) = \dfrac{2^n x}{1+n2^n x^2}$ and compute $\displaystyle\lim_{n\to+\infty}\int_0^1 f_n(t)dt$ and $\displaystyle\int_0^1 \lim_{n\to+\infty} f_n(t)dt$.

4-1-15 Let $(f_n)_n$ be the sequence of functions defined by: $f_n(x) = x^2 \sin\dfrac{1}{nx}$ if $x \neq 0$ and $f_n(0) = 0$

(a) Prove that the sequence $(f_n)_n$ converges uniformly on any interval $[a, b] \subset \mathbb{R}$.

(b) The convergence is it uniform on $\mathbb{R}$?

(c) The sequence $(f_n')_n$ is it uniformly convergent on $\mathbb{R}$.

4-1-16 For $x \in [0,1]$ and $n \in \mathbb{N}$, define $f_n(x) = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k} - \ln(1+x)$

(a) Prove that the sequence $(f_n)_n$ converges uniformly to 0 on $[0,1]$. (We can compute $f_n'(x)$).

(b) Prove that $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{n}{n+1}\right)^k = \ln 2$.

4-1-17 Let $(f_n)_n$ be the sequence defined on $[0,1]$ by:

$$f_1(x) = \begin{cases} n^2 x & \text{if} \quad x \in [0, \frac{1}{n}] \\ -nx^2 + 2x & \text{if} \quad x \in [\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if} \quad x \in [\frac{2}{n}, 1] \end{cases}$$

(a) Study the pointwise and the uniform convergence of the sequence $(f_n)_n$.

(b) Compare $\displaystyle\lim_{n\to\infty}\int_0^1 f_n(x)dx$ and $\displaystyle\int_0^1 \lim_{n\to\infty} f_n(x)dx$.

2 Series of Functions

Definition 2.1.

Let $(f_n)_n$ be a sequence of functions defined on a subset A of $\mathbb{R}$.

1. The series of functions $\displaystyle\sum_{n\geq 1} f_n$ is called pointwise convergent on A if the sequence $\left(S_n = \displaystyle\sum_{k=1}^{n} f_k\right)_n$ is pointwise convergent on A.

2. The series $\sum_{n\geq 1} f_n$ is called uniformly convergent on A if the sequence $\left(S_n = \sum_{k=1}^{n} f_k\right)_n$ converges uniformly on A.

Remark 14 :

1. If the series $\sum_{n\geq 0} f_n$ is pointwise convergent to a function f on an interval I, then $\lim_{n\to+\infty} f_n(x) = 0$, for all $x \in I$.

2. A series $\sum_{n\geq 0} f_n$ is pointwise convergent on J, if and only if, the series $\sum_{n\geq 0} f_n(x)$ fulfills the Cauchy criterion, i.e.

$$\forall x \in I, \forall \varepsilon > 0, \exists N; |\sum_{k=n}^{n+p} f_k(x)| < \varepsilon, \quad \forall n \geq N, p \in \mathbb{N}.$$

Examples 5 :

1. Let $(f_n)_n$ be a sequence of functions defined by: $f_n(x) = x^n$, the series $\sum_{n\geq 0} f_n(x)$ is pointwise convergent on the interval $]0,1[$ to the function $f(x) = \dfrac{1}{1-x}$. If $|x| \geq 1$, $|f_n(x)| \geq 1$, then the series $\sum_{n\geq 0} f_n(z)$ is divergent on $\mathbb{R}\backslash]0,1[$.

2. For $x \geq 0$, we set $f_n(x) = \dfrac{\sin\frac{x}{n}}{n+x}$.

   For all fixed $x > 0$ we have: $\sin\dfrac{x}{n} = \dfrac{x}{n} - \dfrac{x^3}{6n^3} + O(\dfrac{1}{n^3})$, then

$$f_n(x) = \frac{x}{n(x+n)} - \frac{x^3}{6n^3(x+n)} + O(\frac{1}{n^3}).$$

   Then the series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R}^+$.

   Also, the series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R} \setminus \mathbb{Z}_-$.

Remark 15 :

1. If the series $\sum_{n\geq 0} f_n$ is uniformly convergent to f on I, then the series $\sum_{n\geq 0} f_n$ is pointwise convergent to f on I.

2. A series $\sum_{n\geq 0} f_n$ is uniformly convergent on I, if and only if, it fulfills the Cauchy criterion for the uniform convergence i.e.

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \sup_{x\in I} \| \sum_{k=n}^{n+p} U_k(x) \| < \varepsilon, \quad \forall n \geq N,\ p \in \mathbb{N}.$$

Example 2.1 :
The series $\sum_{n\geq 0} x^n$ is pointwise convergent on $]-1,1[$ to the function $f(x) = \frac{1}{1-x}$, but the convergence is not uniform because $\sup_{x\in]-1,1[} f_n(x) = 1$.

Definition 2.2.
A series $\sum_{n\geq 0} f_n$ is called normally convergent on I, if the series $\sum_{n\geq 0} \sup_{x\in I} \|f_n(x)\|$ is convergent.

Proposition 2.3.
If the series $\sum_{n\geq 0} f_n$ is normally convergent on I, then it is uniformly convergent on I.

For the proof we use the Cauchy criterion.

Corollary 2.4.
If $\sup_{x\in I} |f_n(x)| \leq a_n$ and the series $\sum_{n=0}^{+\infty} a_n$ is convergent, then the series $\sum_{n\geq 0} f_n$ is normally converge on I.

Examples 6 :

1. Let $f_n(x) = \frac{e^{inx}}{n^\alpha}$, $(\alpha > 1)$. $|f_n(x)| \leq \dfrac{1}{n^\alpha}$, then the series converges normally on $\mathbb{R}$.

2. For $x \in]0, +\infty[$, we have: $xe^{-x} \leq 1$, then $f_n(x) = \dfrac{e^{-nx}}{n} \leq \dfrac{1}{x.n^2} \leq \dfrac{1}{h.n^2}$ for all $x \in [h, +\infty[$. It results that the series $\sum_{n\geq 1} f_n$ converges uniformly on $[h, +\infty[, \forall h > 0$.

3. Let $f_n(z) = \frac{1}{n(x+n)}$ defined on $\mathbb{R} \setminus \mathbb{Z}_-^*$. $|f_n(x)| \leq \frac{1}{n|x+n|} \leq \frac{1}{n|n-|x||}$. Let K be any compact of $\mathbb{R} \setminus \mathbb{Z}_-^*$, there exists $R > 0$ such that $K \subset]-R, R[$. Let $n_0 \in \mathbb{N}$ such that $R < n_0$, we have: $|f_n(x)| \leq \frac{1}{n(n-R)}$, $\forall n \geq n_0$, $\forall x \in K$. Then the series $\sum_{n\geq 1} f_n$ converges uniformly on K.

2.1 Abel's Criterion for the Uniform Convergence

Theorem 2.5.
Let $(f_n)_n$ be a sequence of functions defined on a subset $X \subset \mathbb{R}$ and let $(g_n)_n$ be a sequence of functions defined on a subset $Y \subset \mathbb{R}$. The series $\sum_{n\geq 1} f_n(x)g_n(y)$ is uniformly convergent on $X \times Y$ under any one of the following conditions.

1. The series $\sum_{n\geq 1} f_n$ is uniformly convergent on X and the sequence $(g_n)_n$ is bounded and monotone on Y.
2. The partial sums of the series $\sum_{n\geq 1} f_n$ are uniformly bounded on X and the sequence $(g_n)_n$ is monotone and uniformly convergent to 0 on Y.
3. The series $\sum_{n\geq 1} f_n$ is uniformly convergent on X and the series $|g_0| + \sum_{n\geq 1} |g_n - g_{n+1}|$ is bounded on Y.

Proof .

1. We set $S_n(x) = \sum_{p=1}^{n} f_p(x)$ and $S(x) = \sum_{n=1}^{+\infty} f_n(x)$. Assume that the sequence $(S_n)_n$ is uniformly convergent to S on X and the sequence $(g_n)_n$ is decreasing and bounded on Y. Then

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\quad \forall\, n \geq N \quad \sup_{x\in X} |S_n(x) - S(x)| \leq \varepsilon.$$

Let $M > 0$ such that $|g_n(y)| \leq M$ for every $n \in \mathbb{N}$ and every $y \in Y$. If $p \geq N+1$ and $q > p$, then

$$\begin{aligned}\sum_{n=p}^{q} f_n(x)g_n(y) &= \sum_{n=p}^{q-1} (S_n(x) - S(x))\,(g_n(y) - g_{n+1}(y)) \\ &\quad + (S_q(x) - S(x))\,g_q(y) - (S_{p-1}(x) - S(x))\,g_p(y).\end{aligned}$$

Then

$$\begin{aligned}\sup_{x\in X, y\in Y}|\sum_{n=p}^{q} f_n(x)g_n(y)| &\leq \varepsilon \sup_{y\in Y}\sum_{n=p}^{q-1}[g_n(y)-g_{n+1}(y)] \\ &+\varepsilon \sup_{y\in Y}(|g_q(y)|+|g_p(y)|) \leq 2\varepsilon M.\end{aligned}$$

It follows that the series $\sum_{n\geq 1} f_n(x)f_n(y)$ converges uniformly on $X\times Y$.

2. Let $M>0$ such that $|S_n(x)|\leq M,\ \ \forall\ x\in X$ and $\forall n\in\mathbb{N}$. Assume that the sequence $(g_n)_n$ is decreasing:

$$\sum_{n=p}^{q} f_n(x)g_n(y) = \sum_{n=p}^{q-1} S_n(x)(g_n(y)-g_{n+1}(y))+S_q(x)g_q(y)-S_{p-1}(x)g_p(y)$$

We have: $\ \ \forall\ \varepsilon>0,\ \exists\ N\in\mathbb{N}$ such that $\forall\ \ n\geq N,\ \sup_{y\in Y}|g_n(y)|\leq\varepsilon.$

For $p\geq N+1$ and $q>p$

$$\sup_{x\in X, y\in Y}|\sum_{n=p}^{q} f_n(x)g_n(y)| \leq \sup_{y\in Y}\Big[M(g_p(y)-g_q(y))+Mg_q(y)+Mg_p(y)\Big] \leq 2M\varepsilon.$$

3. Let $M>0$ be such that
$|g_0(y)|+\sum_{n=0}^{+\infty}|g_n(y)-g_{n+1}(y)|\leq M,\ \ \forall\ y\in Y.$
Let $n\geq N$,
$$g_n(y)=\sum_{p=1}^{n-1}(g_{p+1}(y)-g_p(y))+g_0(y).$$
It follows that $|g_n(y)|\leq M,\ \ \forall\ n\in\mathbb{N}$ and $\ \forall\ y\in Y$.

$$\begin{aligned}\sum_{n=p}^{q} f_n(x)g_n(y) &= \sum_{n=p}^{q-1}(S_n(x)-S(x))(g_n(y)-g_{n+1}(y)) \\ &+(S_q(x)-S(x))g_q(y)-\big(S_{p-1}(x)-S(x)\big)g_p(y).\end{aligned}$$

Thus

$$\sup_{x\in X, y\in Y} |\sum_{n=p}^{q} f_n(x)g_n(y)| \leq \varepsilon \sup_{y\in Y}(\sum_{n=p}^{q} |g_n(y)-g_{n+1}(y)| + 2M) \leq 3\varepsilon M.$$

□

Examples 7 :

1. Let $(a_n)_n$ be a sequence of non negative decreasing real numbers and convergent to 0. The series $\sum_{n\geq 0} a_n e^{inx}$ is uniformly convergent on any compact subset of $\mathbb{R}\setminus 2\pi\mathbb{Z}$.

2. Consider the series $\sum_{n\geq 0} \dfrac{e^{inx}}{n+x}$ and K a compact of $\mathbb{R}\setminus\mathbb{Z}_-^*$, $\exists R>0$ such that $K\subset[-R,R]$. The sequence $g_n(x)=\frac{1}{n+x}$ is decreasing positive $\forall n\geq n_0$, $(n_0>R)$. The series is pointwise convergent on $\mathbb{R}\setminus\mathbb{Z}_-^*$ and it is uniformly convergent on any compact subset $K\subset\mathbb{R}\setminus(\mathbb{Z}_-\cup 2\pi\mathbb{Z})$. In particular this series converges uniformly on any interval $[\delta, 2\pi-\delta]$; $\forall\delta>0$.

Proposition 2.6.
Let $\left(f_n\colon I\longrightarrow\mathbb{R}\right)_n$ be a sequence on continuous functions at a point $a\in I$. Assume that the series $\sum_{n\geq 0} f_n$ is uniformly convergent on I to a function f. Then f is continuous at a.

Proof .
We apply the theorem (1.5) of the previous section. □

Proposition 2.7.
Let I be an open set in $\mathbb{R}$ and $\left(f_n\colon I\longrightarrow\mathbb{R}\right)_n$ a sequence of continuous functions. Assume that the series $\sum_{n\geq 0} f_n$ is uniformly convergent on any compact of I to a function f. Then f is continuous on I.

Theorem 2.8.
Let $\left(f_n\colon [a,b]\longrightarrow\mathbb{R}\right)_n$ be a sequence of Riemann integrable functions. Assume that the series $\sum_{n\geq 0} f_n$ is uniformly convergent on $[a,b]$ to a function f. Then f is Riemann integrable and we have:

$$\sum_{n=0}^{+\infty}\int_a^b f_n(x)dx = \int_a^b \sum_{n=0}^{+\infty} f_n(x)dx.$$

Proposition 2.9.
Let $\big(f_n\colon [a,b] \longrightarrow \mathbb{R}\big)_n$ be a sequence of continuously differentiable functions ($\mathcal{C}^1$ functions). Assume that

1. the series $\displaystyle\sum_{n\geq 0} f_n$ is pointwite convergent on $[a,b]$ to a function f.
2. the series $\displaystyle\sum_{n\geq 0} f_n'$ converge uniformly on $[a,b]$.

Then f is continuously differentiable on $[a,b]$ and we have:

$$f'(x) = \sum_{n=0}^{+\infty} f_n'(x), \quad \forall x \in [a,b].$$

Moreover the series $\displaystyle\sum_{n\geq 0} f_n$ converges uniformly on $[a,b]$ to f.

Corollary 2.10.
Let I be an interval of $\mathbb{R}$ and let $\big(f_n\colon I \longrightarrow \mathbb{R}\big)_n$ be a sequence of continuously differentiable functions. Assume

1. the series $\sum_{n\geq 0} f_n$ is pointwise convergent on I to f,
2. the series $\sum_{n\geq 0} f_n'$ converges uniformly on any compact of I.

Then f is continuously differentiable and we have:

$$f'(x) = \sum_{n=0}^{+\infty} f_n'(x), \quad \forall x \in I.$$

2.2 Exercises

4-2-1 Study the pointwise, absolute, normally and uniform convergence of the following series of general term:

1) $\sum_{n\geq 1} \frac{\sin(n^2 x)}{n^2}$, $x \in \mathbb{R}$,

2) $\sum_{n\geq 1} \frac{1}{n} \tan^{-1} \frac{x}{n}$, $x \in \mathbb{R}$,

3) $\sum_{n\geq 1} x^{n^2} \sin(n\pi x)$, $x \in [0, a]$, $0 < a < 1$.

4) $\sum_{n\geq 1} \frac{x}{(1+x^2)^n}$, $x \in \mathbb{R}$.

5) $\sum_{n\geq 1} x e^{-nx^2}$, $x \in \mathbb{R}$,

6) $\sum_{n\geq 1} x^2 e^{-x\sqrt{n}}$, $x \in \mathbb{R}_+$,

7) $\sum_{n\geq 1} \frac{nx^2}{1+n^3 x}$, $x \in \mathbb{R}_+$,

8) $\sum_{n\geq 1} \frac{(-1)^n}{n^x}$, $x \in \mathbb{R}$,

9) $\sum_{n\geq 1} \frac{x^{2n}}{1+x^{2n}}$, $x \in \mathbb{R}$,

10) $\sum_{n\geq 1} \frac{(-1)^n}{x^2+n}$, $x \in \mathbb{R}$,

11) $\sum_{n\geq 1} \frac{x}{(1+nx^2)^n}$, $x \in \mathbb{R}$,

12) $\sum_{n\geq 1} \frac{(-1)^n x}{(1+x^2)^n}$, $x \in \mathbb{R}$,

13) $\sum_{n\geq 1} \frac{x}{n^\alpha(1+nx^2)}$, $\alpha > 0$.

4-2-2 (a) Study the pointwise convergence of the series $\sum_{n\geq 1} (-1)^n \ln\left(1+\frac{x}{n}\right)$ on $\mathbb{R}_+$.

(b) Study the uniform and normal convergence of this series on any closed bounded interval in $\mathbb{R}_+$.

4-2-3 Find the domain of definition and the domain of continuity of the function: $f(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n e^{-nx}}{n+1}$.

4-2-4 (a) Find the domain of definition $\mathcal{D}$ of the function $g(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n e^{-nx}}{n^2+1}$.

(b) Prove that g is of class C^1 on $\mathcal{D}$.

4-2-5 (a) Prove that the series $\sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1}$ is uniformly convergent on $[-1, 1]$.

Let $f(x)=\sum_{n=0}^{+\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$, for $x\in[-1,1]$.

(b) Prove that f is differentiable on $]-1,1[$ and compute f'.

(c) Deduce the expression of $f(x)$, for $-1\leq x\leq 1$.

(d) Compute $\displaystyle\int_0^1 \tan^{-1} x dx$ and deduce the value the of the following sum $\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)(2n+2)}$.

4-2-6 Consider the series of functions $\sum_{n\geq 1} f_n$ defined on $\mathbb{R}_+$ by: $f_n(x)=x^n-x^{n-\frac{1}{2}}$.

(a) Prove that the series $\sum_{n\geq 1} f_n$ is pointwise convergent on $[0,1]$.

Denote $f(x)=\sum_{n=1}^{+\infty} f_n(x)$, for $x\in[0,1]$.

(b) Prove that the remains $R_n(x)=\sum_{p=n+1}^{+\infty} u_p(x)=x^n f(x)$.

(c) Prove that the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $[0,1]$.

(d) Prove that there exists $M\in\mathbb{R}_+$ such that:

$$\left|\int_0^1 R_n(x)dx\right|\leq\frac{M}{n+1}$$

(e) Deduce that the series $\sum_{n\geq 1} g_n$, where $g_n=\displaystyle\int_0^1 f_n(x)dx$ is convergent and its sum is $\displaystyle\int_0^1 f(x)dx$.

(f) Compute $\displaystyle\int_0^1 f(x)dx$ and deduce the value of the following sum $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$.

4-2-7 Define the series of functions $\sum_{n\geq 0} f_n$, where f_n is defined by: $f_0(x)=0$ and $f_n(x)=\dfrac{\sin n^2 x}{n^2}$, for $n\geq 1$.

(a) Prove that the series $\sum_{n\geq 0} f_n$ is uniformly convergent on $\mathbb{R}$.

(b) Study the convergence of the series $\sum_{n\geq 0} f'_n$.

4-2-8 Let $f(x) = \sum_{n=1}^{+\infty} f_n(x)$; with $f_n(x) = \dfrac{(-1)^{n-1}}{\sqrt{n^2+x^2}}$.

(a) Prove that f is continuous on $\mathbb{R}$.

(b) Study the uniform convergence of the series $\sum_{n\geq 1} f'_n$ and deduce that f is of class $\mathcal{C}^1$.

4-2-9 (a) Find the set of definition D of the function $f(x) = \sum_{n=1}^{+\infty} \dfrac{(-1)^{n+1}}{n^x}$ and prove that f is of class $\mathcal{C}^\infty$ on D.

(b) For $x > 1$; express $f(x)$ in term of $\sum_{n=1}^{\infty} \dfrac{1}{n^x}$.

4-2-10 (a) Prove that the series $\sum_{n\geq 1} \dfrac{x\sin nx}{2\sqrt{n}+\cos x}$ is pointwise convergent on $]0, 2\pi[$

(b) Prove that the convergence of the series is uniform on any interval of the form: $[\alpha, 2\pi-\alpha]$ $\forall 0<\alpha<2\pi$.

4-2-11 Let $\alpha\in\mathbb{R}$ and $f_n(x) = \dfrac{1}{n^\alpha}\ln(1+n^\alpha x^2)$, for $n\geq 1$ and $x\in\mathbb{R}$.

(a) Prove that the series $\sum_{n\geq 1} f_n(x)$ is pointwise convergent on $\mathbb{R}$ if and only if $\alpha > 1$.

(b) Assume that $\alpha > 1$.

  i. The series $\sum_{n\geq 1} f_n(x)$ is it uniformly convergent on $\mathbb{R}$?

  ii. Prove that the function $f(x) = \sum_{n=1}^{+\infty} f_n(x)$ is continuous on $\mathbb{R}$.

(c) Prove that if $\alpha > 2$; f is differentiable on $\mathbb{R}^*$.

(d) Assume $1 < \alpha \leq 2$.

  i. Prove that f is differentiable on $\mathbb{R}^*$.

ii. Prove that $\forall n \geq 1;\ f(n^{\frac{-\alpha}{2}}) \geq \ln 2. \sum_{k=n}^{\infty} \frac{1}{k^{\alpha}}$.

Deduce that $\forall n \geq 1$;

$$n^{\frac{\alpha}{2}} f(n^{\frac{-\alpha}{2}}) > \frac{\ln 2}{\alpha - 1}.$$

f is it differentiable at 0?

4-2-12 Define $f_n(x) = \dfrac{x}{(1+x^2)^n}$, for $x \in \mathbb{R}$.

Prove that

(a) The series $\sum_{n\geq 0} f_n$ and $\sum_{n\geq 0} (-1)^n f_n$ converge and compute their sum.

(b) $\forall a > 0$, the series $\sum_{n\geq 0} f_n$ converges uniformly on $[a, +\infty[$;

(c) The series $\sum_{n\geq 0} (-1)^n f_n$ converges uniformly on $\mathbb{R}$.

4-2-13 Let $f_n(x) = \dfrac{\ln(1+nx)}{nx^n}$, for $x > 0$. Prove that

(a) the domain of the pointwise convergence of the series $\sum_{n\geq 1} f_n(x)$ is $]1, +\infty[$. Let $f = \sum_{n=1}^{+\infty} f_n$ on $]1, +\infty[$.

(b) the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $]1, +\infty[$ and normally convergent on $[a, +\infty[$, for all $a > 1$.

(c) f is continuous on $]1, +\infty[$ and $\lim_{x\to 1^+} f(x) = +\infty$.

4-2-14 Define the sequence $(f_n)_n$ by: $f_n(x) = \dfrac{e^{-x\sqrt{n}}}{1+\sqrt{n^3}}$.

(a) Determine the domain of convergence of the series $\sum_{n\geq 0} f_n$.

Denote $f = \sum_{n=0}^{+\infty} f_n$.

(b) Prove that f is continuous on $\mathbb{R}_+$.

(c) Prove that f is differentiable on $\mathbb{R}_+^*$.

4-2-15 Let $f\colon]-1,+\infty[$ defined by:

$$f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n+x}.$$

Prove that f is continuous on $]-1,+\infty[$ and compute $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to(-1)^+} f(x)$.

4-2-16 Study the pointwise and uniform convergence of the series of functions $\sum_{n\geq 0} \frac{e^{-nx}}{1+n^2}$. Let $f(x) = \sum_{n=0}^{+\infty} \frac{e^{-nx}}{1+n^2}$.

Prove that f is of class $\mathcal{C}^1$ on $\mathbb{R}_+^*$.

4-2-17 Define the series of functions $\sum_{n\geq 1} f_n$, where $f_n(x) = \frac{\sin nx \sin n^2x}{n}$.

Recall that $2\sin kx \sin k^2x = \cos k(k-1)x - \cos k(k+1)x$.

Prove that the series $\sum_{n\geq 1} f_n(x)$ converges uniformly on $\mathbb{R}$.

4-2-18 Let $f_n(x) = \frac{\ln(1+n^\beta x^2)}{n^\alpha}$; with α and β two positive numbers.

Under what conditions the series $\sum_{n\geq 1} f_n(x)$ and $\sum_{n\geq 1} f_n'(x)$ are pointwise convergent on $\mathbb{R}$?

4-2-19 Define by induction the sequence of functions $(f_n(x))_n$ on the interval $[0,1]$ by:

$$f_0(x) = 1 \quad \text{and} \quad f_n(x) = 1 + \int_0^x f_{n-1}(t-t^2)dt.$$

(a) Prove that for each $n \in \mathbb{N}$, the function f_n is a polynomial and that $f_n(x) + f_n(1-x)$ is constant.

(b) Prove that for any $n \in \mathbb{N}$ and any $x \in [0,1]$

$$0 \leq f_n(x) - f_{n-1}(x) \leq \frac{x^n}{n!}.$$

(c) Deduce that the sequence $(f_n)_n$ converges uniformly on $[0,1]$ to a function f of class $\mathcal{C}^1$ on $[0,1]$ and fulfills $f'(x) = f(x - x^2)$.

4-2-20 Consider the sequence of functions $(f_n)_n$ defined on $]0,+\infty[$ by: $f_n(x) = \dfrac{1}{(nx+1)^2}$.

(a) Prove that the series $\sum_{n\geq 0} f_n$, $\sum_{n\geq 0} f'_n$ and $\sum_{n\geq 0} f''_n$ are uniformly convergent on $[a,+\infty[$, with $a > 0$.

(b) Let $F(x) = \sum_{n=1}^{+\infty} f_n(x)$. Recall that $\sum_{n=1}^{+\infty} \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$. Compute $F(\frac{1}{2})$, $F(1)$ and $F(2)$.

(c) Prove that F is $\mathcal{C}^2$ on $]0,+\infty[$ and give the sign of F' and F'' on $]0,+\infty[$.

(d) Determine $\lim_{x\to+\infty} F(x)$ and $\lim_{x\to 0^+} F(x)$.

4-2-21 Consider the series $\sum_{n\geq 0} e^{-n^2x}$ and $\sum_{n\geq 0} xe^{-n^2x}$ and denote $f(x) = \sum_{n=0}^{+\infty} e^{-n^2x}$ and $g(x) = \sum_{n=0}^{+\infty} xe^{-n^2x}$ in the domains of convergence respective D_f andD_g.

(a) Determine D_f and D_g

(b) i. Prove that f is decreasing on D_f.

ii. Give $\lim_{x\to 0^+} f(x)$.

(c) The function f is it continuous on D_f?

(d) i. Compute $\sup_{x\geq 0} xe^{-n^2x}$.

ii. The series $\sum_{n\geq 0}^{+\infty} xe^{-n^2x}$ is it uniformly convergent on D_g?

4-2-22 (a) Prove that the series $\sum_{n\geq 0} (-1)^n \dfrac{e^{-nx}}{n+1}$ defines a continuous function on its domain of definition D.

(b) Prove that the series $\sum_{n\geq 0} (-1)^n \dfrac{e^{-nx}}{n^2+1}$ defines a function g of class C^∞ on its domain of convergence.

4-2-23 Let $(f_n)_n$ be the sequence of functions defined on $\mathbb{R}$ by: $f_n(x) = nxe^{-nx^2}$.

(a) Study the pointwise convergence of the series $\sum_{n\geq 0} f_n$.

(b) Prove that the series $\sum_{n\geq 0} f_n$ is not normally convergent on $\mathbb{R}$.

(c) Prove that it is normally convergent on $[a, +\infty[$, for all $a > 0$.

(d) Let $f(x) = \sum_{n=0}^{+\infty} f_n(x)$. Prove that f is the derivative of a well known function. Deduce the expression of f.

4-2-24 Let $f_n(x) = \dfrac{x}{1 + n^2x^2}$.

(a) Prove that the series $\sum_{n\geq 0} f_n$ is pointwise convergent on $\mathbb{R}$.

(b) For $a > 0$, prove that the series $\sum_{n\geq 0} f_n'$ converges normally on $]-\infty, -a] \cup [a, +\infty[$.

(c) The series $\sum_{n\geq 0} f_n'$ is it uniformly convergent on $\mathbb{R}$?

(d) Determine the set where the function $F(x) = \sum_{n=0}^{+\infty} f_n(x)$ is differentiable.

4-2-25 (a) Prove that $\forall x, y \in \mathbb{R}_+^*$, $x^{\ln(y)} = y^{\ln(x)}$.

(b) Let $x \in \mathbb{R}_+^*$, we set: $f_n(x) = x^{\ln(n)}$.

Prove that the series $\sum_{n\geq 1} f_n(x)$ is convergent if and only if $x < \dfrac{1}{e}$.

(c) i. Let a, b such that $0 < a < b < \dfrac{1}{e}$.

Prove that the series $\sum_{n\geq 1} f_n$ is normally convergent on $[a, b]$.

ii. Let $f(x)$ be the sum of the series $\sum_{n\geq 1} f_n(x)$. $\left(f(x) = \sum_{n=1}^{+\infty} f_n(x)\right)$

Deduce that f is continuous on $]0, \dfrac{1}{e}[$.

(d) Compare the function f the sum of the series $\sum_{n\geq 1} f_n(x)$ with an integral and prove that:

$\forall x \in]0, \frac{1}{e}[, \qquad \frac{-1}{1+\ln(x)} \leq f(x) \leq \frac{\ln(x)}{1+\ln(x)}$

The function f is it bounded on $]0, \frac{1}{e}[$?

4-2-26 Consider the series of functions $\sum_{n\geq 0} f_n$, with $f_n(x) = \frac{(-1)^n}{n!}\frac{1}{x+n}$ for $x \in \mathbb{R}$.

(a) Give the domain of definition of f_n.

(b) Give the set D where the series $\sum_{n\geq 0} f_n$ is convergent.

(c) Denote for $x \in D$, $\quad f(x) = \sum_{n=0}^{+\infty} f_n(x)$.

i. Compute $f(1)$ in term of $e = \sum_{n=0}^{+\infty} \frac{1}{n!}$.

ii. Prove that for any $x \in D$, the function $xf(x) - f(x+1)$ is constant. Give its value.

(d) Study the uniform convergence of the series $\sum_{n\geq 0} f_n'$, and $\sum_{n\geq 0} f_n''$ and deduce that f is two times differentiable on D.

4-2-27 Define the sequence $(f_n)_n$ with $f_n\colon]0,+\infty[\longrightarrow \mathbb{R}$ defined by: $f_n(x) = \frac{(-1)^n \ln n}{n^x}$ and set

$$f(x) = \sum_{n=1}^{+\infty} \frac{(-1)^n \ln n}{n^x}$$

(a) i. Prove that the series $\sum_{n\geq 1} f_n'(x)$ converges normally on any closed interval $[a,b] \subset]1,+\infty[$.

ii. Deduce that f is of class $\mathcal{C}^1$ on $]1,+\infty[$.

(b) i. Prove that the series $\sum_{n\geq 1} f_n'(x)$ converges uniformly on any interval $[\alpha,+\infty[$, with $\alpha > 0$.

ii. Deduce that f is of class $\mathcal{C}^1$ on $]0,+\infty[$.

(c) Prove by the same method that the function f is of class C^∞ on $]0,+\infty[$.

4-2-28 (a) i. Prove that the series $\sum_{n\geq 1} \frac{e^{-2nx}}{4n^2-1}$ converges uniformly on $[0,+\infty[$.

We set $f(x) = \sum_{n=1}^{+\infty} \frac{e^{-2nx}}{4n^2-1}$.

ii. Prove that $\forall x \in]0,+\infty[; |3e^{2x} f(x) - 1| \leq 3e^{-2x} \sum_{n=2}^{+\infty} \frac{1}{4n^2-1}$.

iii. Deduce that $f(x) \underset{+\infty}{\approx} \frac{1}{3} e^{-2x}$.

(b) Let $g(x) = \sum_{n=1}^{+\infty} \frac{e^{-(2n+1)x}}{2n-1}$.

i. Prove that the series $\sum_{n\geq 2} \frac{e^{-(2n+1)x}}{2n-1}$ is pointwise convergent on $]0,+\infty[$ and uniformly convergent on $[a,+\infty[$ for any $a>0$.

ii. Let $a>0$. Prove that:

$$\forall x \in [a,+\infty[\quad |e^{3x} g(x) - 1| \leq e^{-x} \sum_{n=2}^{+\infty} \frac{e^{-(2n-3)a}}{2n-1}.$$

iii. Deduce that $g(x) \underset{+\infty}{\approx} e^{-3x}$.

(c) Let $u(x) = \sum_{n=1}^{+\infty} e^{-(2n-1)x}$.

i. Prove that the series $\sum_{n\geq 1} e^{-(2n-1)x}$ is pointwise convergent on $]0,+\infty[$ and uniformly convergent on $[a,+\infty[$, $\forall a>0$.

ii. Prove that $\forall x \in]0,+\infty[, u(x) = \frac{1}{2\sinh x}$.

(d) Let $F(x) = e^{-x} f(x)$ and $G(x) = e^{2x} g(x)$.

i. Prove that F and G are differentiable on $]0,+\infty[$ and $F'(x) = -g(x)$ and $G'(x) = -u(x)$.

ii. Let $x \in]0,+\infty[$. Compute the integral: $\int_x^{+\infty} \frac{dt}{\sinh t}$ and $\int_x^{+\infty} \frac{1}{t^3} \ln\left(\frac{e^t-1}{e^t+1}\right) dt$.

(e) Deduce the values of $g(x)$ and $f(x)$.

4-2-29 Let f be a continuous function on $[0,1]$. Define the sequence of polynomials $(B_n)_n$ called Bernstein polynomials associated to f,

$$B_n(x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}.$$

(a) Let $\varphi_n(x,t) = \sum_{k=0}^{n} C_n^k e^{\frac{kt}{n}} x^k (1-x)^{n-k} = \sum_{k=0}^{n} C_n^k \left(e^{\frac{t}{n}} x\right)^k (1-x)^{n-k}$.

i. Compute $\dfrac{\partial \varphi_n}{\partial t}(x,t)$ and $\dfrac{\partial^2 \varphi_n}{\partial t^2}(x,t)$.

ii. Prove that

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1,$$

$$\sum_{k=0}^{n} k C_n^k x^k (1-x)^{n-k} = nx$$

and

$$\sum_{k=0}^{n} k^2 C_n^k x^k (1-x)^{n-k} = nx + n(n-1)x^2.$$

(b) Deduce that all $0 < \alpha < 1$,

$$\sum_{|x-\frac{k}{n}| \geq \alpha} C_n^k x^k (1-x)^{n-k} \leq \frac{1}{\alpha^2} \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 \leq \frac{1}{4n\alpha^2}.$$

(c) Using the uniform continuity of f, prove that the sequence $(B_n)_n$ converges uniformly to f.

(d) Deduce that any continuous function on a interval $[a,b]$ is uniform limit of a sequence of polynomials.

3 Approximation Theorems

In this section, we prove the Weierstrass theorem on the density of the space of polynomials on the space of continuous functions on the interval $[a,b]$.

Definition 3.1.
A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is called a step function if there exist a partition $\sigma = (a_j)_{0\le j\le n}$ of $[a,b]$ such that f is constant on any interval $]a_{j-1}, a_j[$, for all $1 \le j \le n$.
A function $f\colon [a,b] \longrightarrow \mathbb{R}$ is called piecewise continuous function, if there exist a partition $\sigma = (a_j)_{0\le j\le n}$ of $[a,b]$ such that f is continuous on any interval $]a_{j-1}, a_j[$, for all $1 \le j \le n$ and f has a finite limit at the right on any point of $[a,b[$ and a finite limit at the left on any point of $]a,b]$.

Theorem 3.2.
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a piecewise continuous function, then there exist a sequence of step functions on $[a,b]$ which converges uniformly to f. (A regulated function f is a uniform limit of a sequence of step function.)

Proof .
If f is continuous, it is uniformly continuous on $[a,b]$, then $\forall\ \varepsilon > 0$, $\exists \alpha > 0$ such that if $|x - x'| < \alpha$, $|f(x) - f(x')| \le \varepsilon$. For all $n \in \mathbb{N}$, we consider the uniform partition $\sigma_n = (a_0, \ldots, a_n)$, with $a_k = a + k\frac{b-a}{n}$ for all $0 \le k \le n$ and we consider the step functions f_n defined by: $f_n(x) = f(a_k)$, if $x \in [a_k, a_{k+1}[$ and $f(a_n) = f(b)$. If $n \ge \frac{b-a}{\alpha}$, we have:

$$||f_n - f||_\infty = \max_{0\le j\le n-1}\Big(\sup_{x\in[a_j,a_{j+1}[} |f_n(x) - f(x)|\Big) \le \varepsilon.$$

If f is piecewise continuous and $\sigma = (a_0, \ldots, a_n)$ a partition associated to f, i.e. f is continuous on $]a_j, a_{j+1}[$ for all $0 \le j \le n-1$. Let f_j be a continuous function on $[a_j, a_{j+1}]$ such that $f_j = f$ on $]a_j, a_{j+1}[$. For every f_j there exist a sequence of step functions $(f_{n,j})_n$ which converges uniformly to f on $]a_j, a_{j+1}[$. Then the sequence $(f_n)_n$ defined by: $f_n(a_j) = f(a_j)$ and $f_n(x) = f_{n,j}(x)$ for $x \in]a_j, a_{j+1}[$, converges uniformly to f on $[a,b]$. $\square$

Theorem 3.3. [Weierstrass Theorem]
Let f be a continuous function on an interval $[a,b]$. There exists a sequence of polynomials $(P_n)_n$ which converges uniformly to f on $[a,b]$. (i.e. $\mathbb{R}[X]$ is dense in $\mathcal{C}([a,b])$ for the norm of uniform convergence.)

Proof .
Without loss of generality, we can assume that $[a,b] = [0,1]$.
Since f is continuous on $[0,1]$, it is uniformly continuous. Then $\forall \varepsilon > 0$, $\exists \alpha > 0$; if $|x - y| \le \alpha$, $|f(x) - f(y)| \le \varepsilon$.
We consider the Bernstein polynomials sequence $(B_n)_n$ defined by:

$$B_n(x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}.$$

$$
\begin{aligned}
|f(x) - B_n(x) &= \Big|\sum_{k=0}^{n} C_n^k (f(x) - f(\frac{k}{n}))x^k(1-x)^{n-k}\Big| \\
&\leq \sum_{k=0}^{n} C_n^k \Big|f(x) - f(\frac{k}{n})\Big| x^k(1-x)^{n-k} \\
&= \sum_{|x-\frac{k}{n}|<\alpha} C_n^k \Big|f(x) - f(\frac{k}{n})\Big| x^k(1-x)^{n-k} \\
&\quad + \sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k |f(x) - f(\frac{k}{n})| x^k(1-x)^{n-k} \\
&\leq \varepsilon + 2||f||_\infty \sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k(1-x)^{n-k}
\end{aligned}
$$

$$
\sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k(1-x)^{n-k} \leq \frac{1}{\alpha^2}\sum_{k=0}^{n} C_n^k x^k(1-x)^{n-k}(x-\frac{k}{n})^2.
$$

$$
\sum_{k=0}^{n} C_n^k x^k(1-x)^{n-k}(x-\frac{k}{n})^2 = x^2 - \frac{2x}{n}\sum_{k=0}^{n} C_n^k k x^k(1-x)^{n-k} + \frac{1}{n^2}\sum_{k=0}^{n} C_n^k k^2 x^k(1-x)^{n-k}.
$$

Since $\sum_{k=0}^{n} C_n^k x^k(1-x)^{n-k} = 1$, then by derivative with respect to x and if we set $h(x) = \sum_{k=0}^{n} C_n^k k x^k(1-x)^{n-k}$, we have: $h(x) = nx$. We iterate this process, we find:

$$
\sum_{k=0}^{n} C_n^k x^k(1-x)^{n-k}(x-\frac{k}{n})^2 = \frac{x(1-x)}{n}.
$$

Then

$$
\frac{1}{\alpha^2}\sum_{k=0}^{n} C_n^k x^k(1-x)^{n-k}(x-\frac{k}{n})^2 \leq \frac{1}{4n\alpha^2}.
$$

The sequence $(B_n)_n$ converge uniformly to f on $[0,1]$.
We give another proof of this theorem in the chapter of Fourier series. We give now another proof.

Theorem 3.4. Weierstrass Theorem
Let f be a continuous function on an interval I, there exist a sequence $(f_n)_n$ of polynomials which converges uniformly on any interval compact of I to f.

Proof .
Assume in the first case that f is continuous on $\mathbb{R}$ and equal to 0 on the complement of the interval $[-\frac{1}{2}, \frac{1}{2}]$. We set

$$P_n(x) = c_n(1 - x^2)^n,$$

with c_n a constant such that $\displaystyle\int_{-1}^{1} P_n(x)dx = 1$. We define the sequence

$$f_n(x) = \int_{-\infty}^{+\infty} f(y)P_n(x - y)dy = \int_{-\infty}^{+\infty} f(x - y)P_n(y)dy. \tag{4.3}$$

Lemma 3.5.
The functions f_n are polynomials and the sequence $(f_n)_n$ converges uniformly to f on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Proof .
By the left side of (4.3), f is a polynomial and by the right side of (4.3) we have for $|x| \leq \frac{1}{2}$:

$$f(x) - f_n(x) = \int_{-1}^{1} (f(x) - f(x - y))P_n(y)dy. \tag{4.4}$$

Let $\varepsilon > 0$, M the maximum of f on $\mathbb{R}$ and $\delta > 0$ such that $|f(x) - f(x - y)| < \varepsilon$ if $|y| < \delta$. It results from the formula (4.4) that

$$|f(x) - f_n(x)| \leq \int_{|y|<\delta} \varepsilon P_n(y)dy + \int_{\delta \leq |y| \leq 1} M\ P_n(y)dy.$$

We have to prove now that $\displaystyle\int_{\delta \leq |y| \leq 1} P_n(y)dy$ tends to 0 when n tends to infinity.
Let $0 < r < 1$.

$$\frac{1}{c_n} = \int_{-1}^{1} (1 - x^2)^n dx \geq \int_{-r}^{r} (1 - r^2)^n dx = 2r(1 - r^2)^n.$$

Then $c_n \leq \dfrac{1}{2r(1 - r^2)^n}$. Thus

$$\int_{\delta \leq |y| \leq 1} P_n(y)dy \leq \frac{1}{2r(1 - r^2)^n} \int_{-1}^{1} (1 - \delta^2)^n dy = \frac{(1 - \delta^2)^n}{r(1 - r^2)^n}.$$

The result is deduced if we take $r < \delta$ and we tends n to infinity.

Proof of theorem (3.4).
If f is zeros on the complement of the interval $[-s, s]$, the function $F(x) = f(2sx)$ is zeros on the complement of the interval $[-\frac{1}{2}, \frac{1}{2}]$. By the previous lemma there exist a sequence $(f_n)_n$ of polynomials which converges uniformly to F on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The sequence of polynomials $g_n(x) = f_n(\frac{x}{2s})$ converges uniformly to f on the interval $[-s, s]$.
If now f is continuous on the interval $I = (a, b)$. For all $n \in \mathbb{N}$ such that $n > \dfrac{2}{b-a}$, there exists a function φ_n continuous on I such that $\varphi_n = 1$ on $[a+\frac{1}{n}, b-\frac{1}{n}]$ and zeros on the complement of the interval $[a+\frac{1}{2n}, b-\frac{1}{2n}]$. There exists a polynomial f_n such that $|f_n(x) - \varphi_n(x)f(x)| < \dfrac{1}{n}$ on I. The sequence $(f_n)_n$ is a solution to the problem.

□

Corollary 3.6.

If f is a continuous function on the interval $[a, b]$ such that $\displaystyle\int_a^b f(x)x^n dx = 0$, for all $n \in \mathbb{N}$, then $f = 0$.

Proof .
It results that for all polynomial P, $\displaystyle\int_a^b f(x)P(x)dx = 0$. Since f is a uniform limit of sequence of polynomial $(P_n)_n$, then

$$\int_a^b f^2(x)dx = \lim_{n\to+\infty} \int_a^b f(x)P_n(x)dx = 0.$$

□

Remark 16 :
The previous result is wrong for the continuous functions on an unbounded interval. For example, let f be the function defined by: $f(x) = e^{-x^{\frac{1}{4}}} \sin(x^{\frac{1}{4}})$, for $x \in [0, +\infty[$. Prove that $\displaystyle\int_0^{+\infty} x^n f(x)dx = 0$, for all $n \in \mathbb{N}$.

Corollary 3.7.
Let $f\colon [a, b] \longrightarrow \mathbb{C}$ be a continuous function. There exist a sequence $(Q_n)_n \in \mathbb{R}[X]$ such that $(Q_n)_n$ converges uniformly to f on $[a, b]$.

5 Power Series

1 Power Series

1.1 Abel's Lemma

Definition 1.1.
Let $(a_n)_n$ be a sequence of real or complex numbers. The series $\sum_{n\geq 0} a_n(x-x_0)^n$ is called a power series centered at x_0.

Let $\sum_{n\geq 0} a_n(x-x_0)^n$ be a power series, we look for its domain of convergence. The series converges at least for $x = x_0$. In which follows, we consider the series centered at 0.

Proposition 1.2. (Abel's lemma)
If the power series $\sum_{n\geq 0} a_n x_0^n$ is convergent for $x_0 \neq 0$, then

1. the series $\sum_{n\geq 0} a_n x^n$ is absolutely convergent on the interval $]-|x_0|, |x_0|[$,
2. for every $r < |x_0|$, the power series $\sum_{n\geq 0} a_n x^n$ is uniformly convergent on $[-r, r]$.

Proof .

1. Let $x \in]-|x_0|, |x_0|[$, $\sum_{n=0}^{+\infty} |a_n x^n| \leq \sum_{n=0}^{+\infty} |a_n x_0^n| |\frac{x}{x_0}|^n$. Since the series $\sum_{n\geq 0} a_n x_0^n$ is convergent, the sequence $(a_n x_0^n)_n$ is bounded. Moreover the series $\sum_{n\geq 0} |\frac{x}{x_0}|^n$ is convergent, then the series $\sum_{n\geq 0} a_n x^n$ is absolutely convergent on $]-|x_0|, |x_0|[$.

2. Let $r < |x_0|$ and $x \in [-r, r]$, $|a_n x^n| \le |a_n| r^n$ and $\sum_{n=0}^{+\infty} |a_n| r^n < +\infty$, thus the series $\sum_{n\ge 0} a_n x^n$ is uniformly convergent on $[-r, r]$.

$\square$

Corollary 1.3.
If the power series $\sum_{n\ge 0} a_n x_0^n$ is divergent then it is divergent for every x such that $|x| > |x_0|$.

1.2 Radius of Convergence of Power Series

Theorem 1.4.
For every power series $\sum_{n\ge 0} a_n x^n$, there exists a unique $R \in [0, +\infty]$ such that:

1. For every $|x| < R$, the series $\sum_{n\ge 0} a_n x^n$ is absolutely convergent.
2. For every $|x| > R$, the sequence $(a_n x^n)_n$ is not bounded and then the series $\sum_{n\ge 0} a_n x^n$ is divergent.

   The number R is called the radius of convergence of the power series and $]-R, R[= \{x \in \mathbb{R};\ |x| < R\}$ is called the open interval of convergence of the power series.

Proof .
The uniqueness results from Abel's lemma. We set

$$R = \sup\{r \ge 1;\ \sum_{n=0}^{+\infty} |a_n| r^n < +\infty\}.$$

If $|x| < R$, the series $\sum_{n\ge 0} a_n x^n$ is absolutely convergent.
If there exists $|x| > R$ such that the series $\sum_{n\ge 0} |a_n| r^n$ is convergent. Then the series $\sum_{n\ge 0} |a_n| r^n$ is convergent for every $R < r < |x|$ which is absurd. $\square$

Remark 17 :

From the proof of the theorem (1.4), we deduce that if R is the radius of convergence of the series $\sum_{n\geq 0} a_n x^n$, then the series is uniformly convergent on any interval $[-r, r]$ with $0 < r < R$.

Theorem 1.5. (Cauchy 1821, used by Hadamard) (Cauchy-Hadamard Rule)
Let $\sum_{n\geq 0} a_n x^n$ be a power series with R its radius of convergence. Then

1. $R = \sup\{r \geq 0;\ \sum_{n=0}^{+\infty} |a_n| r^n < +\infty\} = \sup\{r \geq 0;$ the sequence $(a_n r^n)_n$is bounded $\}$.

2. If $\lim_{n\to+\infty} |\frac{a_n}{a_{n+1}}| = \beta \in [0, +\infty]$, then $R = \beta$.

3. $R = \dfrac{1}{\overline{\lim}_{n\to+\infty} \sqrt[n]{|a_n|}}$. (With $R = +\infty$ if $\overline{\lim}_{n\to+\infty} \sqrt[n]{|a_n|} = 0$ and $R = 0$ if $\overline{\lim}_{n\to+\infty} \sqrt[n]{|a_n|} = +\infty$.)

Theorem 1.6.
Let $\sum_{n\geq 0} a_n x^n$ be a power series with radius of convergence $R > 0$. Define $f(x) = \sum_{n=0}^{+\infty} a_n x^n$. Then the power series $\sum_{n\geq 1} n a_n x^{n-1}$ has R as radius of convergence and the function f is differentiable on $]-R, R[$ and $f'(x) = g(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$.

For the proof, we need the following lemma:

Lemma 1.7.
Let $x \in \mathbb{R}$ and $h \in \mathbb{R}$ such that $0 < |h| \leq r$, then for any $n \in \mathbb{N}$

$$|(x+h)^n - x^n - nhx^{n-1}| \leq \frac{|h|^2}{r^2}(|x| + r)^n \tag{5.1}$$

and

$$n|x|^{n-1} \leq \frac{1}{r}\big(2(|x| + r)^n + |x|^n\big). \tag{5.2}$$

Proof .
From the inequality (7.4)

$$\left|(x+h)^n - x^n - nhx^{n-1}\right| = \left|\sum_{k=0}^{n} C_n^k h^k x^{n-k} - x^n - nhx^{n-1}\right| = \left|\sum_{k=2}^{n} C_n^k h^k x^{n-k}\right|$$

$$\begin{aligned}
&\leq \quad |h|^2 \sum_{k=2}^{n} C_n^k |x|^{n-k} |h|^{k-2} \leq \frac{|h|^2}{r^2} \sum_{k=2}^{n} C_n^k |x|^{n-k} r^k \\
&\leq \quad \frac{|h|^2}{r^2} (|x|+r)^n.
\end{aligned}$$

We have: $|(x+h)^n - x^n - nhx^{n-1}| \geq nr|x|^{n-1} - |x|^n - (|x|+r)^n$. From the relation (7.4), we deduce:

$$nr|x|^{n-1} \leq |x|^n + (|x|+r)^n + |(x+r)^n - x^n - nrx^{n-1}| \leq |x|^n + 2(|x|+r)^n.$$

$\square$

Proof of the theorem (1.6).
We denote R' the radius of convergence of the power series $\sum_{n \geq 1} n a_n x^{n-1}$. It is obvious that $R' \leq R$. Let $r > 0$ such that $|x| + r < R$. From the lemma (1.7); we have: $|n a_n x^{n-1}| \leq \frac{1}{r}\big(2|a_n|(|x|+r)^n + |a_n||x|^n\big)$ and thus $\sum_{n \geq 1} n a_n x^{n-1}$ is absolutely convergent on $]-R, R[$. Thus the radius of convergence of the series defining g is greater than R. Thus $R = R'$.
From the inequality (7.4) we have:

$$|\frac{f(x+h) - f(x)}{h} - g(x)| \leq \frac{|h|}{} \sum_{n=1}^{+\infty} |a_n|(|x|+r)^n.$$

This proves that when h tends to 0; $f'(x) = g(x)$; for any $x \in]-R, R[$. $\square$

Corollary 1.8.
If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, then f is infinitely continuously differentiable on $]-R, R[$ if $R > 0$, $a_n = \frac{f^{(n)}(0)}{n!}$ and $f(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(0)}{n!} x^n$. (This series is called the Taylor's series of f at 0 or the Mac-Laurent series of F.)

Example 1.1 :

1. For $x \in \mathbb{R}$,

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \qquad e^{-x} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^n}{n!},$$

$$\cosh x = \sum_{n=0}^{+\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh x = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

$$\cos x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \sin x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

2. For $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n, \qquad \ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{(n+1)},$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \text{ and } \tan^{-1} x = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)},$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{(2n+1)}.$$

3. Let α be a real number, $\alpha \notin \mathbb{N}$ and $f(x) = (1+x)^\alpha$ for $x \in]-1, 1[$. $f'(x) = \alpha(1+x)^{\alpha-1}$, then f satisfies the following differential equation

$$(1+x)y' - \alpha y = 0. \tag{5.3}$$

We look for a power series $\sum_{n\geq 0} a_n x^n$ solution of the differential equation (5.3).

If $S = \sum_{n=0}^{+\infty} a_n x^n$ is a solution, we have:

$$(1+x)\sum_{n=0}^{+\infty} n a_n x^{n-1} - \alpha \sum_{n=0}^{+\infty} a_n x^n = 0,$$

then $(n+1)a_{n+1} + na_n - \alpha a_n = 0 \iff a_{n+1} = \frac{\alpha-n}{n+1} a_n \ \forall n \geq 0$, which yields that

$$a_n = \frac{\alpha(\alpha-1)\ldots(\alpha-n)}{2.3\ldots(n+1)} a_0.$$

Then

$$S(x) = a_0\Big(1 + \sum_{n=1}^{+\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n\Big).$$

By the uniqueness of the solution of the differential equation

$$(1-x)^\alpha = \sum_{n=0}^{+\infty} a_n x^n, \qquad \text{for } |x| < 1,$$

where $a_n = \dfrac{\alpha(\alpha-1)\dots(\alpha-n)}{2.3\dots(n+1)}$.

For $\alpha = \frac{-1}{2}$, we have:

$$\frac{1}{\sqrt{1-x}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^n, \qquad \sqrt{1+x} = 1 + \frac{1}{2}\sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} \frac{x^{n+1}}{n+1}.$$

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^{2n}, \qquad \sin^{-1} x = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$

$$\cos^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}, \qquad \sinh^{-1} x = \sum_{n=0}^{+\infty} \frac{(-1)^n C_{2n}^n}{4^n} \frac{x^{2n+1}}{2n+1}.$$

1.3 Exercises

5-1-1 Find the sums of the following series and compute their radius of convergence:

1) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{2n-1}$,

2) $\displaystyle\sum_{n=1}^{+\infty} n^2 x^n$,

3) $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2+1}{n!} x^n$,

4) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)(n+3)}$,

5) $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{4n^2-1}$,

6) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n}{n} \cosh(na), a > 0$

7) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n \sin n\theta}{2^n}$,

8) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n2^n}$,

9) $\displaystyle\sum_{n=1}^{+\infty} \frac{nx^n \sin^2(n\theta)}{2^n}$,

10) $\displaystyle\sum_{n=0}^{+\infty} \frac{n^2+1}{n+1} x^n$,

11) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^n}{(2n)!}$,

12) $\displaystyle\sum_{n=0}^{+\infty} \frac{\sin^2(n\theta)}{n!} x^{2n}$,

13) $\displaystyle\sum_{n\geq 0} (2n+1)x^n$,

14) $\displaystyle\sum_{n=0}^{+\infty} \frac{x^{3n}}{(3n)!}$,

15) $\displaystyle\sum_{n=0}^{+\infty} (n^2+1)\frac{x^n}{n!}$,

16) $\displaystyle\sum_{n=0}^{+\infty} \frac{nx^n}{3^n(n+1)}$,

17) $\displaystyle\sum_{n=0}^{+\infty} (-1)^n \frac{(n^2+1)x^n}{n!}$,

18) $\displaystyle\sum_{n=0}^{+\infty} \frac{nx^n}{3^n(n+1)}$,

19) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n x^n}{3n+1}$.

5-1-2 (a) Define the sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ by:
$\begin{cases} u_0 = 1 \\ v_0 = 0 \end{cases}$ and $\begin{cases} u_{n+1} = u_n + 2v_n \\ v_{n+1} = u_n + v_n. \end{cases}$
Determine the radius of convergence and the sum of the power series $\displaystyle\sum_{n\geq 0} u_n x^n$.

(b) Determine the radius of convergence of the power series:

$$\sum_{n\geq 0} a_n x^n;\ \text{with } a_{2n} = 0 \text{ and } a_{2n+1} = \frac{(-1)^n}{(2n-1)(2n+1)}.$$

Let $f(x) = \sum_{n=1}^{+\infty} a_n x^n$, give a simple expression of the derivative $f'(x)$ in term of x and $\tan^{-1} x$.
Deduce $f(x)$.

5-1-3 Say if the following affirmations are true or false.

(a) The series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} (-1)^n a_n x^n$ have the same radius of convergence.

(b) The series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} |a_n| x^n$ have the same radius of convergence.

(c) The series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} (-1)^n a_n x^n$ have the same domain of convergence.

(d) If the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is infinite, then the series is uniformly convergent on $\mathbb{R}$.

(e) If the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is infinite and if a_n are positives, then for any integer p, $\lim_{x\to+\infty} \frac{f(x)}{x^p} = +\infty$, with $f(x) = \sum_{n=0}^{+\infty} a_n x^n$.

5-1-4 Give the expansion in power series in a neighborhood of 0 of the following functions

(a) $x \longmapsto \frac{\ln(1+x)}{1+x}$.

(b) $f(x) = (\sin^{-1} x)^2$. (We will be able to show that f fulfills a differential equation of order 2.)

(c) $\frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}$.

(d) $\ln(1 - 2x\cos\alpha + x^2)$.

(e) $e^{2x}\cos x$.

5-1-5 Give the expansion in power series of the function $f(x) = \frac{x}{1-x-x^2}$.

5-1-6 Give the expansion in power series of the following functions in a neighborhood of 0 and determine the corresponding radius of convergence:

1) $\dfrac{1}{(1-x)^2}$,

2) $\dfrac{1}{(x-2)(x-3)}$,

3) $\ln(1+x+x^2)$

4) $\sin^3 x$,

5) $\sinh^3 x$,

6) $(x-1)\ln(x^2-5x+6)$,

7) $x\ln(x+\sqrt{x^2+1})$,

8) $\dfrac{x-2}{x^3-x^2-x+1}$,

9) $\dfrac{1}{1+x-2x^3}$,

10) $\dfrac{1-x}{(1+2x-x^2)^2}$,

11) $\tan^{-1}(x+1)$,

12) $\tan^{-1}(x+\sqrt{3})$,

13) $\displaystyle\int_0^x \frac{\ln(t^2-\frac{5}{2}t+1)}{t}\,dt$,

14) $\left(\dfrac{(1+x)\sin x}{x}\right)^2$,

15) $\displaystyle\int_x^{2x} e^{-t^2}\,dt$,

16) $\displaystyle e^{-2x^2}\int_0^x e^{2t^2}\,dt$,

17) $\dfrac{e^x}{1-x}$,

18) $\dfrac{e^{x^2}}{1-x}$,

19) $\displaystyle\int_0^x \frac{\cos t-1}{t^2}dt$,

20) $\ln\left(\dfrac{1+x}{2-x}\right)$

21) $\ln\sqrt{1-2x\cosh a+x^2}$,

5-1-7 Define $f(x)=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$.

(a) Prove that f has an expansion in power series in a neighborhood of 0 and precise the radius of convergence.

(b) Prove that f fulfills a differential equation.
Deduce the coefficients of the expansion in power series of f.

(c) Give the expansion in power series of $(\sin^{-1})^2(x)$.

5-1-8 Give the expansion in power series the following functions at the corresponding point x_0.

(a) $f(x)=\cos x$, $(x_0=\dfrac{\pi}{4})$,

(b) $f(x)=(1-x^3)^{-\frac{1}{2}}$, $(x_0=0)$,

5-1-9 Assume that the power series $\displaystyle\sum_{n\geq 0} a_{2n}x^n$ and $\displaystyle\sum_{n\geq 0} a_{2n+1}x^n$ have radius of convergence R and R' respectively.

Determine the radius of convergence of the power series $\displaystyle\sum_{n\geq 0} a_nx^n$.

5-1-10 Let $(a_n)_n$ be a decreasing sequence and $\lim_{n\to+\infty} a_n = 0$ and the series $\sum_{n\geq 0} a_n$ diverges.

(a) Prove that the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is 1.

(b) Study the convergence for $|x| = 1$.

5-1-11 (a) Let $(a_n)_n$ be a sequence of real numbers such that the series $\sum_{n\geq 0} a_n$ is convergent.

We claim to prove that the power series $\sum_{n\geq 0} a_n x^n$ is uniformly convergent on $[0,1]$.

Define $R_n = \sum_{k=n+1}^{+\infty} a_k$ and $S_n = \sum_{k=0}^{n} a_k x^k$.

i. Prove that for $p > n$; $S_p(x) - S_n(x) = R_n x^{n+1} - R_p x^p + \sum_{k=n+1}^{p-1} (x^{k+1} - x^k) R_k$.

ii. Deduce that the series $\sum_{n\geq 0} a_n x^n$ fulfills the Cauchy criterion for the uniform convergence on $[0,1]$.

(b) Let $\sum_{n\geq 0} b_n x^n$ be a power series of radius of convergence R and let $f(x)$ its sum. Let $x_0 \in \mathbb{R}$ such that $|x_0| = R \neq 0$. Assume that the series $\sum_{n\geq 0} b_n x_0^n$ is convergent.

i. Prove that $\lim_{\substack{x\longmapsto x_0\\ x\in[0,x_0]}} f(x) = \sum_{n=0}^{+\infty} b_n x_0^n$ $([0,x_0] = \{t x_0,\ t\in[0,1]\})$.

ii. Deduce the value of the following sum $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n}$.

5-1-12 For each of the following power series, determine the interval of convergence of this series and prove that its sum is a solution of the suitable differential equation.

$$f(x)=\sum_{n=0}^{\infty}\frac{x^{4n}}{(4n)!},\quad y^{(4)}=y$$

$$f(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{(n!)^2},\quad xy^{''}+y'-y=0$$

$$f(x)=\sum_{n=0}^{+\infty}\frac{(-1)^n2^{2n}x^{2n}}{(2n)!},\quad y^{''}+4y=0$$

5-1-13 (a) Prove that there exists a solution as power series of the following differential equation

$$x(x-1)y^{''}+3xy'+y=0.$$

(b) Determine the radius of convergence of the obtained series.

5-1-14 For any $\lambda\in\mathbb{R}$, consider the following differential equation

$$y^{''}(x)-2xy'(x)+2\lambda y(x)=0 \tag{5.4}$$

(a) Prove that the equation (6.6) has a unique even solution P_λ as a power series on $\mathbb{R}$ and fulfills $P_\lambda(0)=1$.

(b) Prove that (6.6) has a unique odd solution Q_λ as a power series on $\mathbb{R}$ and fulfills $Q'_\lambda(0)=1$.

(c) Determine all the values of λ for that the equation (6.6) has a non vanishing polynomial solution.

5-1-15 (a) Find the solutions as power series of the following differential equations:

i. $y'-2xy=0; y(0)=1$

ii. $y^{''}+xy'+y=0$

iii. $4xy^{''}+2y'-y=0,\ x>0$

(b) Give the expansion in power series the function $f(x)=e^{\frac{-x^2}{2}}\displaystyle\int_0^x e^{\frac{t^2}{2}}\,dt.$

5-1-16 Define $u_n(x)=(-1)^n\dfrac{x^n}{n(n-1)}$, for $n\geq 2$.

(a) Determine the interval of convergence of the series $\displaystyle\sum_{n\geq 2}(-1)^n\frac{x^n}{n(n-1)}$ and study this series to the endpoints of this interval.

(b) Study the series $\sum_{n\geq 2} u_n'(x)$ and the series $\sum_{n\geq 2} u_n''(x)$.

(c) Deduce the sum of the series $\sum_{n\geq 2} u_n(x)$.

5-1-17 (a) Consider the sequence (a_n) defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 7a_{n+1} + 12a_n = 0$.

i. Compute $F(x) = \sum_{n=0}^{+\infty} a_n x^n$.

ii. Deduce the expression of a_n.

(b) Consider the sequence (a_n) defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 7a_{n+1} + 12a_n = n$.
Compute the expression of a_n.

(c) Consider the sequence $(a_n)_n$ defined by: $a_0 = 1, a_1 = 2, a_{n+2} - 8a_{n+1} + 16a_n = 0$.
Find the expression of a_n.

5-1-18 Let $(a_n)_n \in \mathbb{R}^*$ be a convergent sequence of real numbers and let $a = \lim_{n\to+\infty} a_n$.

(a) Find the radius of convergence of the power series $\sum_{n\geq 0} \frac{a_n x^n}{n!}$.

Define $f(t) = \sum_{n=0}^{+\infty} \frac{a_n}{n!} t^n$, for $t \in \mathbb{R}$.

(b) Compute $\lim_{t\longmapsto+\infty} e^{-t} f(t)$.

5-1-19 Prove that the equation $3xy' + (2-5x)y = x$ has a solution as a power series in a neighborhood of 0 and give its radius of convergence.

5-1-20 Consider the following differential equation

$$x^2 y'' + xy' - (x^2 + x + 1)y = 0. \tag{5.5}$$

(a) Find a solution of the equation (6.1) $\varphi(x) = \sum_{n=0}^{+\infty} a_n x^n$ with $a_1 = 1$.

(b) Prove that, for $n \geq 1$, $|a_n| \leq \frac{1}{(n-1)!}$ and deduce the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$.

(c) Solve the equation (6.1) in putting $y = \dfrac{e^{-x}}{x} z$.

$\boxed{\text{5-1-21}}$ We claim to prove that the following differential equation

$$x^2 y'(x) = y(x) - x^2 \tag{5.6}$$

has no solution as sum of a power series.

Assume that this equation has a solution $y = \sum_{n=0}^{+\infty} a_n x^n$.

(a) Give the values of a_0, a_1 and a_2?

(b) Give the relation between a_{n+1} and a_n for $n \geq 2$.

(c) Prove that the relations stated in 1) and 2) give the uniqueness of the power series $\sum_{n \geq 0} a_n x^n$. Compute its coefficients and prove that it diverges.

6 Fourier Series

In this chapter, we consider the locally Riemann integrable functions. The reader can always take the piecewise continuous functions.
The aim of this chapter is the study the expansion of function (in physics we said a signal) of one real variable then of the synthase or reconstitution of this function has from of the its composite elements.

1 Fourier Series Expansion

1.1 Preliminary

1. Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a locally Riemann-integrable function and $T-$periodic with $T > 0$, then

$$\int_a^{a+T} f(t)dt = \int_0^T f(t)dt \quad \forall a \in \mathbb{R}.$$

Indeed, $\displaystyle\int_a^{a+T} f(t)dt = \int_a^0 f(t)dt + \int_0^T f(t)dt + \int_T^{a+T} f(t)dt$. Taking the change of variable $u = t - T$ in the last integral, we get the result. This means that the integral of a $T-$periodic function on an interval of length T does not depends of the chosen interval.

2. For $n, m \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{int}\ dt = \begin{cases} 0 & \text{if} \quad n \neq 0 \\ 1 & \text{if} \quad n = 0 \end{cases},$$

$$\frac{1}{\pi}\int_0^{2\pi} \sin(mt)\cos(nt)\ dt = 0,$$

$$\frac{1}{\pi}\int_0^{2\pi} \cos(mt)\cos(nt)\ dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \neq 0 \end{cases},$$

$$\frac{1}{\pi}\int_0^{2\pi} \sin(mt)\sin(nt)\ dt = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \neq 0 \\ 0 & \text{if} \quad n = m = 0 \end{cases}$$

Definition 1.1.
We consider the space $\mathscr{E}$ of continuous functions 2π-periodic defined on $\mathbb{R}$ with complex values. The map defined on $E \times E$ by:

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt = \frac{1}{2\pi}\int_{a-\pi}^{a+\pi} f(t)\overline{g(t)}dt$$

is a inner product. It defines a norm called the Euclidean norm denoted by $\| \ \|_2$.

Remark 18 :
The system $\{1, \cos(nt), \sin(nt), \quad n \in \mathbb{N}\}$ is an orthogonal system. Also the system $\{e^{int}, \quad n \in \mathbb{Z}\}$ is orthogonal.

1.2 Bessel Inequality

Definition 1.2.

1. A trigonometric polynomial of degree $\leq N$ is a complex linear combination of $\{1, \cos(kx), \sin(kx), \quad 1 \leq k \leq N\}$, i.e. a trigonometric polynomial P of degree $\leq N$ has the form

$$P(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)), \tag{6.1}$$

   with $a_n, b_n \in \mathbb{C}$. In particular a trigonometric polynomial is a function of class C^∞ and 2π-periodic.

2. A trigonometric series is a series of functions in the form

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos(nx) + b_n\sin(nx)),$$

   with a_n and $b_n \in \mathbb{C}$.

Remark 19 :
Let $P(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx))$ a trigonometric polynomial of degree $\leq N$, then

$$P(x) = \frac{a_0}{2} + \sum_{n=1}^{N} e^{inx}(\frac{a_n}{2} - \mathrm{i}\frac{b_n}{2}) + \sum_{n=1}^{N} e^{-inx}(\frac{a_n}{2} + \mathrm{i}\frac{b_n}{2}) = \sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}, \quad (6.2)$$

with

$$C_n = (\frac{a_n}{2} - \mathrm{i}\frac{b_n}{2}), \quad C_{-n} = (\frac{a_n}{2} + \mathrm{i}\frac{b_n}{2})$$

for $n \geq 1$ and $C_0 = \frac{a_0}{2}$. This form is called the exponential form of P, and the form (6.1) is called trigonometric form of P.
If P is a trigonometric polynomial of degree $\leq N$ in the form (6.1) or (6.1), then

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} P(t)e^{-int}\, dt, \ \forall n \in \mathbb{Z},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} P(t)\cos(nt)\, dt, \ \forall n \in \mathbb{N} \cup \{0\},$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} P(t)\sin(nt)\, dt, \ \forall n \in \mathbb{N}.$$

Theorem 1.3.
Let $f\colon [0, 2\pi] \longrightarrow \mathbb{C}$ be a Riemann-integrable function. define

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt, \quad n \in \mathbb{Z}$$

$$S_N(x) = \sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}, \qquad N \in \mathbb{N} \cup \{0\}.$$

Then:

1. For any trigonometric polynomial P of degree $\leq N$,

$$\int_0^{2\pi} |f(t) - S_N(t)|^2\, dt \leq \int_0^{2\pi} |f(t) - P(t)|^2\, dt. \quad (6.3)$$

2. The series $\sum_{n\in\mathbb{Z}} |C_n|^2$ is convergent and

$$\sum_{n=-\infty}^{+\infty} |C_n|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt \quad \text{(Bessel Inequality)}. \quad (6.4)$$

The property (6.3) shows that S_N realized the best approximation in quadratic mean of f by a trigonometric polynomial of degree $\leq N$.

Proof .

1. Let $P(x) = \sum_{n=-N}^{N} d_n e^{inx}$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} |f(t) - P(t)|^2\, dt &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt - \frac{1}{2\pi}\int_0^{2\pi} f(t)\bar{P}(t)dt \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} P(t)\bar{f}(t)dt + \frac{1}{2\pi}\int_0^{2\pi} |P(t)|^2 dt.\end{aligned}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{P(t)}dt = \sum_{n=-N}^{N} \bar{d}_n \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}\, dt = \sum_{n=-N}^{N} \bar{d}_n C_n.$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \bar{f}(t)P(t)\, dt = \sum_{n=-N}^{N} d_n\bar{C}_n, \qquad \frac{1}{2\pi}\int_0^{2\pi} |P(t)|^2\, dt = \sum_{n=-N}^{N} |d_n|^2.$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t)-P(t)|^2\, dt = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt - \sum_{n=-N}^{N} |C_n|^2 + \sum_{n=-N}^{N} |d_n - C_n|^2.$$

If the polynomial P is the polynomial S_N, we have:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(t) - S_N(t)|^2\, dt = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt - \sum_{n=-N}^{N} |C_n|^2,$$

this yields the result.

2. $\frac{1}{2\pi}\int_0^{2\pi} |f(t) - P(t)|^2\, dt = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt - \sum_{n=-N}^{N} |C_n|^2$, thus $\sum_{n=-N}^{N} |C_n|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\, dt$ and we take the limit when $N \longrightarrow +\infty$.

$\square$

Corollary 1.4.
If $f\colon [0,2\pi] \longrightarrow \mathbb{C}$ is a Riemann-integrable function, then

$$\lim_{n\to+\infty} \int_0^{2\pi} f(t)\cos(nt)\ dt = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_0^{2\pi} f(t)\sin(nt)\ dt = 0.$$

Proof .
As the series $\sum_{n\in\mathbb{Z}} |C_n|^2$ converges, then $\lim_{n\to\infty} |C_n|^2 = 0$. If we set $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\cos(nt)\ dt$ and $b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(t)\sin(nt)\ dt$, for $n \in \mathbb{N}$ we have: $a_n = C_n + C_{-n}$ and $b_n = \mathrm{i}(C_n - C_{-n})$, and we have the result.

$\square$

Theorem 1.5. (Riemam-Lebesgue Lemma)
Let $f\colon [a,b] \longrightarrow \mathbb{C}$ be a Riemann-integrable function, then

$$\lim_{\lambda\to+\infty} \int_a^b f(t)\cos(\lambda t)\ dt = 0 \text{ and } \lim_{\lambda\to+\infty} \int_a^b f(t)\sin(\lambda t)\ dt = 0.$$

Proof .
As $\displaystyle\int_a^b f(t)\cos(\lambda t)\quad dt = \int_a^b \operatorname{Re} f(t)\cos(\lambda t)\quad dt + \int_a^b \operatorname{Im} f(t)\cos(\lambda t)\quad dt$, it suffices to prove the theorem for f real .
• If $f = \chi_{[\alpha,\beta]}$ is the characteristic function of an interval $[\alpha,\beta]$, we have:

$$\int_a^b f(t)\cos(\lambda t)\ \ dt = \int_\alpha^\beta \cos(\lambda t)\ \ dt = \frac{\sin(\lambda\alpha)}{\lambda} - \frac{\sin(\lambda\beta)}{\lambda} \underset{\lambda\to+\infty}{\longrightarrow} 0.$$

• If f is a step function on $[a,b]$, there exists a partition $\sigma = \{x_0 = a < x_1 < \ldots < x_n = b\}$ of $[a,b]$ such that $f = c_j$ on $]x_j, x_{j+1}[$. In this case

$$\int_a^b f(t)\cos(\lambda t)dt = \sum_{j=0}^{n-1} c_j \int_{x_j}^{x_{j+1}} \cos(\lambda t)dt.$$

Thus

$$\Big|\int_a^b f(t)\cos(\lambda t)dt\Big| \leq \frac{2}{\lambda}\sum_{j=0}^{n-1} |c_j| \underset{\lambda\to+\infty}{\longrightarrow} 0.$$

In the general case: as f is Riemann-integrable on $[a,b]$, for $\varepsilon > 0$, there exists a step function f_ε such that $f_\varepsilon \leq f$ and $\int_a^b (f(t) - f_\varepsilon(t))\ dt < \varepsilon$. Then

$$\int_a^b f(t)\cos(\lambda t)\ dt = \int_a^b (f(t) - f_\varepsilon(t))\cos(\lambda t)\ dt + \int_a^b f_\varepsilon(t)\cos(\lambda t)\ dt.$$

We deduce that

$$\Big|\int_a^b f(t)\cos(\lambda t)\ dt\Big| \leq \Big|\int_a^b (f(t) - f_\varepsilon(t))\ dt\Big| + \Big|\int_a^b f_\varepsilon(t)\cos(\lambda t)\ dt\Big|.$$

As f_ε is a step function, $\lim\limits_{\lambda\to+\infty} |\int_a^b f_\varepsilon(t)\cos(\lambda t)\ dt| = 0$ and the result is deduced. $\square$

1.3 Fourier Series

1. Let f be a complex $2\pi-$periodic function, Riemann-integrable on $[0, 2\pi]$. We set

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-\mathrm{i}nt}\ dt, \quad n \in \mathbb{Z},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\ dt, \quad n \in \mathbb{N}_0,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\ dt, \quad n \in \mathbb{N}$$

The coefficients $(C_n)_n$ will be called the exponential Fourier coefficients of f and a_n and b_n will be called the trigonometric Fourier coefficients of f. We recall that:

$$a_0 = 2C_0, \quad a_n = C_n + C_{-n}, \quad b_n = \mathrm{i}(C_n - C_{-n}), \qquad \forall n \geq 1.$$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n\cos(nx) + b_n\sin(nx)) = \sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}.$$

$$\lim_{n\to+\infty} S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n\cos(nx) + b_n\sin(nx)) = \sum_{n=-\infty}^{+\infty} C_n e^{\mathrm{i}nx}$$

The series $\displaystyle\sum_{n\in\mathbb{Z}} C_n e^{\mathrm{i}nx} = \frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos(nx) + b_n\sin(nx))$ will be called the Fourier series of f. We will denote formally $\tilde{f}(x)$ the sum of this series.

We say that the Fourier series of f converges at $x_0 \in \mathbb{R}$ if the sequence $(S_N)_N,\ S_N(x) = \displaystyle\sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}$ converges at x_0.

2. If f is $T-$periodic, the function $g(x) = f(\frac{Tx}{2\pi})$ is 2π-periodic on $\mathbb{R}$. Moreover the function f is locally Riemann integrable on $\mathbb{R}$, we associate to f the Fourier coefficients defined from the Fourier coefficients of g by:

$$C_n = \frac{1}{T}\int_0^T f(t)e^{-\mathrm{i}n\frac{2\pi}{T}t}\,dt, \qquad \forall n \in \mathbb{Z},$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi}{T}nt\,dt, \qquad \forall n \in \mathbb{N}_0,$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi}{T}nt\,dt,\ n \in \mathbb{N}.$$

The exponential Fourier series of f is

$$\sum_{n\in\mathbb{Z}} C_n e^{-\mathrm{i}n\frac{2\pi}{T}t}$$

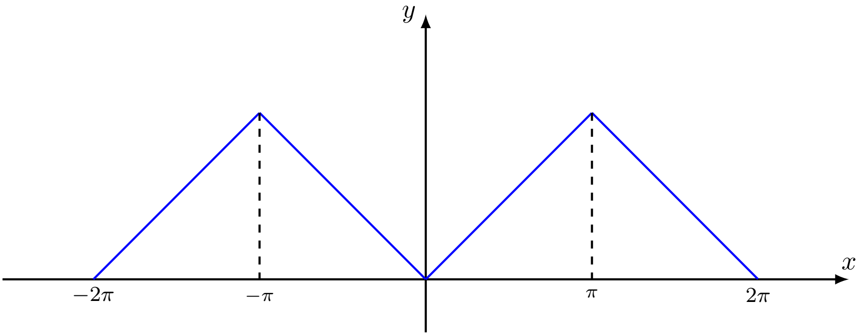
and the trigonometric Fourier series is

$$\frac{a_0}{2} + \sum_{n\geq 1}(a_n\cos\frac{2\pi}{T}nx + b_n\sin\frac{2\pi}{T}nx).$$

Definition 1.6.
Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a $2\pi-$periodic function and Riemann-integrable on $[0, 2\pi]$. Develop f in Fourier series, means that find Fourier trigonometric or exponential series of f, study the convergence of the series $\tilde{f}$ of f and give its value.

Examples 8 :

1. $f(x) = |x|$ if $|x| \leq \pi$ and f $2\pi-$periodic. The curve of f on $[-2\pi, 2\pi]$ has the following form:



We have:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \, dt = \pi, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \cos(nt) \, dt = \frac{2}{n^2\pi}((-1)^n - 1), \quad n \geq 1.$$

As f is even $b_n = 0$. The Fourier series of f converges uniformly on $\mathbb{R}$.

2. Let $f(x) = \sin x$, for $x \in [0, \pi]$ even and 2π-periodic. Thus $b_n = 0$ and $a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos(nx) \, dx$. $a_{2n+1} = 0$ and $a_{2n} = \frac{-4}{\pi(4n^2 - 1)}$. The Fourier series of f converges uniformly on $\mathbb{R}$.

3. Let $\alpha \in \mathbb{C} \setminus (\mathrm{i}\mathbb{Z})$, $f(x) = e^{\alpha . x}$ on $]-\pi, \pi[$ and $2\pi-$periodic.

$$C_n = \frac{(-1)^n \sinh \alpha\pi}{\pi(\alpha - \mathrm{i}n)}, \qquad \tilde{f}(x) = \frac{\sinh \pi\alpha}{\pi}\sum_{-\infty}^{+\infty}(-1)^n \frac{e^{\mathrm{i}nx}}{\alpha - \mathrm{i}n}.$$

4. Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ be a power series ($z \in \mathbb{C}$) of radius of convergence $R > 0$. For $r \in [0, R[$, the map $\theta \xmapsto{F} f(re^{\mathrm{i}\theta})$ is $2\pi-$periodic and we have:

$$f(re^{\mathrm{i}\theta}) = \sum_{n=0}^{+\infty}(a_n r^n) e^{\mathrm{i}n\theta} \qquad (6.5)$$

and the trigonometric series converges uniformly on $\mathbb{R}$.

Thus $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(re^{\mathrm{i}\theta})e^{-\mathrm{i}p\theta}d\theta = \sum_{n=0}^{+\infty}(a_n r^n)\int_0^{2\pi} e^{\mathrm{i}(n-p)\theta}d\theta.$

The series (6.5) is the Fourier series of f. Moreover $a_p = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(re^{\mathrm{i}\theta})e^{-\mathrm{i}p\theta}d\theta,$

then $|a_p| \leq \dfrac{M(r)}{r^p}$, with $M(r) = \sup_{|z|=r}|f(z)|$.

If we take the function $f(z) = \frac{1}{1-z}$, we know that for $|z| < 1$, $\dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{+\infty} z^n$. Thus for any $\theta \in \mathbb{R}$ and any $r \in [0,1[$, $\dfrac{1}{1-re^{\mathrm{i}\theta}} = \displaystyle\sum_{n=0}^{+\infty} r^n e^{\mathrm{i}n\theta}$ and in taking the real part of each member we get:

$$\frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta} = \sum_{n=0}^{+\infty} r^n \cos(n\theta).$$

1.4 The Dirichlet Theorem

The natural question in Fourier analysis is: "In what condition the Fourier series of a function f is convergent and the relation between the limit and the function f.

Definition 1.7. [Dirichlet Kernel]
The Dirichlet kernel of degree $N \in \mathbb{N}_0$ is the trigonometric polynomial D_N defined by:

$$D_N(x) = \sum_{n=-N}^{N} e^{\mathrm{i}nx} = \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}.$$

The function D_N is even and $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} D_N(t)dt = 1.$

Theorem 1.8. (Dirichlet Theorem)
Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a 2π-periodic function and Riemann-integrable on $[0, 2\pi]$. Let $x \in \mathbb{R}$ such that $f(x+) = \displaystyle\lim_{t\to x,\ t>x} f(t)$ and $f(x-) = \displaystyle\lim_{t\to x,\ t<x} f(t)$ exist in $\mathbb{C}$. We assume also that there exists $\delta_x > 0$ (depends of x) and $M_x \geq 0$ (depends of x) such that: $\forall t,\ 0 < |t| < \delta_x,$:

$$\frac{|f(x+t) + f(x-t) - f(x+) - f(x-)|}{|t|} \leq M_x \tag{6.6}$$

then the Fourier series of f at x converges to $\dfrac{f(x+) + f(x-)}{2}$, i.e.

$$\lim_{N\to+\infty} \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)) = \frac{f(x+) + f(x-)}{2}. \tag{6.7}$$

The condition (6.6) is called the Dirichlet condition at x.

Proof .
Let C_n be the Fourier exponential coefficients of f, with $n \in \mathbb{Z}$.

$$\begin{aligned}
S_N(x) &= \sum_{n=-N}^{N} C_n e^{inx} = \sum_{n=-N}^{N} \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}e^{inx}dt \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(t)D_N(t-x)\ dt. \\
&\overset{u=t-x}{=} \frac{1}{2\pi}\int_0^{2\pi} f(u+x)D_N(u)\ du.
\end{aligned}$$

If we denote $y = \dfrac{f(x+) + f(x-)}{2}$ we have:

$$\begin{aligned}
S_N(x) - y &= \frac{1}{2\pi}\int_0^{\pi} (f(x-u) + f(x+u) - 2y)D_N(u)du \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{f(x-u)+f(x+u) - f(x+) - f(x-)}{u}\frac{u}{\sin\frac{u}{2}}\sin(\frac{(2N+1)u}{2})du.
\end{aligned}$$

The function φ defined on $]0,\pi[$ by:

$$\varphi(u) = \frac{f(x-u) + f(x+u) - f(x+) - f(x-)}{u}\frac{u}{\sin\frac{u}{2}}$$

is Riemann-integrable on $]0,\pi]$. Moreover $\forall\ u \in]0,\delta_x[$, we have: $\varphi(u)| \leq M_x\pi$. $(\frac{2}{\pi} \leq \frac{\sin t}{t} \leq 1,\ \forall t \in [0,\frac{\pi}{2}])$ and by the Riemman-Lebesgue lemma (1.5), $\lim\limits_{N\to+\infty}\int_0^{\pi}\varphi(u)\sin(N+\frac{1}{2})u\ du = 0$. Thus $\lim\limits_{N\to+\infty} S_N(x) = \dfrac{f(x+) + f(x-)}{2}$.

□

Theorem 1.9.

1. Let $x \in \mathbb{R}$ such that $f(x+)$, $f(x-)$, $f'(x+) = \lim\limits_{t\to0,\ t>0}\dfrac{f(x+t) - f(x+)}{t}$ and $f'(x-) = \lim\limits_{t\to0,\ t>0}\dfrac{f(x-t) - f(x-)}{t}$, exist in $\mathbb{C}$. Then the Dirichlet condition is realized at x and the Fourier series of f at x converges to $\frac{f(x+)+f(x-)}{2}$.

2. If f is also continuous at x, then the Fourier series of f at x converges to $f(x)$.

3. If f is $2\pi-$periodic and of class piecewise continuously differentiable $[0, 2\pi]$, then $\forall x \in \mathbb{R}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_n \cos(nx) + b_n \sin(nx)) = \sum_{n=-\infty}^{+\infty} C_n e^{\mathrm{i}nx}.$$

Examples 9 :

1. Let f be the function defined by: $f(x) = |x|$ if $x \in [-\pi, \pi]$ and $2\pi-$periodic. f is continuous at the left of π and at the right of $-\pi$, by parity and periodicity, f is continuous at π and at $-\pi$. f is continuously differentiable on $[-\pi, \pi]$, thus by Dirichlet theorem, the Fourier series of f coincides with f at any point $x \in \mathbb{R}$. Thus for $|x| \leq \pi$, we have:

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{+\infty}\frac{\cos(2k+1)x}{(2k+1)^2}.$$

   For $x = 0$, we have:

$$\frac{\pi^2}{8} = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2}.$$

   The Fourier series of f converges uniformly to f on $\mathbb{R}$.

2. Let f be the function defined by: $f(x) = x$ on $]-\pi, \pi[$ and $2\pi-$periodic. (we associate an arbitrary value at π). f is continuously differentiable on $]-\pi, \pi[$ and has a derivative at the left and at the right at any point on $\mathbb{R}$. By Dirichlet theorem, we have for any $x \in \mathbb{R} \setminus \{(2k+1)\pi, k \in \mathbb{Z}\}$,

$$f(x) = 2\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\sin(nx).$$

   In particular for $x = \frac{\pi}{2}$

$$\frac{\pi}{4} = \sum_{p=0}^{+\infty}\frac{(-1)^p}{2p+1}.$$

1.5 The Parseval Theorem

Definition 1.10. (The Cesaro Summation)
Let $(U_n)_n$ be a sequence of complex numbers. We define the sequence $S_N = \sum_{k=0}^{N} U_k$. We say that the series $\sum_{n\geq 0} U_n$ is Cesaro summable if the sequence $T_N = \dfrac{S_0 + \ldots S_N}{N+1}$ converges in $\mathbb{C}$.

Examples 10 :

1. If $U_n = (-1)^n$, $S_{2p} = 1$ and $S_{2p+1} = 0$, $T_{2n} = \frac{n}{2n+1}$ and $T_{2n+1} = \frac{n}{2n+2}$, thus the series $\sum_{n\geq 0} U_n$ is Cesaro summable and has $\frac{1}{2}$ as sum, but the series $\sum_{n\geq 0} U_n$ diverges.

2. If the series $\sum_{n\geq 0} U_n$ converges to ℓ, then it is Cesaro summable and has ℓ as sum.

Definition 1.11. [Fejer Kernel]
For $N \in \mathbb{N}_0$, we set $F_N(x) = \sum_{n=0}^{N} D_N(x)$, $x \in \mathbb{R}$, with D_N the Dirichlet kernel.
F_N is a polynomial trigonometric called the Fejer kernel of degree N.
F_N is even function and $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F_N(t)dt = N + 1$.

Notations
Let $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ be a Riemann-integrable function on $[0, 2\pi]$ and $2\pi-$periodic. Let $(a_n)_n$ and $(b_n)_n$ its trigonometric Fourier coefficients. We define for all $N \in \mathbb{N}_0$

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)),$$

and

$$\Lambda_N(f, x) = \frac{S_0(x) + \ldots S_N(x)}{N+1},$$

then as in the proof of Dirichlet theorem, we have:

$$S_N(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+u)D_N(u)\, du = \frac{1}{2\pi}\int_0^{2\pi} f(x-u)D_N(u)\, du.$$

$$\Lambda_N(f,x)=\frac{1}{2\pi(N+1)}\int_0^{2\pi} f(x+u)F_N(u)\,du=\frac{1}{2\pi(N+1)}\int_0^{2\pi} f(x-u)F_N(u)\,du.$$

The real expression of F_N is

$$F_N(x)=\frac{\sin^2\frac{N+1}{2}x}{\sin^2\frac{x}{2}}.$$

Theorem 1.12.
Let $f\colon\mathbb{R}\longrightarrow\mathbb{C}$ be a Riemann-integrable function on $[0,2\pi]$ and $2\pi-$periodic.

1. Let $x\in\mathbb{R}$ such that $f(x+)$ and $f(x-)$ exist, then

$$\lim_{N\to+\infty}\Lambda_N(f,x)=\frac{f(x+)+f(x-)}{2}.$$

2. The sequence $(\Lambda_N)_N$ converges uniformly on any compact K on which f is continuous.

Proof .

1. We know that $\Lambda_N(f,x)=\dfrac{1}{2\pi(N+1)}\displaystyle\int_0^{2\pi} f(x+u)F_N(u)\,du.$

   Let y be a constant, as $\dfrac{1}{2\pi(N+1)}\displaystyle\int_0^{2\pi} F_N(u)\,du=1$, we have:

$$\begin{aligned}\Lambda_N(f,x)-y &= \frac{1}{2\pi(N+1)}\int_0^{2\pi}(f(x+u)-y)F_N(u)\,du\\ &= \frac{1}{2\pi(N+1)}\int_0^{\pi}(f(x+u)+f(x-u)-2y)F_N(u)\,du.\end{aligned}$$

   We take $y=\frac{f(x+)+f(x-)}{2}$. Let $\varepsilon>0$, $\exists\delta_x>0$ such that $\forall u\in]0,\delta_x[$, $|f(x+u)-f(x+)|<\dfrac{\varepsilon}{2}$ and $|f(x-u)-f(x-)|<\dfrac{\varepsilon}{2}$. There it results that

$$\begin{aligned}|\Lambda_N(f,x)-y| &\le (\frac{\varepsilon}{2}+\frac{\varepsilon}{2})\frac{1}{2\pi(N+1)}\int_0^{\delta_x}F_N(u)\,du\\ &+ \frac{1}{2\pi(N+1)}\int_{\delta_x}^{\pi}|f(x+u)+f(x-u)-2y|F_N(u)\,du\\ &\le \varepsilon+\frac{1}{2\pi(N+1)\sin^2\delta_x/2}\int_0^{\pi}|f(x+u)+f(x-u)-2y|\,du.\end{aligned}$$

f is bounded on $\mathbb{R}$, then there exists $N_0 \in \mathbb{N}_0$ such that for any $N \geq N_0$

$$\frac{1}{2\pi(N+1)\sin^2 \delta_x/2} \int_0^{\pi} |f(x+u)+f(x-u)-2y| \, du \leq \varepsilon.$$

2. We take $\delta > 0$ which does not depends on $x \in K$. (This is possibly, because f is uniformly continuous on K.)

□

Corollary 1.13.
Let $f \colon \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function and $2\pi-$periodic. If the sequence $(S_N)_N$ converges, then its limit is f.

Proof .
Let $g = \lim\limits_{N \to +\infty} S_N$. The sequence $(\Lambda_N)_N$ converges uniformly to f, then $g = f$.
□

Corollary 1.14.
Let $f \colon \mathbb{R} \longrightarrow \mathbb{C}$ be a continuous function and $2\pi-$periodic, then $\forall \varepsilon > 0$, there exists a trigonometric polynomial P_ε such that

$$\sup_{x \in \mathbb{R}} |f(x) - P_\varepsilon(x)| < \varepsilon.$$

Otherwise a continuous function $2\pi-$periodic is limit uniform of trigonometric polynomials.

1.6 The Parseval Identity

Let f be a $2\pi-$periodic function, Riemann-integrable on $[0, 2\pi]$. If

$$C_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-\mathrm{i}nt} \, dt, \quad \text{for } n \in \mathbb{Z}.$$

For $N \in \mathbb{N}_0$,we posed

$$S_N(x) = \sum_{n=-N}^{N} C_n e^{\mathrm{i}nx}, \quad \text{and} \quad \Lambda_N(f,x) = \frac{S_0 + \ldots + S_N(x)}{N+1} = \sum_{k=-N}^{N} \gamma_k e^{\mathrm{i}kx}.$$

$\gamma_0 = C_0$, $\gamma_1 = \frac{N}{N+1} C_1$, $\gamma_{-1} = \frac{N}{N+1} C_{-1}$, $\gamma_p = \frac{N-p+1}{N+1} C_p$ and $\gamma_{-p} = \frac{N-p+1}{N+1} C_{-p}$, $\forall\, p \geq 2$. Then

$$\Lambda_N(f,x) = \sum_{k=-N}^{N} (1 - \frac{|k|}{N+1}) C_k e^{\mathrm{i}kx}.$$

Theorem 1.15. (Parseval Identity)
Let f be a $2\pi-$periodic function and piecewise continuous on $[0, 2\pi]$, then:

$$\sum_{-\infty}^{+\infty} |C_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^2\ dt = \frac{|a_0|^2}{4} + \frac{1}{2}\sum_{n=1}^{+\infty}(|a_n|^2 + |b_n|^2).$$

Lemma 1.16. With the same notations

$$\lim_{N\to+\infty}\int_0^{2\pi} |\Lambda_N(f,x)|^2\ dx = \int_0^{2\pi} |f(t)|^2\ dt.$$

Proof .
Let $x_0 = 0 < x_1 < \ldots < x_s = 2\pi$ such that f is continuous on $]x_i, x_{\mathrm{i}+1}[\ \forall i \in \{0, \ldots, s-1\}$, thus $(\Lambda_N(f))_N$ converges uniformly to f on $I_\eta = [x_i+\eta, x_{\mathrm{i}+1}-\eta]$, for any $\eta > 0$, $\eta \in]0, (x_{\mathrm{i}+1} - x_i)/2[$.
For any $x \in \mathbb{R}$, $|\Lambda_N(f,x)| \leq \dfrac{1}{2\pi(N+1)}\displaystyle\int_{-\pi}^{\pi} |f(x+u)|F_N(u)\ du \leq M$, with $M = \displaystyle\sup_{x\in[0,2\pi]} |f(x)|$, where $(|\Lambda_N(f,.)|^2)_N$ converges uniformly to $|f|^2$ on I_η and then
$\displaystyle\lim_{N\to+\infty}\int_{x_i}^{x_{\mathrm{i}+1}} |\Lambda_N(f,x)|^2\ dx = \int_{x_i}^{x_{\mathrm{i}+1}} |f(x)|^2\ dx$ because for $\varepsilon > 0$,

$$\begin{aligned}
\int_{x_i}^{x_{\mathrm{i}+1}} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx &= \int_{x_i}^{x_{\mathrm{i}}+\varepsilon} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx \\
&\quad + \int_{x_i+\varepsilon}^{x_{\mathrm{i}+1}-\varepsilon} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx \\
&+ \int_{x_{\mathrm{i}+1}-\varepsilon}^{x_{\mathrm{i}+1}} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx \\
&\leq 4\varepsilon M^2 + \int_{x_i+\varepsilon}^{x_{\mathrm{i}+1}-\varepsilon} |\ |\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx
\end{aligned}$$

$$\lim_{N\to+\infty}\int_{x_i+\varepsilon}^{x_{\mathrm{i}+1}-\varepsilon} ||\Lambda_N(f,x)|^2 - |f(x)|^2|\ dx = 0.$$

□

Proof of the theoreme (1.15).

$$\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_N(f,x)|^2\ dx = \sum_{k=-N}^{N}(1 - \frac{|k|}{N+1})|C_k|^2 \leq \sum_{k=-N}^{N} |C_k|^2,$$

$\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_N(f,x)|^2\, dx \leq \sum_{k=-\infty}^{+\infty} |C_k|^2$ and the Bessel inequality yields that

$$\frac{1}{2\pi}\int_0^{2\pi} |\Lambda_N(f,x)|^2\, dx \leq \sum_{k=-\infty}^{+\infty} |C_k|^2 \leq \frac{1}{2\pi}\int_0 |f(x)|^2 dx.$$

$\square$

Corollary 1.17.
Let f be a piecewise continuous function on $\mathbb{R}$ and $2\pi-$periodic. We assume that

$$\langle f, e^{\mathrm{i}nx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathrm{i}nx}dx = 0,\ \forall n \in \mathbb{Z},$$

then f is zero at all its points of continuity and $||f||_2 = 0$.

Remark 20 :
If f and g are two piecewise continuous functions and $2\pi-$periodic. Let C_n (respectively D_n) be the Fourier coefficients of f (respectively g). As the series $\sum_{n\in\mathbb{Z}} |C_n|^2$ and $\sum_{n\in\mathbb{Z}} |D_n|^2$ converge, then the series $\sum_{n\in\mathbb{Z}} C_n\overline{D_n}$ converges absolutely. We consider the map $h(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t+x)}dt$. In using the Fubini formula we prove $\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-\mathrm{i}nx}dx = C_n\overline{D_n}$.
It results that the Fourier series of h converges uniformly and at any point x of continuity of h, $h(x) = \sum_{-\infty}^{+\infty} C_n\overline{D_n}e^{\mathrm{i}nx}$.

1.7 Weierstrass Theorem

Proposition 1.18.
Let $f\colon [a,b] \longrightarrow \mathbb{C}$ be a continuous function. There exists a sequence of polynomials $(Q_n)_n \in \mathbb{C}[X]$ such that $(Q_n)_n$ converges uniformly to f on $[a,b]$.

Proof .
First case: We assume that $a = 0$, $b = 2\pi$ and $f(0) = f(2\pi)$. In this case f can be extended to a continuous function on $\mathbb{R}$ and $2\pi-$periodic. From the corollary (1.14) $\forall\varepsilon > 0$, there exists P_ε a trigonometric polynomial such that

$$\sup_{x\in[0,2\pi]} |f(x) - P_\varepsilon(x)| < \varepsilon.$$

$$P_\varepsilon(x) = \sum_{n=-N}^{N} \alpha_n e^{\mathrm{i}nx}.$$

Moreover, we know that the series $\sum_{n\geq 0} \frac{z^n}{n!}$ converges uniformly on any compact to the function e^z. Thus for any $-N \leq n \leq N$, there exists $d_n \geq 0$ such that

$$\sup_{x\in[0,2\pi]} |e^{\mathrm{i}nx} - \sum_{p=0}^{d_n} \frac{(\mathrm{i}n)^p x^p}{p!}| < \frac{\varepsilon}{\sum_{n=-N}^{N} |\alpha_n|}.$$

We set $R_n(x) = \sum_{p=0}^{d_n} \frac{(\mathrm{i}n)^p x^p}{p!}$ and $H_N = \sum_{-N}^{N} \alpha_n R_n(x)$. H_N is a polynomial.

$$\sup_{x\in[0,2\pi]} |f(x) - H_N(x)| \leq \sup_{x\in[0,2\pi]} |f(x) - P_\varepsilon(x)| + \sup_{x\in[0,2\pi]} |P_\varepsilon(x) - H_N(x)|$$

$$\sup_{x\in[0,2\pi]} |P_\varepsilon(x) - H_N(x)| \leq \sum_{n=-N}^{N} \sup_{x\in[0,2\pi]} |\alpha_n e^{\mathrm{i}nx} - \alpha_n R_n(x)| < \varepsilon.$$

Thus $\sup_{x\in[0,2\pi]} |f(x) - H_N(x)| \leq 2\varepsilon$ and the corollary is proved in this case.

General Case: A from of f one constructed a function which verifies the conditions of the first case.

Define the continuous function g on $[a,b]$ by: $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$, $g(a) = f(a) = g(b)$ and let h be the function defined on $[0,2\pi]$ by: $h(x) = g(x.\frac{b-a}{2\pi} + a)$. h is continuous on $[0,2\pi]$ and $h(0) = h(2\pi)$. Let $\varepsilon > 0$, by the first case, there exists $K_\varepsilon \in \mathbb{C}[x]$ such that $\sup_{x\in[0,2\pi]} |h(x) - K_\varepsilon(x)| < \varepsilon$, thus $\sup_{x\in[a,b]} |g(y) - K_\varepsilon(\frac{2\pi}{b-a}(y-a))| < \varepsilon$.

We set $Q_\varepsilon(y) = K_\varepsilon(\frac{2\pi}{b-a}(y-a))$. This is a polynomial and gives an answer to the corollary.

Other Proof

Theorem 1.19. (Weierstrass Theorem)
Let f be a continuous function on an interval I, there exists a sequence $(f_n)_n$ of polynomials which converges uniformly on any closed and bounded interval I to f.

Proof .
We assume in the first case that f is continuous on $\mathbb{R}$ and identically zero on the complement of the interval $[-\frac{1}{2}, \frac{1}{2}]$. In this case we set

$$P_n(x) = c_n(1-x^2)^n$$

where c_n is chosen such that $\displaystyle\int_{-1}^{1} P_n(x)dx = 1$. We define the sequence

$$f_n(x) = \int_{-\infty}^{+\infty} f(y)P_n(x-y)dy = \int_{-\infty}^{+\infty} f(x-y)P_n(y)dy. \tag{6.8}$$

□

Lemma 1.20.
The functions f_n are polynomials and converge uniformly to f on the interval $[-\frac{1}{2}, \frac{1}{2}]$.

Proof .
From the left side of the formula (6.8), f is a polynomial. From the right side of the formula (6.8), we have for $|x| \leq \frac{1}{2}$

$$f(x) - f_n(x) = \int_{-1}^{1} f(x-y)P_n(y)dy \tag{6.9}$$

Let $\varepsilon > 0$, M the maximum of f on $\mathbb{R}$ and $\delta > 0$ such that $|f(x)-f(x-y)| < \varepsilon$ if $|y| < \delta$. It results from the formula (6.9) that

$$|f(x) - f_n(x)| \leq \int_{|y|<\delta} \varepsilon P_n(y)dy + \int_{\delta \leq |y| \leq 1} M \ P_n(y)dy.$$

We intend to prove that $\displaystyle\int_{\delta \leq |y| \leq 1} P_n(y)dy$ tends to 0 when n tends to infinity.
Let $0 < r < 1$.

$$\frac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n dx \geq \int_{-r}^{r} (1-r^2)^n dx = 2r(1-r^2)^n.$$

Thus $c_n \leq \dfrac{1}{2r(1-r^2)^n}$ and

$$\int_{\delta \leq |y| \leq 1} P_n(y)dy \leq \frac{1}{2r(1-r^2)^n} \int_{-1}^{1} (1-\delta^2)^n dy = \frac{(1-\delta^2)^n}{r(1-r^2)^n}.$$

The result is deduced if we take $r < \delta$ and tends n to infinity.
Proof of the theorem

If f is zero outside the interval $[-s, s]$, the function $F(x) = f(2sx)$ is zero outside the interval $[-\frac{1}{2}, \frac{1}{2}]$. From the previous lemma there exists a sequence $(f_n)_n$ of polynomials which converges uniformly to F on the interval $[-\frac{1}{2}, \frac{1}{2}]$. The sequence of polynomials $g_n(x) = f_n(\frac{x}{2s})$ converges uniformly to f on the interval $[-s, s]$.

If f is continuous on the interval $I = (a, b)$. For any $n \in \mathbb{N}_0$ and $n > \dfrac{2}{b-a}$, there exists a continuous function φ_n on I such that $\varphi_n = 1$ on $[a + \frac{1}{n}, b - \frac{1}{n}]$ and zero outside $[a + \frac{1}{2n}, b - \frac{1}{2n}]$. There exists a polynomial f_n such that $|f_n(x) - \varphi_n(x)f(x)| < \dfrac{1}{n}$ on I. The sequence $(f_n)_n$ is a solution of the problem.

$\square$

1.8 Exercises

6-1-1 Let $t \in \mathbb{R} \setminus \mathbb{Z}$ and $f(x) = \cos tx$, for $-\pi \leq x \leq \pi$ and $2\pi-$periodic.

(a) Give the Fourier series of f.

(b) Deduce that $\cos tx = \dfrac{\sin t\pi}{\pi}\left[\dfrac{1}{t} + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n 2t}{t^2 - n^2}\cos(nx)\right]$, for $x \in [-\pi, \pi]$.

(c) Show that

i. $\dfrac{\pi}{\sin t\pi} = \dfrac{1}{t} + \displaystyle\sum_{n=1}^{\infty} \dfrac{(-1)^n 2t}{t^2 - n^2}$, for $t \notin \mathbb{Z}$.

ii. $\pi \mathrm{cotan} \pi t = \dfrac{1}{t} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2t}{t^2 - n^2}$.

iii. $\dfrac{\pi^2}{\sin^2 \pi t} = \displaystyle\sum_{-\infty}^{+\infty} \dfrac{1}{(t+n)^2}$.

6-1-2 Let $\delta \in]0, \frac{\pi}{2}]$ and let f be the even function 2π-periodic defined by:

$$f(x) = \begin{cases} \frac{2\pi}{\delta}(1 - \frac{x}{2\delta}) & \text{if } 0 \leq x \leq 2\delta \\ 0 & \text{if } 2\delta \leq x \leq \pi \end{cases}$$

(a) Give the Fourier series the function f and prove that this series converges uniformly to f on $\mathbb{R}$.

(b) Compute $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin^2 n\delta}{n^2}$ and $\displaystyle\sum_{n=1}^{+\infty} \frac{\sin^4 n\delta}{n^4}$.

6-1-3 Let f be a continuous function on $\mathbb{R}$ and $2\pi-$periodic.

Prove that if the Fourier series of f is convergent, then f is the sum of its Fourier series.

6-1-4 (a) Prove the following formulas which gives an expansion in trigonometric series of the function $f(x) = x$ in divers intervals, in looking in each case, the periodic function $\varphi(x)$ whose expansion in Fourier series yields the given result.

$$x = \pi - 2\sum_{n=1}^{+\infty} \frac{\sin(nx)}{n} \quad \text{pour} \quad 0 < x < 2\pi.$$

$$x = -2\sum_{n=1}^{+\infty} \frac{(-1)^n \sin(nx)}{n} \quad \text{pour} \quad -\pi < x < \pi.$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} \quad \text{pour} \quad 0 \le x \le \pi.$$

$$x = \frac{4}{\pi}\sum_{n=0}^{+\infty} \frac{(-1)^n \sin(2n+1)x}{(2n+1)^2} \quad \text{for} \quad \frac{-\pi}{2} \le x \le \frac{\pi}{2}.$$

$$x = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{+\infty} \frac{\cos(2n+1)x}{(2n+1)^2} + \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}\sin(nx)}{n} \quad \text{for} \quad 0 \le x < \pi.$$

(b) Deduce

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \quad ; \quad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad ; \quad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

(c) i. In use of the formulas of the question 1) to compute the sum $g(x)$ of the trigonometric series $\displaystyle\sum_{n\ge 0} \frac{\sin(2n+1)\pi x}{(2n+1)^3}$.

ii. Verify the result in compute the Fourier coefficients of g.

6-1-5 Let f be the even function, 2π periodic defined by: $f(x) = \begin{cases} 1 & \text{if} \quad x \in \left[0, \frac{\pi}{2}\right[\\ -1 & \text{if} \quad x \in \left[\frac{\pi}{2}, \pi\right[\end{cases}$

(a) Determine the Fourier coefficients of f.

(b) Deduce the value of the sum $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1}$.

6-1-6 (a) Does there exists a locally Riemann integrable function f such that its Fourier series is $\displaystyle\sum_{n\ge 1} \frac{\sin(nx)}{\sqrt{n}}$?

(b) Same question for the series $\displaystyle\sum_{n\ge 1} \frac{\sin(nx)}{n^3}$.

6-1-7 (a) Determine, for $a > 0$ the expansion in Fourier series of the function $f(x) = \dfrac{1}{\cosh(a) - \cos(x)}$.

(b) Deduce the value of $\displaystyle\int_0^{2\pi} \frac{dx}{\cosh(a) - \cos(x)}$.

$\boxed{6\text{-}1\text{-}8}$ (a) Compute the Fourier series of the following 2π-periodic functions on $\mathbb{R}$ given by:

i. $f(x) = \pi - x$ if $0 \leq x < 2\pi$.

ii. $g(x) = \pi - x$ if $0 \leq x < \pi$, g even.

(b) Deduce that the Fourier series of the $2\pi-$periodic odd function h defined by: $h(x) = x(\pi - \dfrac{x}{2})$ for $0 \leq x \leq \pi$.

$\boxed{6\text{-}1\text{-}9}$ Let φ be the 2π-periodic function on $\mathbb{R}$ defined on $]-\pi,\pi]$ by $\varphi(x) = e^x$.

(a) Compute its Fourier coefficients.

(b) Prove that:

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} = \frac{\pi\cosh\pi + \sinh\pi}{2\sinh\pi}.$$

$\boxed{6\text{-}1\text{-}10}$ (a) Find the Fourier series of the 2π-periodic function

$$f(x) = \begin{cases} 0 & \text{if} \quad -\pi < x \leq 0 \\ x^2 & \text{if} \quad 0 \leq x < \pi \end{cases}$$

(b) Use the first question to compute the following sums:

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} \text{ and } \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}$$

$\boxed{6\text{-}1\text{-}11}$ Let g be the odd 2π-periodic function such that:

$$g(x) = x(\pi - x), \text{ for } 0 \leq x \leq \pi.$$

(a) Give the Fourier series of g.

(b) Use the Parseval identity to compute

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^6}.$$

$\boxed{6\text{-}1\text{-}12}$ (a) Compute the sum of the following series $\displaystyle\sum_{n\geq 1} r^n \cos n\theta$, for $0 < r < 1$.

(b) Deduce the following equality:

$$Q_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = 1 + 2\sum_{n=1}^{+\infty} r^n \cos n\theta = \sum_{-\infty}^{+\infty} r^{|n|} e^{in\theta}.$$

(c) Using the theory of Fourier series, deduce the following value of the integral:

$$I_n(r) = \int_0^{2\pi} \frac{\cos n\theta}{1 - 2r\cos\theta + r^2} d\theta.$$

6-1-13 Let h be the function defined by:

$$h(x) = \frac{x^2 - 1}{x^2 - 4x + 1}.$$

(a) i. Give the power series of h in a neighborhood of 0.
   ii. Compute the radius of convergence of the obtained series.

Let a and z be two complex numbers, such that $|a| \neq |z|$ and $az \neq 0$. Recall that:

$$\frac{1}{z-a} = \begin{cases} \frac{1}{z}\sum_{n=0}^{+\infty}(\frac{a}{z})^n & if \quad |a| < |z|. \\ \frac{-1}{a}\sum_{n=0}^{+\infty}(\frac{z}{a})^n & \text{if} \quad |a| > |z| \end{cases}$$

(b) Prove that there exists a sequence of real numbers $(\lambda_n)_{n\leq 1}$, such that $\forall z \in \mathbb{C}$ such that $(|z| \in]2-\sqrt{3}, 2+\sqrt{3}[)$:

$$h(z) = \sum_{n=1}^{+\infty} \frac{\lambda_n}{z^n} - \sum_{n=1}^{+\infty} \lambda_n z^n$$

Let f be the 2π-periodic function on $\mathbb{R}$ defined by: $f(t) = \dfrac{\sin(t)}{2-\cos(t)}$.

(c) Prove that $h(e^{\mathrm{i}t}) = -\mathrm{i}f(t)$, $\forall t \in \mathbb{R}$.

(d) Deduce the expansion of f in Fourier series.

(e) Deduce the value of the following integral $\displaystyle\int_0^{2\pi} \frac{\sin^2 x}{2-\cos x} dx$.

Let F be the 2π-periodic function defined by: $F(t) = \ln(2 - \cos t)$.

(f) Say why F can has an expansion in Fourier series.

(g) Compute $F'(t)$ and deduce, without compute the Fourier coefficients of F that the Fourier series of F converges normally to F.

(h) Deduce the value of the integral $\displaystyle\int_0^{\pi} \ln(2-\cos x)dx$.

6-1-14 Define the sequence $(f_n)_n$ by: $f_n(x) = \dfrac{1}{a^2 + (x + 2n\pi)^2}$ and $a > 0$.

(a) Prove that the series $\sum_{n\geq 1} f_n$ converges normally on any interval $[-A, A] \subset \mathbb{R}$.

(b) i. Prove that for any $n \geq 1$ and any $t \in \mathbb{R}$, $|f'_n(t)| \leq \dfrac{f_n(t)}{a}$.

ii. Deduce that the series $\sum_{n=1}^{+\infty} f'_n$ converges normally on any interval $[-A, A] \subset \mathbb{R}$.

(c) Deduce that the function

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{a^2 + (x + 2n\pi)^2}$$

is even, 2π-periodic and equal in each point to its Fourier series on $\mathbb{R}$.

(d) i. For any $k \in \mathbb{Z}$ compute the integral

$$I_k(a) = \int_{-\infty}^{+\infty} \frac{\cos kx}{a^2 + x^2} dx.$$

ii. Prove that $\int_0^{2\pi} f(x) \cos kx dx = I_k(a)$.

iii. Give the expression of f.

7 Lebesgue Integral

In this chapter, we present the Lebesgue measure theory and compare it with the Riemann integral.

1 Classes of Subsets of $\mathbb{R}$

1.1 Algebra and $\sigma-$Algebra

Definition 1.1.

1. A non empty collection of subsets $\mathcal{A}$ of $\mathbb{R}$ is called an algebra or a field if:

   (a) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,

   (b) If $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

2. An algebra $\mathscr{A}$ in $\mathscr{P}(\mathbb{R})$ is called a $\sigma-$algebra if every countable intersection of a collection of elements of $\mathscr{A}$ is again in $\mathscr{A}$. That is if $(A_j)_j$ is a sequence in $\mathscr{A}$ then $\bigcap_{j=1}^{+\infty} A_j \in \mathscr{A}$.
   If $\mathscr{A}$ is a $\sigma-$algebra. The pair $(\mathbb{R}, \mathscr{A})$ is called a **measurable space**, and the elements of $\mathscr{A}$ are called measurable subsets.

Properties 1.2.
Let $\mathcal{A}$ be an algebra, then

1. $\emptyset, \mathbb{R} \in \mathcal{A}$;

2. $\mathcal{A}$ is closed under finite union and finite intersection.
   (i.e. if $A_1, \ldots, A_n \in \mathscr{A}$, then $\bigcap_{j=1}^{n} A_j \in \mathscr{A}$ and $\bigcup_{j=1}^{n} A_j \in \mathscr{A}$).

3. Let $\mathscr{A}$ be a $\sigma-$algebra then: if $(A_j)_j$ is a sequence in $\mathscr{A}$, then $\displaystyle\bigcup_{j=1}^{+\infty} A_j \in \mathscr{A}$.

Proof .

1. Since $\mathcal{A}$ is non empty there exists $A \in \mathcal{A}$. So $A^c \in \mathcal{A}$, hence $\emptyset = A \cap A^c \in \mathcal{A}$ and $\mathbb{R} = \emptyset^c \in \mathcal{A}$.

2. Let $A, B \in \mathcal{A}$, then $A^c, B^c \in \mathcal{A}$ and $A^c \cap B^c \in \mathcal{A}$. Since $(A \cup B)^c = A^c \cap B^c \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. By induction we prove that if $A_1, \dots, A_n \in \mathcal{A}$ then $\bigcup_{j=1}^{n} A_j \in \mathcal{A}$ and $\bigcap_{j=1}^{n} A_j \in \mathcal{A}$.

3. We have $A_j^c \in \mathscr{A}$ and $\bigcap_{j=1}^{+\infty} A_j^c \in \mathscr{A}$, hence $\left(\bigcap_{j=1}^{+\infty} A_j^c\right)^c = \bigcup_{j=1}^{+\infty} A_j \in \mathscr{A}$.

$\square$

Example 1.1 :

1. $\mathscr{A} = \{\emptyset, \mathbb{R}\}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$.

2. The power set $\mathscr{P}(\mathbb{R})$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$.

3. Let $\{A, B, C\}$ be a partition of $\mathbb{R}$. The set $\mathcal{A} = \{\emptyset, \mathbb{R}, A, B, C, A^c, B^c, C^c\}$ is an algebra. ($A \cup B = C^c, A \cup C = B^c, B \cup C = A^c$.)

4. Let $\mathcal{A}$ be the collection of subsets A of $\mathbb{R}$ such that either A or A^c is finite. $\mathcal{A}$ is an algebra. but not a σ-algebra.

5. Let $\mathscr{A}$ be the collection of subsets A of $\mathbb{R}$ such that either A or A^c is countable or $\emptyset$. $\mathscr{A}$ is a $\sigma-$algebra. Indeed: let $(A_j)_j$ be a sequence of elements of $\mathscr{A}$. If there exists p such that A_p is countable, then $\cap_{j=1}^{+\infty} A_j \subset A_p$ is countable and $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$. If the sets A_j are all not countable, then the sets A_j^c are countable. The set $\cup_{j=1}^{+\infty} A_j^c$ is countable and $\cap_{j=1}^{+\infty} A_j \in \mathscr{A}$.

Theorem 1.3.
Any intersection of algebras (resp $\sigma-$ algebra) is an algebra (resp $\sigma-$ algebra) i.e. if $(\mathcal{A}_j)_{j \in J}$ is a family of algebras (resp $\sigma-$ algebra) on $\mathbb{R}$, then $\bigcap_{j \in J} \mathcal{A}_j$ is an algebra (resp $\sigma-$ algebra).

Proof .
Consider the case where $\mathcal{A}_j$ are algebra.
$\mathbb{R} \in \mathcal{A}_j$ for all $j \in J$, then $\mathbb{R} \in \bigcap_{j \in J} \mathcal{A}_j$.
If $A \in \bigcap_{j \in J} \mathcal{A}_j$, as $A \in \mathcal{A}_j$ for all $j \in J$, then $A^c \in \bigcap_{j \in J} \mathcal{A}_j$.
Let $A_1, \dots, A_n$ in $\bigcap_{j \in J} \mathcal{A}_j$, then $A_1, \dots, A_n$ are in $\mathcal{A}_j$ for all $j \in J$. Thus $\bigcap_{k=1}^{n} A_k \in \bigcap_{j \in J} \mathcal{A}_j$.

Now, if $\mathscr{A}_j$ are $\sigma-$ algebra.
If $(A_n)_n$ is a sequence in $\bigcap\limits_{j\in J} \mathscr{A}_j$, then $(A_n)_n \in \mathscr{A}_j$ for all $j \in J$. Thus $\bigcap_{n=1}^{+\infty} A_n \in \bigcap_{j\in J} \mathscr{A}_j$.

$\square$

Theorem 1.4.
Let $(\mathscr{A}_j)_{j\in J}$ be a family of $\sigma-$algebras on $\mathbb{R}$, then $\bigcap\limits_{j\in J} \mathscr{A}_j$ is a $\sigma-$ algebra.

Proof .
$\bigcap\limits_{j\in J} \mathscr{A}_j$ is an algebra. Let $(A_n)_n$ be a sequence in $\bigcap\limits_{j\in J} \mathscr{A}_j$. Since each $\mathscr{A}_j$ is a $\sigma-$ algebra then $\bigcap\limits_{n=1}^{+\infty} A_n \in \mathscr{A}_j$ for all $j \in J$. Thus $\bigcap\limits_{n=1}^{+\infty} A_n \in \bigcap\limits_{j\in J} \mathscr{A}_j$. $\square$

Definition 1.5.
Let $\mathcal{B} \subset \mathscr{P}(\mathbb{R})$. The intersection of the algebras (resp $\sigma-$ algebra) on $\mathbb{R}$ that contain $\mathcal{B}$ is the smallest algebra (resp $\sigma-$ algebra) denoted by $\mathcal{A}(\mathcal{B})$ (rep $\sigma(\mathcal{B})$) that contain $\mathcal{B}$. This algebra (resp $\sigma-$ algebra) is called the algebra (resp the $\sigma-$ algebra) generated by $\mathcal{B}$.

Example 1.2 :
Let $\mathscr{A}$ be the $\sigma-$ algebra of subsets $A \subset \mathbb{R}$ such that either A or A^c is countable.
$\mathscr{A}$ is the σ-algebra generated by the singleton sets $S = \{\{x\} : x \in \mathbb{R}\}$.
It is evident that if A or A^c is countable then $A \in \sigma(S)$. Then $\mathscr{A} \subset \sigma(S)$. The other inclusion is evident.

Exercise 1.4 :
Let $\mathcal{A}$ and $\mathcal{B}$ two family of subsets of $\mathbb{R}$.
Prove that

$$\sigma(\mathcal{A}) = \sigma(\mathcal{B}) \iff \begin{cases} \forall A \in \mathcal{A}, & A \in \sigma(\mathcal{B}) \\ \quad \& & \\ \forall B \in \mathcal{B}, & B \in \sigma(\mathcal{A}) \end{cases}$$

Solution:
Il suffices to prove that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B}) \iff A \in \sigma(\mathcal{B}),\ \forall A \in \mathcal{A}$.
Assume that $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$. If $A \in \mathcal{A}$, then $A \in \mathcal{A} \subset \sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$.
Assume that $A \in \sigma(\mathcal{B}),\ \forall A \in \mathcal{A}$. Then $\mathcal{A} \subset \sigma(\mathcal{B})$. Since $\sigma(\mathcal{A})$ is the smallest $\sigma-$ algebra that contain $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \sigma(\mathcal{B})$.

1.2 The Borelian $\sigma-$Algebra

Definition 1.6. [The Borelian $\sigma-$Algebra on $\mathbb{R}$]

Let $\mathscr{B}_{\mathbb{R}}$ be the $\sigma-$algebra generated by the family $\{[a,b[: (a,b) \in \mathbb{R}^2\}$. This $\sigma-$algebra is called the Borel $\sigma-$algebra on $\mathbb{R}$. The elements of $\mathscr{B}_{\mathbb{R}}$ are called Borel subsets of $\mathbb{R}$.

We have the following theorem:

Theorem 1.7.

1. The open and the closed subsets of $\mathbb{R}$ are Borel subsets;
2. $\mathscr{B}_{\mathbb{R}}$ is generated by the family of open subsets in $\mathbb{R}$;
3. $\mathscr{B}_{\mathbb{R}}$ is generated by the family of closed subsets in $\mathbb{R}$;
4. $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a,+\infty[: \ a \in \mathbb{R}\}$;
5. $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]-\infty, a] : \ a \in \mathbb{R}\}$.

Proof .
For the proof we use the exercises (1.1).

1. As any open subset of $\mathbb{R}$ is countable union of open intervals. It suffices to prove that the open intervals are Borel sets. We have $]a,b[= \cup_{n=1}^{+\infty}[a+\dfrac{1}{n}, b[$. Then $]a,b[\in \mathscr{B}_{\mathbb{R}}$.
2. Since $[a,b[= \cap_{n=1}^{+\infty}]a-\dfrac{1}{n}, b[$, then $\mathscr{B}_{\mathbb{R}}$ is generated by the family of open subsets in $\mathbb{R}$;
3. Since $[a,b[= \cup_{n=1}^{+\infty}[a,b-\dfrac{1}{n}]$ and $[a,b] = \cap_{n=1}^{+\infty}[a,b+\dfrac{1}{n}[$, then $\mathscr{B}_{\mathbb{R}}$ is generated by the family of closed subsets in $\mathbb{R}$;
4. The $\sigma-$Algebra generated by the family $\{]a,+\infty[: \ a \in \mathbb{R}\}$ is a subset of the $\sigma-$Algebra generated by open sets. To prove that $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a,+\infty[: \ a \in \mathbb{R}\}$, it suffices to prove that any open interval $]a,b[$ is in the $\sigma-$Algebra generated by the family $\{]a,+\infty[: \ a \in \mathbb{R}\}$.
   We have $]a,b] =]a,+\infty[\cap (]b,+\infty[)^c$ and $]a,b[= \cup_{n=1}^{+\infty}]a,b-\dfrac{1}{n}]$. Then $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]a,+\infty[: \ a \in \mathbb{R}\}$.
5. With the same arguments as in the previous property, $\mathscr{B}_{\mathbb{R}}$ is generated by $\{]-\infty, a] : \ a \in \mathbb{R}\}$.

□

1.3 Exercises

$\boxed{\text{7-1-1}}$ Find all $\sigma-$algebras that contain three elements in $\mathscr{P}(\mathbb{R})$.
Find all $\sigma-$algebras that contain four elements in $\mathscr{P}(\mathbb{R})$.

$\boxed{\text{7-1-2}}$ Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Prove that the set $\mathscr{A} = \{A \subset \mathbb{R} : f^{-1}(f(A)) = A\}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$.

$\boxed{\text{7-1-3}}$ Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a bijective function.
Prove that the set

$$\mathscr{A} = \{A \subset X : \ f(A) \subset A \& \ f^{-1}(A) \subset A\}.$$

is a $\sigma-$algebra.

$\boxed{\text{7-1-4}}$ Let E be a non empty subset of $\mathbb{R}$.
Find all the $\sigma-$algebras generated by the set $\mathscr{C} = \{F : \mathcal{E} \subset F \subset \mathbb{R}\}$.

$\boxed{\text{7-1-5}}$ Let E be infinite subset of $\mathbb{R}$ and $S = \{\{x\} : x \in E\}$.
Find the $\sigma-$algebra generated by S. (Discuss the case of E countable and not countable)

$\boxed{\text{7-1-6}}$ Let A be non-empty subset of $\mathbb{R}$.

(a) Find the $\sigma-$algebra generated by the set $\mathscr{C} = \{B \subset \mathbb{R} : A \subset B\}$.

(b) In which case this $\sigma-$algebra is equal to $\mathscr{P}(\mathbb{R})$?

2 The Lebesgue Measure on $\mathbb{R}$

2.1 Lebesgue Outer Measure

Definition 2.1.

A set function $\mu^*\colon \mathscr{P}(\mathbb{R}) \longrightarrow [0, \infty]$ is called an outer measure or exterior measure on $\mathbb{R}$ if:

1. $\mu^*(\emptyset) = 0$;

2. μ^* is increasing (i.e. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$);

3. $\mu^*(\bigcup\limits_{n=1}^{+\infty} A_n) \leq \sum\limits_{n=1}^{+\infty} \mu^*(A_n)$, for any sequence $(A_n)_n$ of subsets of $\mathbb{R}$.

We give an example of an outer measure on $\mathbb{R}$ which helps us to construct the Lebesgue measure on $\mathbb{R}$.

Proposition 2.2.
Let $\mathcal{A} \subset \mathscr{P}(\mathbb{R})$ be a family of subsets of $\mathbb{R}$ such that $\emptyset, \mathbb{R} \in \mathcal{A}$. Consider a function $\rho\colon \mathcal{A} \longrightarrow [0, +\infty]$ such that $\rho(\emptyset) = 0$. For all subset $A \subset \mathbb{R}$, define

$$\mu^*(A) = \inf\{\sum_{n=1}^{+\infty} \rho(A_n) : \ A_n \in \mathcal{A}, \ A \subset \cup_{n=1}^{+\infty} A_n\}. \tag{7.1}$$

The function μ^* is an outer measure on $\mathbb{R}$.

Proof .
For each subset $A \subset \mathbb{R}$, there exists a sequence $(A_n)_n \in \mathcal{A}$ such that $A \subset \cup_{n=1}^{+\infty} A_n$. (We can take $A_n = \mathbb{R}$). So the function μ^* is well-defined.
It is obvious that $\mu^*(\emptyset) = 0$ and that $\mu^*(A) \leq \mu^*(B)$ if it was $A \subset B$.
Let $(A_n)_n$ be a sequence in $\mathscr{P}(\mathbb{R})$ such that $A \subset \cup_{n=1}^{+\infty} A_n$.
If there exists A_n such that $\rho(A_n) = +\infty$, then $\mu^*(A) \leq \displaystyle\sum_{k=1}^{+\infty} \mu^*(A_k) = +\infty$.
Now assume that $\rho(A_n) < +\infty$ for every $n \in \mathbb{N}$.
For $\varepsilon > 0$, and for each $n \in \mathbb{N}$, there is a sequence $(A_{n,k})_k$ in $\mathcal{A}$ such that $A_n \subset \cup_{k=1}^{+\infty} A_{n,k}$ and

$$\sum_{k=1}^{+\infty} \rho(A_{n,k}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}.$$

We have $A \subset \cup_{n,k=1}^{+\infty} A_{n,k}$ and $\displaystyle\sum_{n,k=1}^{+\infty} \rho(A_{n,k}) \leq \sum_{n=1}^{+\infty} \mu^*(A_n) + \varepsilon.$ □

Remark 21 :
If we take $\mathcal{I}$ is the family of open intervals in $\mathbb{R}$ and the function $\rho(I) = \mathscr{L}(I)$, where $\mathscr{L}(I)$ is the length of I.
In this case, we denote the outer measure defined by this function by λ^*. It is called a the Lebesgue outer measure.

$$\lambda^*(A) = \inf\{\sum_{n=1}^{+\infty} \mathscr{L}(I_n) : \ I_n \in \mathcal{I}, \ A \subset \cup_{n=1}^{+\infty} I_n\}.$$

This outer measure fulfills the following properties:

Lemma 2.3.
For any interval I in $\mathbb{R}$, $\lambda^*(I) = \mathscr{L}(I)$.

Proof .
The result is obvious if the interval is not bounded, and if the interval is bounded I and a and b are its limits, then for any $\varepsilon > 0$, $I \subset]a - \varepsilon, b + \varepsilon[$.

Then $\lambda^*(I) \le \mathscr{L}(I) + 2\varepsilon$ and $\lambda^*(I) \le \mathscr{L}(I)$.
Inversely if $(I_k)_k$ is open covering of I, then $[a+\varepsilon, b-\varepsilon] \subset \cup_{k=1}^{+\infty} I_k$. As the interval $[a+\varepsilon, b-\varepsilon]$ is compact, there is a finite covering $(I_k)_{1\le k\le n}$ of $[a+\varepsilon, b-\varepsilon]$. Therefore $b-a-2\varepsilon \le \displaystyle\sum_{k=1}^{n} \mathscr{L}(I_k) \le \sum_{k=1}^{+\infty} \mathscr{L}(I_k)$. Then $b-a-2\varepsilon \le \lambda^*(I)$ for every $\varepsilon > 0$. Therefore $\lambda^*(I) = \mathscr{L}(I)$.

□

Lemma 2.4.
Let Ω be an open subset of $\mathbb{R}$ and let $(I_n)_n$ the connected components of Ω. Then

$$\lambda^*(\Omega) = \sum_{n=1}^{+\infty} \mathscr{L}(I_n).$$

Proof .
Using the definition of the outer measure λ^*, we have $\lambda^*(\Omega) \le \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$.
Inversely, let $(J_k)_k$ be a covering of Ω by open intervals. As $I_n = \cup_{k=1}^{+\infty} J_k \cap I_n$, then

$$\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \le \sum_{n=1}^{+\infty}\sum_{k=1}^{+\infty} \mathscr{L}(I_n \cap J_k) = \sum_{k=1}^{+\infty}\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k).$$

On the other hand, since the intervals $(I_n)_n$ are disjoint, then $\displaystyle\bigcup_{n=1}^{m} (J_k \cap I_n) \subset J_k$ for every m. Therefore $\displaystyle\sum_{n=1}^{m} \mathscr{L}(J_k \cap I_n) \le \mathscr{L}(J_k)$ and $\displaystyle\sum_{n=1}^{+\infty} \mathscr{L}(I_n \cap J_k) \le \sum_{k=1}^{+\infty} \mathscr{L}(J_k)$. Hence $\displaystyle\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \le \lambda^*(\Omega)$ and therefore $\lambda^*(\Omega) = \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$. □

Theorem 2.5.
For any subset $A \subset \mathbb{R}$, $\lambda^*(A) = \displaystyle\inf_{O\in\mathcal{O}_A} \lambda^*(O)$, where $\mathcal{O}_A$ the collection of open sets that contain the subset A.

Proof .
Let $(I_n)_n$ be any countable covering of $A \subset \mathbb{R}$ formed by open intervals. If $\omega = \bigcup_{n=1}^{+\infty} I_n$, then $\lambda^*(A) \le \lambda^*(\omega) \le \sum_{n=1}^{+\infty} \mathscr{L}(I_n)$. Then $\lambda^*(A) \le \displaystyle\inf_{O\in\mathcal{O}_A} \lambda^*(O)$.
The converse inequality is evident if $\lambda^*(A) = +\infty$.
Assume that $\lambda^*(A) < +\infty$. For $\varepsilon > 0$, there exist a countable covering $(I_n)_n$ of A by open intervals so that $\displaystyle\sum_{n=1}^{+\infty} \mathscr{L}(I_n) \le \lambda^*(A) + \varepsilon$. The open interval $\Omega =$

$\cup_{n=1}^{+\infty} I_n$ contains A and $\lambda^*(\Omega) \leq \sum_{n=1}^{+\infty} \mathscr{L}(I_n) \leq \lambda^*(A) + \varepsilon$. Then $\inf_{O\in\mathcal{O}_A} \lambda^*(O) \leq \lambda^*(A)$. $\square$

Corollary 2.6.
If A is countable subset of $\mathbb{R}$, then $\lambda^*(A) = 0$.

As $\lambda^*\{a\} = \mathscr{L}([a,a]) = 0$, then if $A = \{a_n : n \in \mathbb{N}\}$, $\lambda^*(A) \leq \sum_{n=1}^{+\infty} \lambda^*\{a_n\} = 0$.

Corollary 2.7.
$\mathbb{R}$ and any interval $[a,b]$ are not countable, for $a \neq b$.

Theorem 2.8.
Let $A \subset \mathbb{R}$ and $r \in \mathbb{R}$, then $\lambda^*(A+r) = \lambda^*(A)$ and $\lambda^*(rA) = |r|\lambda^*(A)$.

Proof .
If $A = (a,b)$, then $A+r = (a+r, b+r)$ and if $r \geq 0$, $rA = (ra, rb)$ and if $r \leq 0$, $rA = (rb, ra)$. Therefore $\lambda^*(A+r) = b-a = \lambda^*(A)$ and $\lambda^*(rA) = |r|(b-a) = |r|\lambda^*(A)$.
If A is an open subset, then $A = \cup_{n=1}^{+\infty}(a_j, b_j)$ with $(a_j,b_j) \cap (a_k, b_k) = \emptyset$ for every $j \neq k$ and $\lambda^*(A) = \cup_{n=1}^{+\infty}(b_j - a_j)$. Therefore $\lambda^*(A+r) = \lambda^*(A)$ and $\lambda^*(rA) = |r|\lambda^*(A)$.
In the general case since, for any subset $A \subset \mathbb{R}$, $\lambda^*(A) = \inf_{O\in\mathcal{O}_A} \lambda^*(O)$, where $\mathcal{O}_A$ is the collection of open subsets that contain A, then $\lambda^*(A+r) = \lambda^*(A)$ and $\lambda^*(rA) = |r|\lambda^*(A)$. $\square$

2.2 The Lebesgue $\sigma-$algebra

Definition 2.9.
Let μ^* be an outer measure on $\mathbb{R}$. We say that a subset A of $\mathbb{R}$ is measurable with respect to the outer measure μ^* If

$$\forall X \subset \mathbb{R} : \quad \mu^*(X) = \mu^*(X \cap A) + \mu^*(X \cap A^c).$$

Theorem 2.10.
The set $\mathscr{B}$ of measurable subsets in $\mathbb{R}$ with respect to the outer measure μ^* is a $\sigma-$Algebra.

Proof .

1. As $\mu^*(X \cap \emptyset) + \mu^*(X \cap \emptyset^c) = \mu^*(\emptyset) + \mu^*(X) = \mu^*(X)$ for any subset X in $\mathbb{R}$, then $\emptyset$ is measurable.

2. Let $A \in \mathscr{B}$, the for any subset X in $\mathbb{R}$, $\mu^*(X) = \mu^*(X\cap A) + \mu^*(X \cap A^c)$. This definition is symmetric with respect to A and A^c. Then A^c is also measurable.

3. Let $A, B \in \mathscr{B}$ and X a subset in $\mathbb{R}$. As A is measurable

$$\begin{aligned}
\mu^*(X\cap(A\cup B)) &= \mu^*(X\cap(A\cup B)\cap A)+\mu^*(X\cap(A\cup B)\cap A^c) \\
&= \mu^*(X\cap A)+\mu^*(X\cap B\cap A^c).
\end{aligned}$$

Then

$$\begin{aligned}
\mu^*(X\cap(A\cup B))+\mu^*(X\cap(A\cup B)^c) &= \mu^*(X\cap A)+\mu^*(X\cap B\cap A^c) \\
&\quad +\mu^*(X\cap A^c\cap B^c) \\
&= \mu^*(X\cap A)+\mu^*(X\cap A^c) \\
&= \mu^*(X).
\end{aligned}$$

We deduce that $A\cup B$ is measurable.

4. Let A_1, A_2 be two disjoint measurable sets and X a subset in $\mathbb{R}$. Let $B = X\cap(A_1\cup A_2)$. As $B\cap(A_1\cup A_2)^c = \emptyset$, then

$$\begin{aligned}
\mu^*(B) &= \mu^*(B\cap(A_1\cup A_2))+\mu^*(B\cap(A_1\cup A_2)^c) \\
&= \mu^*(B\cap A_1)+\mu^*(B\cap A_1^c) \\
&= \mu^*(X\cap A_1)+\mu^*(X\cap A_2).
\end{aligned}$$

Therefore $\mu^*(X\cap(A_1\cup A_2)) = \mu^*(X\cap A_1)+\mu^*(X\cap A_2)$.
Let $(A_n)_n$ be disjoint sequence in $\mathscr{B}$ and $X\subset\mathbb{R}$.

$$\begin{aligned}
\mu^*(X) &= \mu^*(X\cap\bigcup_{j=1}^{n}A_j)+\mu^*(X\cap(\bigcup_{j=1}^{n}A_j)^c) \\
&\geq \mu^*(X\cap\bigcup_{j=1}^{n}A_j)+\mu^*(X\cap(\bigcup_{j=1}^{+\infty}A_j)^c) \\
&\geq \sum_{j=1}^{n}\mu^*(X\cap A_j)+\mu^*(X\cap(\bigcup_{j=1}^{+\infty}A_j)^c).
\end{aligned}$$

Then

$$\begin{aligned}
\mu^*(X) &\geq \sum_{n=1}^{+\infty}\mu^*(X\cap A_n)+\mu^*(X\cap(\bigcup_{n=1}^{+\infty}A_n)^c) \qquad (7.2) \\
&\geq \mu^*(X\cap\bigcup_{n=1}^{+\infty}A_n)+\mu^*(X\cap(\bigcup_{n=1}^{+\infty}A_n)^c).
\end{aligned}$$

The inverse inequality results from the outer measure property.

So that to complete the proof, consider a sequence $(B_n)_n$ in $\mathscr{B}$. We define the sequence $(A_n)_n$ as follows: $A_1 = B_1$, $A_n = B_n \setminus \bigcup_{j=1}^{n-1} B_j$. Hence $\bigcup_{n=1}^{+\infty} A_n = \bigcup_{n=1}^{+\infty} B_n$.

Since $\bigcup_{n=1}^{+\infty} A_n \in \mathscr{B}$ then $\bigcup_{n=1}^{+\infty} B_n \in \mathscr{B}$ Therefore $\mathscr{B}$ $\sigma-$algebra.

□

Theorem 2.11.
The Borel sets are measurable with respect to the outer measure λ^*, i.e. $\mathscr{B}_{\mathbb{R}} \subset \mathscr{B}$.

Proof .
It suffice to prove that $]a, +\infty[\in \mathscr{B}$ for any $a \in \mathbb{R}$.
Let X be a subset in $\mathbb{R}$, We want to prove that:

$$\lambda^*(X) = \lambda^*(X \cap]a, +\infty[) + \lambda^*(X \cap] - \infty, a]).$$

As λ^* is an outer measure

$$\lambda^*(X) \leq \lambda^*(X \cap]a, +\infty[) + \lambda^*(X \cap] - \infty, a]).$$

For the inverse inequality, the result is evident if $\lambda^*(X) = +\infty$.
Suppose that $\lambda^*(X) < +\infty$. So for any $\varepsilon > 0$, there exists an open set Ω_ε such that $X \subset \Omega_\varepsilon$ and $\lambda^*(\Omega_\varepsilon) \leq \lambda^*(X) + \varepsilon$.
Assume first that $a \notin \Omega_\varepsilon$.

$$\lambda^*(\Omega_\varepsilon) = \sum_{I \in \mathcal{C}} \mathscr{L}(I) = \sum_{I \in \mathcal{C} \cap]a, +\infty[} \mathscr{L}(I)) + \sum_{I \in \mathcal{C} \cap]-\infty, a[} \mathscr{L}(I),$$

where $\mathcal{C}$ is the set of component connected of Ω_ε. Then

$$\begin{aligned} \lambda^*(\Omega_\varepsilon) &= \lambda^*(\Omega_\varepsilon \cap [a, +\infty[) + \lambda^*(\Omega_\varepsilon \cap] - \infty, a[) \\ &\geq \lambda^*(X \cap [a, +\infty[) + \lambda^*(X \cap] - \infty, a[). \end{aligned}$$

Therefore $\lambda^*(X) \geq \lambda^*(X \cap [a, +\infty[) + \lambda^*(X \cap] - \infty, a])$.
If $a \in \Omega_\varepsilon$, we use the first case, by considering the open set $\Omega'_\varepsilon = \Omega_\varepsilon \setminus \{a\}$ instead of Ω_ε. $(\lambda^*(\Omega'_\varepsilon) = \lambda^*(\Omega_\varepsilon).)$ □

Exercise 2.1 :

We say that a subset $A \subset \mathbb{R}$ is a zero set with respect to outer measure λ^* if there exists a measurable subset B so that $A \subset B$ and $\lambda^*(B) = 0$.
Prove that each zero set is measurable.
Solution
If A is a zero set, there is $B \in \mathscr{B}$ such that $A \subset B$ and $\lambda^*(B) = 0$. If X is a subset of $\mathbb{R}$, then $\lambda^*(X \cap A) = 0$ and

$$\lambda^*(X) \geq \lambda^*(X \cap A^c) = \lambda^*(X \cap A) + \lambda^*(X \cap A^c).$$

The inverse inequality results from the definition of the outer measure λ^*. So the set A is measurable.

2.3 The Lebesgue Measure

2.3.1 Measure Theory

Definition 2.12.
Let $\mathscr{A}$ be a $\sigma-$algebra on $\mathbb{R}$. We say that a function $\mu \colon \mathscr{A} \to [0, \infty]$ is a measure (positive measure) on $\mathscr{A}$ if the following conditions are satisfied:

1. $\mu(\emptyset) = 0$,
2. For any disjoint sequence $(A_n)_n \in \mathscr{A}$, $\mu(\cup_{n=1}^{+\infty} A_n) = \displaystyle\sum_{n=1}^{+\infty} \mu(A_n)$

The set $(\mathbb{R}, \mathscr{A}, \mu)$ is called a measure space.

Examples 11 :

1. If $\mathscr{A} = \mathscr{P}(\mathbb{R})$ and $\mu(A) = \#A$ (number of elements of A if A is finite and $+\infty$ otherwise). The function μ is a measure on $\mathscr{A}$. This measure is called a the counting measure on $\mathbb{R}$.
2. Let $a \in \mathbb{R}$ and $\delta_a(A) = 1$ if $a \in A$ and 0 if $a \notin A$.
   δ_a is a measure called a point measure at a or the Dirac measure at a.
3. Let μ be the function defined on $\mathscr{P}(\mathbb{R})$ as follows: $\mu(A) = 0$ if the set A is finite and $\mu(A) = +\infty$ if the set A is infinite.
   The function μ is not a measure since $\mathbb{N} = \cup_{n=1}^{+\infty}\{n\}$, but $\mu(\mathbb{N}) = +\infty \neq \displaystyle\sum_{n=1}^{+\infty} \mu(\{n\}) = 0$.

Theorem 2.13.
Let $\mathscr{A}$ be a $\sigma-$algebra on $\mathbb{R}$ and μ a measure on $\mathscr{A}$. The measure μ satisfies the following properties:

1. If $A_1, \ldots, A_n \in \mathscr{A}$ are disjoint, then
$$\mu(\cup_{j=1}^n A_j) = \sum_{j=1}^n \mu(A_j).$$
2. If $A, B \in \mathscr{A}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$. (μ is increasing)
3. If $(A_n)_n \in \mathscr{A}$ and $A = \cup_{n=1}^{+\infty} A_n$, then
$$\mu(A) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$
4. If $(A_n)_n$ is increasing sequence in $\mathscr{A}$ and $A = \cup_{n=1}^{+\infty} A_n$, then
$$\mu(A) = \lim_{n \to +\infty} \mu(A_n).$$
5. If $A, B \in \mathscr{A}$ and $A \subset B$ and $\mu(B) < +\infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$ (The result remains true if $\mu(A) < \infty$).
6. If $(A_n)_n$ is a decreasing sequence in $\mathscr{A}$ and $A = \cap_{n=1}^{+\infty} A_n = \lim\limits_{n \to +\infty} A_n$. If $\mu(A_1) < \infty$, then $\mu(A) = \lim\limits_{n \to +\infty} \mu(A_n)$.

Proof .

1. We prove this property by induction.
2. Since $B = A \cup (B \setminus A)$, then $\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A)$.
3. Let $B_1 = A_1$, and $B_n = A_n \setminus \cup_{j=1}^{n-1} B_j$, for every $n \geq 2$. The sets $(B_n)_n$ are disjoint and $A = \cup_{n=1}^{+\infty} B_n = \cup_{n=1}^{+\infty} A_n$. Therefore
$$\mu(A) = \sum_{n=1}^{+\infty} \mu(B_n) \leq \sum_{n=1}^{+\infty} \mu(A_n).$$
4. Let $(B_n)_n$ the sequence defined previously. As $\cup_{j=1}^n A_j = \cup_{j=1}^n B_j$, then
$$\begin{aligned}
\mu(A) &= \mu(\cup_{n=1}^{+\infty} A_n) = \mu(\cup_{n=1}^{+\infty} B_n) \\
&= \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n \to \infty} \sum_{j=1}^{n} \mu(B_j) \\
&= \lim_{n \to \infty} \mu(\cup_{j=1}^n B_j) = \lim_{n \to \infty} \mu(\cup_{j=1}^n A_j) = \lim_{n \to \infty} \mu(A_n).
\end{aligned}$$

5. $\mu(B \setminus A) + \mu(A) = \mu(B)$. If $\mu(A) < \infty$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

6. We apply property (3) to the sequence $(A_1 \setminus A_n)_n$.

$\square$

Example 2.1 :
Let $\mathscr{A}$ be a $\sigma-$algebra on $\mathbb{R}$ and $\mu \colon \mathscr{A} \longrightarrow [0, +\infty]$ a function on $\mathscr{A}$. μ is a measure if and only if:

1. $\mu(\emptyset) = 0$

2. $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$.

3. If $(A_n)_n$ is an increasing sequence in $\mathscr{A}$, then $\mu(\cup_{n=1}^{+\infty} A_n) = \lim\limits_{n \to +\infty} \mu(A_n)$.

If μ is a measure, it fulfills the properties (1) and (2).
Let $(A_n)_n$ be an increasing sequence in $\mathscr{A}$. Define $B_1 = A_1$ and $B_n = A_n \setminus \cup_{j=1}^{n-1} A_j$ for every $n \in \mathbb{N}$. The sequence $(B_n)_n$ is disjoint and $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$. Then

$$\begin{aligned}
\mu\left(\cup_{n=1}^{+\infty} A_n\right) &= \sum_{n=1}^{+\infty} \mu(B_n) = \lim_{n \to +\infty} \sum_{j=1}^{n} \mu(B_j) \\
&= \lim_{n \to +\infty} \mu(\cup_{j=1}^{n} B_j) = \lim_{n \to +\infty} \mu(A_n)
\end{aligned}$$

Inversely, if μ is a function satisfying the properties (1), (2) and (3). If $(A_n)_n$ is a disjoint sequence of measurable sets. So the sequence $\left(B_n = \cup_{j=1}^{n} A_j\right)_n$ is increasing and $\cup_{n=1}^{+\infty} A_n = \cup_{n=1}^{+\infty} B_n$. Therefore

$$\mu(\cup_{n=1}^{+\infty} A_n) = \lim_{n \to +\infty} \mu(B_n) = \lim_{n \to +\infty} \sum_{j=1}^{n} \mu(A_j) = \sum_{n=1}^{+\infty} \mu(A_n).$$

2.3.2 The Uniqueness Theorem

Theorem 2.14.
Let μ and ν two measure on the measurable space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Assume that there exists a class $\mathscr{C} \subset \mathscr{B}_{\mathbb{R}}$ that satisfies the following properties:

1. $\mathbb{R} \in \mathscr{C}$ and if $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$

2. $\mathscr{C}$ generates the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$. ($\sigma(\mathscr{C}) = \mathscr{B}_{\mathbb{R}}$)

3. $\mu(C) = \nu(C) < +\infty$ for every $C \in \mathscr{C}$.

Then $\mu = \nu$.

Remarks 22 :
Let μ and ν two measures that fulfill the hypotheses of the theorem (2.14). Define the family $\mathscr{F} = \{A \in \mathscr{B}_{\mathbb{R}} : \ \mu(A) = \nu(A)\}$. The class $\mathscr{F}$ verifies the following properties:

1. If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$.
   This is because $\mu(A^c) = \mu(\mathbb{R}) - \mu(A) = \nu(\mathbb{R}) - \nu(A) = \nu(A^c)$.

2. If $A, B \in \mathscr{F}$ and $A \subset B$, then $B \cap A^c \in \mathscr{F}$:
   $\mu(B) = \mu(A) + \mu(B \cap A^c) = \nu(B) = \nu(A) + \nu(B \cap A^c)$. Therefore $\mu(B \cap A^c) = \nu(B \cap A^c)$

3. If $(A_n)_n$ is a monotone sequence in $\mathscr{F}$, then $\lim\limits_{n\to+\infty} A_n \in \mathscr{F}$.

Theorem 2.15.
Let $A \in \mathscr{F}$, the set : $\tilde{A} = \{B \in \mathscr{B}_{\mathbb{R}} : \ A \cup B, B \cap A^c, A \cap B^c \in \mathscr{F}\}$ is a $\sigma-$algebra.

Proof .
We have $\emptyset \in \tilde{A}$. Moreover from the definition of $\tilde{A}$, we have $B \in \tilde{A} \iff A \in \tilde{B}$. Also if $A \in \mathscr{F}$ and $B \in \tilde{A}$, then $A \cap B \in \mathscr{F}$. Therefore $\tilde{A} \subset \mathscr{F}$.
We want to prove first that $\mathbb{R} \in \tilde{A}$. We have
$\mu(\mathbb{R} \cup A) = \mu(\mathbb{R}) = \nu(\mathbb{R}) = \nu(\mathbb{R} \cup A)$, $\mu(\mathbb{R} \cap A^c) = \mu(A^c) = \nu(A^c) = \nu(\mathbb{R} \cap A^c)$ and $\mu(\mathbb{R}^c \cap A) = \mu(\emptyset) = \nu(\emptyset) = 0 = \nu(\mathbb{R}^c \cap A)$. Then $\mathbb{R} \in \tilde{A}$.
In this step we want to prove that $A^c \in \tilde{A}$.
$\mu(A \cup A^c) = \mu(\mathbb{R}) = \nu(\mathbb{R}) = \nu(A \cup A^c)$, $\mu(A \cap (A^c)^c) = \mu(A) = \nu(A) = \nu(A \cap (A^c)^c)$, $\mu(A^c \cap A^c) = \mu(A^c) = \nu(A^c) = \nu(A^c \cap A^c)$. Then $A^c \in \tilde{A}$.
Let $B \in \tilde{A}$. We want to prove that $B^c \in \tilde{A}$

$$
\begin{aligned}
\mu(A \cup B^c) &= \mu\left((A \cap B) \cup B^c\right) = \mu(A \cap B) + \mu(B^c) \\
&= \nu(A \cap B) + \nu(B^c) = \nu(A \cup B^c)
\end{aligned}
$$

$\mu(B^c \cap A^c) = \mu(A \cup B)^c = \nu(B \cup A)^c$. Since $B \in \tilde{A}$, then $A \cap B \in \mathscr{F}$. Then $B^c \in \tilde{A}$.
If $(B_n)_n$ is an increasing sequence in $\tilde{A}$ and $B = \lim\limits_{n\to+\infty} B_n$, the sequences $(B_n \cup A)_n$ and $(B_n \cap A^c)_n$ are increasing, so $A \cup B$ and $B \cap A^c$ are elements of $\mathscr{F}$. But the sequence $(A \cap B_n^c)_n$ is decreasing and since $\mu(\mathbb{R}) = \nu(\mathbb{R}) < +\infty$, then $A \cap B^c \in \mathscr{F}$. $\square$

Corollary 2.16.
For every $A \in \mathscr{C}$, $\tilde{A} = \mathscr{B}_{\mathbb{R}}$.

Proof .
If $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$. Therefore $\mu(A \cap B) = \nu(A \cap B)$. On the other hand, since $\mu(A) = \nu(A)$, then $\mu(A \cap B^c) = \nu(A \cap B^c)$ and so $\mu(A^c \cap B) = \nu(A^c \cap B)$. Therefore $\mu(A \cup B) = \nu(A \cup B)$. Since $\tilde{A}$ is a $\sigma-$algebra and since it contains $\mathscr{C}$ then $\tilde{A} = \mathscr{B}_{\mathbb{R}}$. $\square$

Proof of the theorem (2.14).
If $A \in \mathscr{B}_{\mathbb{R}}$, then $A \in \tilde{\mathbb{R}}$. Therefore $A \in \mathscr{F}$. $\square$

Theorem 2.17.
Let μ and ν be two measures on the measurable space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and suppose there is a class $\mathscr{C}$ of measurable sets verifying the following properties:

1. If $A, B \in \mathscr{C}$, then $A \cap B \in \mathscr{C}$.
2. $\mathscr{C}$ generates the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$.
3. $\mu(C) = \nu(C) < +\infty$ for every $C \in \mathscr{C}$.
4. There is an increasing sequence $(X_n)_n$ in $\mathscr{C}$ such that $\mathbb{R} = \lim\limits_{n \to +\infty} X_n$.

Then $\mu = \nu$.

Proof .
Define μ_n and ν_n the measures $\mathscr{B}_{\mathbb{R}}$ as follows: $\mu_n(A) = \mu(A \cap X_n)$ and $\nu_n(A) = \nu(A \cap X_n)$. We deduce from the theorem (2.14) that $\mu_n = \nu_n$ and since the measures $(\mu_n)_n$ and $(\nu_n)_n$ are increasing, then $\mu = \nu$, where μ and ν are the limits respectively of $(\mu_n)_n$ and $(\nu_n)_n$. $\square$

2.3.3 The Lebesgue Measure

Theorem 2.18.
The restriction of the outer measure λ^* on the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ is a measure. We denote this measure by λ and called the Lebesgue measure on $\mathbb{R}$.
λ is the unique measure on $\mathscr{B}_{\mathbb{R}}$ which verifies the following properties:

1. $\lambda([0, 1]) = 1$
2. $\lambda(A + x) = \lambda(A)$, for all $x \in \mathbb{R}$ and for all $A \in \mathscr{B}_{\mathbb{R}}$. (we say that λ is invariant by translation)

Proof .
The restriction of the outer measure λ^* on the $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ is a measure results from the inequality (7.2) if we take the set $X = \cup_{n=1}^{+\infty} A_n$.
The uniqueness: Suppose there are two measures μ and ν on $\mathscr{B}_{\mathbb{R}}$ that they achieve the proof.

As $\nu[0,\frac{1}{n}[\leq \frac{1}{n}$, then $\nu\{0\}=0$ and any finite or countable set is a zero set. Also the intervals $[a,b]$, $]a,b]$, $[a,b[$ and $]a,b[$ has the same measure $b-a$.
Let $\mathscr{C}$ be set of finite union of intervals $[a,b[$, where $a,b\in\mathbb{R}$.
The set $\mathscr{C}$ closed under finite intersection and $\mathbb{R}=\bigcup_{n=1}^{+\infty}[-n,n[$. Then $\mu=\nu$ on $\mathscr{C}$ and using the theorem (2.17), we have $\mu=\nu$ on $\mathscr{B}_{\mathbb{R}}$. □

Remark 23 :
The Lebesgue measure λ can be defined on the $\sigma-$algebra $\mathscr{B}^*=\mathscr{B}\cup\mathscr{N}$, where $\mathscr{N}$ is the set null sets. We proved that $\mathscr{B}_{\mathbb{R}}\subset\mathscr{B}\subset\mathscr{B}^*$.

2.4 Measurable Functions

In which follow, Ω is a measurable set in $\mathbb{R}$.

Definition 2.19.
We say that a function $f\colon\Omega\longrightarrow\mathbb{R}$ is measurable if $f^{-1}(A)\in\mathscr{B}$ for any Borel set A, $(A\in\mathscr{B}_{\mathbb{R}})$.
The of measurable functions on Ω will be denoted by $\mathscr{M}(\Omega)$ and the set of non negative measurable functions on Ω will be denoted by $\mathscr{M}^+(\Omega)$.

Theorem 2.20.
Let $f\colon\Omega\longrightarrow\mathbb{R}$ be a function. The following properties are equivalent:

1. The function f is measurable,
2. $f^{-1}[a,+\infty[\in\mathscr{B}$ for every $a\in\mathbb{R}$,
3. $f^{-1}]-\infty,a[\in\mathscr{B}$, for every $a\in\mathbb{R}$,
4. $f^{-1}]-\infty,a]\in\mathscr{B}$, for every $a\in\mathbb{R}$,
5. $f^{-1}]a,b[\in\mathscr{B}$, for every $a,b\in\mathbb{R}$,
6. $f^{-1}[a,b[\in\mathscr{B}$, for every $a,b\in\mathbb{R}$.

This theorem results from the definition of the Borel $\sigma-$algebra $\mathscr{B}_{\mathbb{R}}$ which generated by any of the following family of sets:

1. $\{[a,+\infty[:\ a\in\mathbb{R}\}$,
2. $\{]a,+\infty[:\ a\in\mathbb{R}\}$,
3. $\{]-\infty,a[:\ a\in\mathbb{R}\}$,
4. $\{]-\infty,a]:\ a\in\mathbb{R}\}$,
5. $\{]a,b[:\ a,b\in\mathbb{R}\}$,
6. $\{[a,b[:\ a,b\in\mathbb{R}\}$,
7. $\{]a,b]:\ a,b\in\mathbb{R}\}$,
8. $\{[a,b]:\ a,b\in\mathbb{R}\}$.

□

Remark 24 :
Let Ω be an open set. Any continuous function $f\colon\Omega\longrightarrow\mathbb{R}$ is measurable.

Theorem 2.21.

1. If $f\in\mathscr{M}(\Omega)$, then the function $|f|\in\mathscr{M}(\Omega)$.

2. If $(f_n)_n$ is a sequence in $\mathscr{M}(\Omega)$, then the following functions are measurable

   (a) $g = \sup_{n\in\mathbb{N}} f_n$,

   (b) $h = \overline{\lim}_{n\to+\infty} f_n$,

   (c) $k = \underline{\lim}_{n\to+\infty} f_n$.

Proof .

1. If $a < 0$, then $\Omega = \{x \in \Omega : \ |f(x)| > a\}$
   If ≥ 0, then

$$\begin{aligned}\{x\in\Omega : \ |f(x)|>a\} &= \{x\in\Omega : \ f(x)>a\}\cup\{x\in\Omega : \ f(x)<-a\} \\ &= f^{-1}(]a,+\infty]) \cup f^{-1}([-\infty,-a[) \in \mathscr{B}.\end{aligned}$$

2. $h(x) = \inf_{n\in\mathbb{N}} (\sup_{j\geq n} f_j(x))$
   $\{x \in \Omega : \ g(x) > a\} = \bigcup_{n\in\mathbb{N}}\{x \in \Omega : \ f_n(x) > a\} \in \ BBB,$

$$\{x \in \Omega : \ h(x) > a\} = \bigcap_{n=1}^{+\infty} \bigcup_{j=n}^{\infty} \{x \in \Omega : \ f_j(x) > a\} \in \mathscr{B}$$

3. $k(x) = \sup_{n\in\mathbb{N}}(\inf_{j\geq n} f_j(x))$.

$$\{x \in \Omega : \ k(x) > a\} = \bigcup_{n=1}^{+\infty} \bigcap_{j=n}^{\infty} \{x \in \Omega : \ f_j(x) > a\} \in \mathscr{B}$$

□

Corollary 2.22.

1. If $f \in \mathscr{M}(\Omega)$, then the functions $f^+ = \sup(f, 0)$ and $f^- = \inf(f, 0)$ are measurable.

2. If $(f_n)_n$ is a pointwise convergent sequence of measurable functions. The limit function f, is measurable.

3. Let $(f_n)_n$ be a sequence of measurable functions. The set C of points $x \in \Omega$ where the sequence $(f_n)_n(x)$ has a limit in $\overline{\mathbb{R}}$ is measurable.

Proof .

1. The proof results from the theorem (2.21).

2. The function $f = \underline{\lim}_{n\to+\infty} f_n$ is measurable.

3. Let $g = \underline{\lim}_{n\to+\infty} f_n$ and $h = \overline{\lim}_{n\to+\infty} f_n$. The set $D = C^c = \{x \in \Omega : \underline{\lim}_{n\to+\infty} f_n(x) < \overline{\lim}_{n\to+\infty} f_n(x)\}$. For every number r, the set

$$D_r = \{x \in \Omega : \ g(x) < r < h(x)\} = \{g(x) < r\} \cap \{h(x) > r\}$$

is measurable, so the set $D = \bigcup_{r\in\mathbb{Q}} D_r$ is also measurable.

$\square$

2.5 Exercises

7-2-1 Let μ be a measure on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$. Prove that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$$

for every $A, B \in \mathscr{B}_{\mathbb{R}}$.

7-2-2 Give an example of measure μ on $(\mathbb{R}, \mathscr{B}_{\mathbb{R}})$ and a decreasing sequence $(A_n)_n$ such that $\lim\limits_{n\to+\infty} \mu(A_n) \neq \mu(\lim\limits_{n\to+\infty} A_n)$.

7-2-3 Let $\varepsilon > 0$. Give a dense open subset of $\mathbb{R}$ and its measure is less than ε.

7-2-4 Let A be a measurable set in $\mathbb{R}$ of finite measure.
Prove that the function $f(x) = \lambda(A \cap]-\infty, x])$ is continuous.

7-2-5 Prove that for each increasing function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is measurable.

7-2-6 Let $f\colon \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be a measurable function.
Prove that the set $\{x \in \mathbb{R} : \ f(x) \neq 0\}$ is measurable.

7-2-7 Let $(\mathbb{R}, \mathscr{B}, \lambda)$ be the measure space where λ is the Lebesgue measure and $\mathscr{B}$ the Lebesgue $\sigma-$algebra.
For every measurable set A, we define the function μ as follows:

$$\mu(A) = \int_A \frac{1}{1+x^2} d\lambda(x).$$

Prove that μ is a measure.

7-2-8 Let f be an integrable function on the measure space $(\mathbb{R}, \mathscr{B}_{\mathbb{R}}, \lambda)$.
Prove that the set $\{x \in \mathbb{R} : \ f(x) = \pm\infty\}$ is a null set.

7-2-9 Let f be an integrable function such that $\displaystyle\int_E f(x) d\mu(x) \ = 0$ for all measurable set E.
Prove that $f = 0$ a.e.

7-2-10 Prove that the two functions $\sin(x^2)$ and $\cos(x^2)$ are not integrable on $[0, +\infty[$.

3 The Lebesgue Integration

3.1 Simple Functions

Definition 3.1.
A function $f\colon \Omega \longrightarrow \mathbb{R}$ is called simple if it is measurable and takes a infinite number of values.

If $f\colon \Omega \longrightarrow \bar{\mathbb{R}}$ is a simple function and if $\{c_1, \ldots, c_m\}$ are the different values of f, then $f = \sum_{j=1}^{m} c_j \chi_{A_j}$, where $A_j = f^{-1}\{c_j\}$ and the function f is measurable if and only if the sets A_j are measurable for each $j = 1, \ldots, m$.

Theorem 3.2.
Let $f\colon \Omega \longrightarrow \overline{\mathbb{R}}$

1. If f is a bounded measurable function, there exists a sequence of simple functions which converges uniformly on Ω to f.

2. If f is a non-negative measurable function, there exists a sequence of non-negative simple functions which increases to f.

Proof .

1. Let $M > 0$ such that $|f(x)| < M$ for every $x \in \Omega$. For $(n, k) \in \mathbb{N}_0 \times \mathbb{Z}$ and $-2^n \leq k \leq 2^n - 1$, consider the measurable subsets

$$A_{n,k} = \{x \in \Omega : \frac{kM}{2^n} \leq f(x) < \frac{(k+1)M}{2^n}\}$$

and the measurable functions $f_n = \sum_{k=-2^n}^{2^n-1} \frac{kM}{2^n} \chi_{A_{n,k}}$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $x_0 \in \Omega$, there exists k_0 such that $x_0 \in A_{n,k_0}$. Then $f_n(x_0) = \frac{Mk_0}{2^n}$ and $|f(x_0) - f_n(x_0)| < \frac{M}{2^n}$. Hence, the sequence $(f_n)_n$ converges uniformly on Ω to f.

2. For $n \in \mathbb{N}$, the function $g_n = \inf(f, n) - \frac{1}{n}$ is bounded and measurable, then from the first case there exists a sequence $(f_m)_m$ of simple functions such that $||f_n - g_n||_\infty < \frac{1}{2^n}$. Therefore

$$\lim_{n\to+\infty} f_n = \lim_{n\to+\infty} g_n = \lim_{n\to+\infty} \inf(f, n) = f.$$

$$f_n \leq g_n + \frac{1}{2^n} = \inf(f,n) - \frac{1}{n} + \frac{1}{2^n} \leq \inf(f,n+1) - \frac{1}{n+1} + \frac{1}{2^{n+1}} \leq f_{n+1}.$$

(It suffices to prove that for n big enough $-\frac{1}{n} + \frac{1}{2^n} < -\frac{1}{n+1} + \frac{1}{2^{n+1}}$.)

So the sequence $(f_n)_n$ increasing.

$\square$

3.2 The Lebesgue Integration

To define the Lebesgue integral of measurable functions, we first define the integral of non negative positive simple functions. Then we define the integral of non-negative measurable functions using the increasing limit. For arbitrary measurable functions f, we use the decomposition $f = f^+ - f^-$ as the difference of two non-negative measurable functions and we extend the definition of the integral to the measurable functions only if one of the integral of f^+ or f^- is finite.

Definition 3.3.

If $f = \sum_{k=1}^{N} c_k \chi_{\{f=c_k\}}$ is a non negative simple function, we define the integral of the function f as follows:

$$\int_\Omega f(x)d\,\lambda(x) = \sum_{k=1}^{N} c_k \lambda(\{f = c_k\}). \tag{7.3}$$

If $A = \{x \in \Omega : \ f(x) = 0\}$ and $\lambda(A) = +\infty$ or if $A = \{x \in \Omega : \ f(x) = +\infty\}$ and $\lambda(A) = 0$, we assume that $0.\infty = 0$.

Theorem 3.4.

Let $\mathscr{E}^+$ be the set of non negative simple functions defined on Ω. The integral defined on $\mathscr{E}^+$ fulfills the following properties:

1. $\int_\Omega \alpha\, f(x)d\,\lambda(x) = \alpha \int_\Omega f(x)d\,\lambda(x)$ for every $\alpha \in \mathbb{R}^+$ and for each $f \in \mathscr{E}^+$.
2. $\int_\Omega (f+g)(x)d\lambda(x) = \int_\Omega f(x)d\,\lambda(x) + \int_\Omega g(x)d\lambda(x)$ for every $f, g \in \mathscr{E}^+$.
3. $\int_\Omega f(x)d\,\lambda(x) \leq \int_\Omega g(x)d\,\lambda(x)$ for every $f, g \in \mathscr{E}^+$ such that $f \leq g$.
4. If $(f_n)_n$ is an increasing sequence in $\mathscr{E}^+$ and if $\lim_{n\to+\infty} f_n = f \in \mathscr{E}^+$, then $\int_\Omega f(x)d\,\lambda(x) = \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x)$.

Proof .
It is obvious that if $\alpha \geq 0$ and f and g are in $\mathscr{E}^+$ then $\alpha f \in \mathscr{E}^+$ and $f+g \in \mathscr{E}^+$.

1. The first property is evident.
2. Let f and g be two elements of $\mathscr{E}^+$ and let F (resp G) be the set of values of f (resp of g). We have:

$$f = \sum_{a\in F} a\chi_{\{f=a\}}, \quad g = \sum_{b\in G} b\chi_{\{g=b\}}.$$

$$\{f = a\} = \bigcup_{b\in G} \{f = a, g = b\}, \qquad \forall\ a \in F$$

$$\{g = b\} = \bigcup_{a\in F} \{f = a, g = b\}, \qquad \forall\ b \in G$$

$$\int_\Omega f(x) d\,\lambda(x) = \sum_{a\in F} a\lambda\{f = a\} = \sum_{(a,b)\in F\ timesG} a\lambda\{f = a, g = b\}$$

$$\int_\Omega g(x) d\,\lambda(x) = \sum_{b\in G} a\lambda\{g = b\} = \sum_{(a,b)\in F\ timesG} b\lambda\{f = a, g = b\}$$

$$\int_\Omega f(x) d\,\lambda(x) + \int_\Omega g(x) d\,\lambda(x) = \sum_{(a,b)\in F\times G} (a + b)\lambda\{f = a, g = b\}$$

$\{f + g = u\} = \bigcup_{(a,b)\in F\times G, a+b=u} \{f = a, g = b\}$. Therefore

$$\lambda\{f + g = u\} = \sum_{(a,b)\in F\times G, a+b=u} \lambda\{f = a, g = b\}.$$

Then

$$\begin{aligned} \int_\Omega f(x) d\,\lambda(x) + \int_\Omega g(x) d\,\lambda(x) &= \sum_u u\lambda\{f + g = u\} \\ &= \int_\Omega (f + g)(x) d\,\lambda(x). \end{aligned}$$

3. If $\int_\Omega f(x)d\,\lambda(x) = +\infty$, then $\int_\Omega g(x)d\,\lambda(x) = +\infty$.
The result is evident if $\int_\Omega f(x)d\,\lambda(x) < +\infty$ and the $\int_\Omega g(x)d\,\lambda(x) = +\infty$
.
Suppose that $\int_\Omega f(x)d\,\lambda(x) < +\infty$ and $\int_\Omega g(x)d\,\lambda(x) < +\infty$.
So the sets $\{x \in \Omega : \ f(x) = +\infty\}$ and $\{x \in \Omega : \ g(x) = +\infty\}$ are zero sets. Let $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_m\}$ the sets of finite values of f respectively of g.
$\tilde{f} = \sum_{j=1}^{n} a_j \chi_{\{x\in\Omega:\ f(x)=a_j\}}$ and $\tilde{g} = \sum_{j=1}^{m} b_j \chi_{\{x\in\Omega:\ g(x)=b_j\}}$. Therefore $\int_\Omega f(x)d\,\lambda(x) = \int_\Omega \tilde{f}(x)d\,\lambda(x)$ and $\int_\Omega g(x)d\,\lambda(x) = \int_\Omega \tilde{g}(x)d\,\lambda(x)$ and $h = \tilde{g} - \tilde{f} \in \mathscr{E}^+$.
We deduce from 2. that

$$\int_\Omega g(x)d\,\lambda(x) = \int_\Omega f(x)d\,\lambda(x) + \int_\Omega h(x)d\,\lambda(x) \geq \int_\Omega f(x)d\,\lambda(x)).$$

□

Lemma 3.5.
Let $(f_n)_n$ be an increasing sequence in $\mathscr{E}^+$. if there exists $g \in \mathscr{E}^+$ such that $g \leq \lim_{n\to+\infty} f_n$, then $\int_\Omega g(x)d\,\lambda(x) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x)$.

Proof .
Let $E_y = \{x \in \Omega : \ g(x) = y\}$ for every $y \in g(\Omega)$. To prove the lemma it therefore suffices to prove that for all $y \in g(X)$

$$\int_\Omega g(x)\chi_{E_y}(x)d\,\lambda(x) = y\lambda(E_y) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x).$$

The result is obvious if $y = 0$.
Now suppose that $y > 0$, for every $0 < t < y$, define the sets $A_n = E_y \cap \{x \in \Omega : \ f_n(x) \geq t\}$.
The sequence $(A_n)_n$ is increasing and measurable and $E_y = \lim_{n\to+\infty} A_n$ because for $x \in E_y$, $f_n(x) > t$ for every n big enough.

$$\begin{aligned} t\lambda\{E_y \cap \{x \in \Omega : \ f_n(x) > t\}\} &= \int_\Omega t\chi_{E_y\cap\{x\in\Omega:\ f_n(x)>t\}}(x)d\,\lambda(x) \\ &\leq \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x). \end{aligned}$$

$t\lambda(E_y) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x).$
This is for every $0 < t < y$. Therefore

$$y\lambda(E_y) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)\chi_{E_y}(x)d\,\lambda(x).$$

To prove (4), we define the function $g = \lim_{n\to+\infty} f_n$.
$f_n \leq g$, for $n \in \mathbb{N}$ and the sequence $\left(\int_\Omega f_n(x)d\,\lambda(x)\right)_n$ is increasing and bounded above by the number $\int_\Omega g(x)\lambda(x)$.
To prove the other inequality, we apply the lemma (3.5). □

Definition 3.6.
Let f be a non negative measurable function, we define the integral of f by:

$$\int_\Omega f(x)d\,\lambda(x) = \sup\{\int_\Omega g(x)d\,\lambda(x) : g \leq f,\ g \in \mathscr{E}^+\}.$$

This is a non negative real number or $+\infty$.

Remark 25 :
If f is a non negative measurable function, by theorem (3.2) there exists an increasing sequence $(f_n)_n$ in $\mathscr{E}^+$ which converges to f. We conclude from which above that $\lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x) \leq \int_\Omega f(x)d\,\lambda(x)$. On the other hand, according to the lemma (3.5) for any function $g \in \mathscr{E}^+$ such that $g \leq f = \lim_{n\to+\infty} f_n$, we have $\int_\Omega g(x)d\,\lambda(x) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x)$. So by definition (3.6) $\int_\Omega f(x)d\,\lambda(x) \leq \lim_{n\to+\infty} \int_\Omega f_n(x)d\lambda(x)$. Therefore

$$\int_\Omega f(x)d\,\lambda(x) = \lim_{n\to+\infty} \int_\Omega f_n(x)d\,\lambda(x).$$

. This result is not related to the sequence $(f_n)_n$ in $\mathscr{E}^+$ which converges to f.

Theorem 3.7.
If f and g are in $\mathscr{M}^+(\Omega)$ and $\alpha \geq 0$, then

1. $\int_\Omega \alpha f(x)d\,\lambda(x) = \alpha \int_\Omega f(x)d\,\lambda(x)$
2. $\int_\Omega (f+g)(x)d\,\lambda(x) = \int_\Omega f(x)d\,\lambda(x) + \int_\Omega g(x))d\,\lambda(x)$

3. If $f \leq g$, then $\displaystyle\int_\Omega f(x)d\,\lambda(x) \leq \int_\Omega g(x)d\,\lambda(x)$.

Proof .
For proof, it suffices to take two increasing sequences $(f_n)_n$ and $(g_n)_n$ in $\mathscr{E}^+$ which converge respectively to f and g, and we apply the theorem (3.4). $\square$

Definition 3.8.
Let f, g two functions. We say that $f = g$ outside a zero set or $f = g$ a.e. If the set $\{x \in \Omega : \ f(x) \neq g(x)\}$ is a null set.
Let A be a measurable set. The function $\chi_A = 0$ a.e. if and only if $\lambda(A) = 0$.

Definition 3.9.
We say that a function f is defined a.e. on Ω, if there exists a null set N so that the function f is defined on $\Omega \setminus N$.

Definition 3.10.
We say that sequence of functions $(f_n)_n$ on Ω is convergent a.e. if there exists a function f such that $\{x \in \Omega : f_n(x) \not\longrightarrow f(x)\}$ is a null set.

Theorem 3.11.
Let f, g be two functions in $\mathscr{M}^+(\Omega)$.

1. $\displaystyle\int_\Omega f(x)d\,\lambda(x) = 0$ If and only if $f = 0$ a.e.

2. If $f = g$ a.e then $\displaystyle\int_\Omega f(x)d\,\lambda(x) = \int_\Omega g(x).\lambda(x)$.

Proof .

1. Suppose that $\displaystyle\int_\Omega f(x)d\,\lambda(x) = 0$. Then for every $n \in \mathbb{N}$, the subsets $A_n = \{x \in \Omega : \ f(x) \geq \frac{1}{n}\}$ are measurable and $\chi_{A_n} \leq nf$. Then

$$\int_\Omega \chi_{A_n}(x)d\,\lambda(x) = \lambda(A_n) \leq n\int_\Omega f(x)d\,\lambda(x) = 0$$

   and $\lambda(A_n) = 0$, for every $n \in \mathbb{N}$. Therefore $\{x : \ f(x) \neq 0\} = \bigcup_{n=1}^{+\infty} A_n$ is a null set.
   If $f = 0$ a.e, the set $A = \{x \in \Omega : \ f(x) \neq 0\}$ is a null set and the function $g = \infty.\chi_A$ is a simple and $f \leq g$. As $\displaystyle\int_\Omega g(x)d\lambda(x) = 0$, then $\displaystyle\int_\Omega f(x)d\,\lambda(x) = 0$.

2. suppose that $f \leq g$. the function $h = g - f$ is defined a.e and equal to 0 a.e.

   If $\displaystyle\int_\Omega f(x) d\,\lambda(x) = \int_\Omega g(x) d\,\lambda(x) = +\infty$, the result is correct.

   If $\displaystyle\int_\Omega f(x) d\,\lambda(x) < +\infty$, and $\displaystyle\int_\Omega g(x) d\,\lambda(x) < +\infty$, then

   $\displaystyle 0 = \int_\Omega h(x) d\,\lambda(x) = \int_\Omega g(x) d\,\lambda(x) - \int_\Omega f(x) d\,\lambda(x).$

   The function $h = \inf(f, g)$ is non negative and measurable and $h = f = g$ a.e. As $h \leq f$ then $\displaystyle\int_\Omega f(x) d\,\lambda(x) = \int_\Omega f(x) d\,\lambda(x)$. Also as $h \leq g$, then $\displaystyle\int_\Omega h(x) d\,\lambda(x) = \int_\Omega g(x) d\,\lambda(x)$. We conclude that $\displaystyle\int_\Omega f(x) d\,\lambda(x) = \int_\Omega g(x) d\,\lambda(x)$.

□

Definition 3.12.

We say that a function $f \colon \Omega \longrightarrow \bar{\mathbb{R}}$ is integrable if the functions f^+ and f^- are integrable, where $f^+ = \sup(f, 0)$ and $f^- = \sup(-f, 0)$. In this case we define the integral of f as:

$$\int_\Omega f(x) d\,\lambda(x) = \int_\Omega f^+(x) d\,\lambda(x) - \int_\Omega f^-(x) d\,\lambda(x).$$

Also if the function f is measurable and $\displaystyle\int_\Omega f^+(x) d\,\lambda(x) < \infty$ or $\displaystyle\int_\Omega f^-(x) d\,\lambda(x) < \infty$ We define the integral of the function f on Ω by:

$$\int_\Omega f(x) d\,\lambda(x) = \int_\Omega f^+(x) d\,\lambda(x) - \int_\Omega f^-(x) d\,\lambda(x).$$

The set of integrable functions on Ω is denoted by $\mathcal{L}^1(\Omega)$.

Theorem 3.13.

The set $\mathcal{L}^1(\Omega)$ is a vector space on $\mathbb{R}$ and the map $f \longmapsto \displaystyle\int_\Omega f(x) d\,\lambda(x)$ is linear on the space $\mathcal{L}^1(\Omega)$ and

$$\Big| \int_\Omega f(x) d\,\lambda(x) \Big| \leq \int_\Omega \big| f(x) \big| d\,\lambda(x),$$

for every $f \in \mathcal{L}^1(\Omega)$.

Proof .
As $\left|f+g\right| \leq |f|+|g|$, for every $f, g \in \mathscr{M}(\Omega)$, then

$$\int_\Omega \left|f(x)+g(x)\right| d\,\lambda(x)| \leq \int_\Omega \left|f(x)\right| d\,\lambda(x) + \int_\Omega \left|g(x)\right|\, d\,\lambda(x).$$

If $f+g \in \mathcal{L}^1(\Omega)$.
$f+g = (f+g)^+ - (f+g)^- = f^+ - f^- + g^+ - g^-$.
Then $(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$, and

$$\begin{aligned}
\int_\Omega (f+g)^+(x) d\,\lambda(x) &+ \int_\Omega f^-(x) d\,\lambda(x) + \int_\Omega g^-(x)\; d\,\lambda(x) \\
&= \int_\Omega (f+g)^-(x) d\,\lambda(x) + \int_\Omega f^+(x)\; d\,\lambda(x) \\
&\quad + \int_\Omega g^+(x) d\,\lambda(x)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega (f+g)(x) d\,\lambda(x) &= \int_\Omega (f+g)^+(x) d\,\lambda(x)) - \int_\Omega (f+g)^-(x)\; d\,\lambda(x) \\
&= \int_\Omega f^+(x) d\,\lambda(x) - \int_\Omega f^-(x)\; d\,\lambda(x) \\
&\quad + \int_\Omega g^+(x) d\,\lambda(x) - \int_\Omega g^-(x) d\,\lambda(x) \\
&= \int_\Omega f(x) d\,\lambda(x) + \int_\Omega g(x) d\,\lambda(x).
\end{aligned}$$

The other properties are evident. □

Corollary 3.14.

1. If the function is f measurable and $a \leq f \leq b$ and $\lambda(\Omega) < +\infty$, then $f \in \mathcal{L}^1(\Omega)$ and $a\lambda(\Omega) \leq \displaystyle\int_\Omega f(x)\; d\,\lambda(x) \leq b\lambda(\Omega)$.
2. If $f \leq g$, where $f \in \mathscr{M}(\Omega)$ and $g \in \mathcal{L}^1(\Omega)$, then $\displaystyle\int_\Omega f(x) d\,\lambda(x) \leq \int_\Omega g(x) d\,\lambda(x)$.
3. If E is a measurable null set, $\displaystyle\int_E f(x) d\,\lambda(x) = 0$ for every measurable function f.

Remarks 26 :

1. If f is an integrable function, then the set $\{x \in \Omega : \; f(x) = \pm\infty\}$ is a null set.

2. We introduce the equivalence relation $\sim$ on $\mathcal{L}^1(X, \mathscr{A}, \mu)$ by setting $f \sim g \iff f = g$ a.e. Thus we may consider the quotient space $L^1(X, \mathscr{A}, \mu) = \mathcal{L}^1(X, \mathscr{A}, \mu)/_{\sim}$. This space is often abbreviated to $L^1(\mu)$.

3.3 The Monotone Convergence Theorem

Theorem 3.15. [Monotone Convergence Theorem]
(The theorem is called also the Beppo-Levi's Theorem)
Let $(f_n)_n$ be an increasing sequence of non-negative measurable functions on Ω, then

$$\int_\Omega \lim_{n\to+\infty} f_n(x) d\,\lambda(x) = \lim_{n\to+\infty} \int_\Omega f_n(x) d\,\lambda(x).$$

.

Proof .
For every $n \in \mathbb{N}$, there exists a non-negative increasing sequence $(\varphi_{n,j})_j$ in $\mathscr{E}^+$ which converge to f_n. For every j, define the function $\psi_j = \sup\limits_{1\le n\le j} \varphi_{n,j}$. The sequence $(\psi_j)_j \in \mathscr{E}^+$ is increasing because

$$\psi_j = \sup_{1\le n\le j} \varphi_{n,j} \le \sup_{1\le n\le j} \varphi_{n,j+1} \le \sup_{1\le n\le j+1} \varphi_{n,j+1} = \psi_{j+1}.$$

for every $j \ge n$, $\varphi_{n,j} \le \psi_j$, therefore $f_n = \lim\limits_{j\to+\infty} \varphi_{n,j} \le \lim\limits_{j\to+\infty} \psi_j$. Then $f = \lim\limits_{n\to+\infty} f_n \le \lim\limits_{j\to+\infty} \psi_j$. on the other side inequalities $\varphi_{n,j} \le f_n \le f$ prove that $\psi_j \le f$ and $\lim\limits_{j\to+\infty} \psi_j \le f$. The sequence $(\psi_j)_j$ is increasing in $\mathscr{E}^+$ with limit f. Then $\displaystyle\int_\Omega f(x) d\,\lambda(x) = \lim_{j\to+\infty} \int_\Omega \psi_j(x) d\,\lambda(x)$. Moreover $\psi_j \le f_j$, then

$$\lim_{j\to+\infty} \int_\Omega \psi_j(x) d\,\lambda(x) \le \lim_{j\to+\infty} \int_\Omega f_j(x) \; d\,\lambda(x) \le \int_\Omega f(x) d\,\lambda(x).$$

$\square$

Corollary 3.16.
Let $(f_n)_n \in \mathscr{M}^+(\Omega)$ be a sequence, then

$$\int_\Omega \sum_{n=1}^{+\infty} f_n(x) d\,\lambda(x) = \sum_{n=1}^{+\infty} \int_\Omega f_n(x) d\,\lambda(x).$$

Corollary 3.17.
Let $f \in \mathscr{M}^+(\Omega)$, then for every $A \in \mathscr{B}_{\mathbb{R}}$, the function

$$\mu(A) = \int_{\Omega} f(x)\chi_A(x)d\,\lambda(x)$$

is a measure on $\mathscr{B}_{\mathbb{R}}$.

Proof .
Let $(A_n)_n$ be a disjoint sequence of measurable sets ($A_j \cap A_k = \emptyset$ for every $j \neq k$). Then $f\chi_{\cup_n A_n} = \sum_{n=1}^{+\infty} f\chi_{A_n}$ and

$$\begin{aligned}
\mu\,(\bigcup_n A_n) &= \int_{\Omega} f(x)\chi_{\cup_n A_n}(x)d\,\lambda(x) \\
&= \int_{\Omega} \sum_{n=1}^{+\infty} f(x)\chi_{A_n}(x)d\,\lambda(x) \\
&= \sum_{n=1}^{+\infty} \int_{\Omega} f(x)\chi_{A_n}(x)d\,\lambda(x).
\end{aligned}$$

The second part of the result is true if the function f is the characteristic function of a measurable set, and therefore is true for every simple function. So if f is a non negative measurable function, there exists an increasing sequence of simple functions which increases to f. We get the result using the monotone convergence theorem. □

3.4 Fatou's Lemma

Lemma 3.18. [Fatou's Lemma]
If $(f_n)_n \in \mathscr{M}^+(\Omega)$, then

$$\int_{\Omega} \underline{\lim}_{n\to+\infty} f_n(x)d\,\lambda(x) \leq \underline{\lim}_{n\to+\ infty} \int_{\Omega} f_n(x)d\,\lambda(x).$$

Proof .
$\underline{\lim}_{n\to+\infty} f_n = \lim_{n\to+\infty}(\inf_{j\geq n} f_j)$. Therefore $\int_{\Omega} \inf_{j\geq n} f_j(x)\,d\,\lambda(x) \leq \inf_{j\geq n} \int_{\Omega} f_j(x)d\,\lambda(x)$ and we get the result using the monotone convergence theorem. □

Example 3.1 :
Let $f_n = n^2\chi_{[0,\frac{1}{n}]}$, $\int_{\mathbb{R}} \underline{\lim}_{n\to+\infty} f_n(x)d\lambda(x) = 0$ and $\underline{\lim}_{n\to+\infty} \int_{\mathbb{R}} f_n(x)d\lambda(x) = +\infty$

3.5 Dominate Convergence Theorem

Theorem 3.19. [Dominate Convergence Theorem or Lebesgue's theorem]
Let $(f_n)_n \in \mathscr{M}(\Omega)$ such that

1. $(f_n)_n$ converges a.e. to a function f defined a.e.

2. There exists a non negative integrable function g so that: $|f_n| \leq g$ a.e. for every n.
   Then the sequence $(f_n)_n$ and the function f is integrable and

$$\int_\Omega f(x)\, d\,\lambda(x) \;=\; \lim_{n\to+\infty} \int_\Omega f_n(x) d\,\lambda(x).$$

.

Theorem 3.20.
Let $(f_n)_n \in \mathscr{M}(\Omega)$. Assume that there is a non negative integrable function g such that for every n, $|f_n| \leq g$ a.e. Then

$$\int_\Omega \underline{\lim} f_n(x) d\,\lambda(x) \leq \underline{\lim} \int_\Omega f_n(x) d\,\lambda(x) \tag{7.4}$$

$$\int_\Omega \overline{\lim} f_n d\,\lambda(x) \geq \overline{\lim} \int_\Omega f_n(x) d\,\lambda(x) \tag{7.5}$$

If the sequence $(f_n)_n$ converges a.e. on Ω and its limit is a measurable function f defined a.e., then $f \in L^1(\Omega)$ and

$$\int_\Omega f(x) d\,\lambda(x) \;=\; \lim_{n\to+\infty} \int_\Omega f_n(x) d\,\lambda(x). \tag{7.6}$$

Proof .
As the function g is Integral, the set $\{x \in \Omega : \; |f(x)| = +\infty\}$ is a null set. So we can be substitute the function g by the function $g\chi_{\{x:\; g(x)<+\infty\}}$. This substitution does not change anything about the inequality: $|f_n| \leq g$ a.e.. The sequence $(f_n)_n$ can also be substituted by the sequence $f_n\chi_{\{|f_n|\leq g\}}$. This substitution does not change the value of the integral $\displaystyle\int_\Omega f_n(x) d\,\lambda(x)$ and not in the limit $\displaystyle\lim_{n\to+\infty} f_n$ a.e. So we can assume that $|f_n| \leq g$ on Ω. So the functions $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are integrable on Ω. Using Fatou's lemma on the sequence $f_n + g$, we get

$$\int_\Omega \underline{\lim}(f_n + g)(x) d\,\lambda(x) \leq \underline{\lim} \int_\Omega (f_n + g)(x) d\;, \lambda(x).$$

As $\underline{\lim}_{n\to+\infty}(f_n + g) = (\underline{\lim}_{n\to+\infty} f_n\;) \;+ g$ on Ω, then

$$\int_{\Omega} \underline{\lim}_{n\to+\infty} f_n(x) d\,\lambda(x) \leq \underline{\lim}_{n\to+\infty} \int_{\Omega} f_n(x) d\,\lambda(x).$$

.
and using Fatou's lemma on the sequence $(-f_n + g)_n$, we get

$$\int_{\Omega} \underline{\lim}_{n\to+\infty} (-f_n)(x) d\,\lambda(x) \leq \underline{\lim}_{n\to+\infty} \int_{\Omega} -f_n(x) d\,\lambda(x).$$

Then

$$\int_{\Omega} \overline{\lim}_{n\to+\infty} f_n(x) d\,\lambda(x) \geq \overline{\lim}_{n\to+\infty} \int_{\Omega} f_n(x) d\,\lambda(x).$$

□

Example 3.2 :
Let f be an Integrable function on $[0, +\infty[$. We want to prove that

$$\lim_{n\to+\infty} \int_0^{+\infty} e^{-n\sin^2 x} f(x) dx = 0.$$

Consider the sequence $(f_n)_n$ defined on $[0, \infty[$ by: $f_n(x) = e^{-n\sin^2 x} f(x)$.
Let $A = \{x : \ f(x) = \pm\infty\} \cup \mathbb{N}_0$. For every $x \notin A$, $\lim\limits_{n\to+\infty} f_n(x) = 0$ and $|f_n| \leq |f|$ and the function f is Integrable. Then

$$\lim_{n\to+\infty} \int_0^{+\infty} e^{-n\sin^2 x} f(x) dx = 0.$$

3.6 Exercises

7-3-1 Find the following limits:

(a) $\lim\limits_{n\longrightarrow+\infty}\int_0^n \sqrt{x}\ln x(1-\frac{x}{n})^n dx$,

(b) $\lim\limits_{n\longrightarrow+\infty}\int_{\mathbb{R}} \frac{|\sin x|^{\frac{2}{n}}}{1+x^2}dx$,

(c) $\lim\limits_{n\longrightarrow+\infty}\int_0^{+\infty} \frac{dx}{(1+x^p)^n}, p>0$,

(d) $\lim\limits_{n\longrightarrow+\infty}\int_0^{+\infty} e^{-n\sin^2 x}f(x)\ dx,\ f\in L^1([0,+\infty[)$,

(e) $\lim\limits_{h\longrightarrow0^+}\int_\alpha^{+\infty} \frac{hf(x)}{h^2+x^2}dx$, and $\lim\limits_{h\longrightarrow0^+}\int_0^{+\infty} \frac{hf(x)}{h^2+x^2}dx$, Where f is an integrable function on the interval $[0,+\infty[$ and continuous at 0 and $\alpha>0$.

(f) $\lim\limits_{n\rightarrow+\infty}\int_0^{+\infty} \frac{\sin(e^x)}{1+nx^2}dx$,

(g) $\lim\limits_{n\rightarrow+\infty}\int_0^{n} (1+\frac{x}{n})^{-n}\cos x dx$,

(h) $\lim\limits_{n\rightarrow+\infty}\int_0^{n} (1+\frac{x}{n})^{n}e^{-2x} dx$,

(i) $\lim\limits_{n\rightarrow+\infty}\int_0^{n} (1-\frac{x}{n})^{-n}e^{\frac{x}{2}} dx$,

(j) $\lim\limits_{n\rightarrow+\infty}\int_0^{n} (1-\frac{x}{n})^{n}\frac{1+nx}{n+x}\cos x dx$,

(k) $\lim\limits_{n\rightarrow+\infty}\int_0^{+\infty} (1+\frac{x}{n})^{n^2}e^{-nx} dx$.

7-3-2 Prove that

$$\int_0^{+\infty}\frac{e^{-2x}dx}{1+e^x} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{3+n}$$

and find the value of the series.

7-3-3 (a) Let $f\in L^1(\mathbb{R})$ and $\alpha>0$.

Prove that $\lim\limits_{n\rightarrow+\infty}\frac{f(nx)}{n^\alpha}=0$ a.e. $x\in\mathbb{R}$. (We can integrate the series $\sum\limits_{n=1}^{+\infty}\frac{f(nx)}{n^\alpha}$ on $\mathbb{R}$.)

(b) Let $f\colon \mathbb{R} \longrightarrow \mathbb{C}$ be a measurable function and $T-$periodic and $\displaystyle\int_0^T |f(t)|\, dt < +\infty$.

i. Prove that $\displaystyle\lim_{n\to+\infty} \frac{f(nx)}{n^2} = 0$ a.e.

ii. Prove that $\displaystyle\lim_{n\to+\infty} (|\cos nx|)^{\frac{1}{n}} = 1$ a.e. (We can use the function $(\ln|\cos x|)^2$.)

$\boxed{\text{7-3-4}}$ Consider the sequence $(I_n)_n$ defined on $]1, +\infty[$ as follows:

$$I_n(x) = \int_x^{+\infty} \frac{dt}{t^2 \ln^n(t)}.$$

Prove that the sequence $(I_n)_n$ is well defined and find its limit.

$\boxed{\text{7-3-5}}$ Let $f(x) = \dfrac{xe^{-ax}}{1 - e^{-bx}}$, with $a > 0$ and $b > 0$.

Prove that the function f is integrable on $[0, +\infty[$ and $\displaystyle\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \frac{1}{(a+nb)^2}$.

$\boxed{\text{7-3-6}}$ Consider the sequence $(I_n)_n$ where $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n(x)\, dx$.

Find the limit of the sequence $(I_n)_n$ and deduce the sum of the following sequence: $U_n = \dfrac{(-1)^n}{2n+1}$ and $U_n = \dfrac{(-1)^n}{n}$.

4 Riemann Integral and Lebesgue Integral

4.1 The Riemann and Lebesgue Integral

Let λ be the Lebesgue measure to on the $\sigma-$algebra $\mathscr{B}$ of measurable functions on the interval $[a, b]$.

If $f\colon [a, b] \longrightarrow \mathbb{R}$ is a Riemann integrable function, then $\displaystyle\int_a^b f(x)dx$ symbolizes the Riemann integral of f on the interval $[a, b]$, and if the function is Lebesgue integrable on $[a, b]$, then $\displaystyle\int_{[a,b]} f(x)d\,\lambda(x)$ symbolizes the Lebesgue integral of f on the interval $[a, b]$.

Theorem 4.1.
Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a Riemann integrable function, then f Lebesgue integrable on $[a,b]$ and

$$\int_{[a,b]} f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

Proof .
As the function f is Riemann integrable on $[a,b]$, there exists a sequence $(\sigma_n = \{x_0 = a, \dots, x_{p_n} = b\})_n$ of partitions of $[a,b]$ such that

$$U(f) = \lim_{n\to+\infty} U(\sigma_n, f) = L(f) = \lim_{n\to+\infty} L(\sigma_n, f).$$

We define two sequences $(g_n)_n$ and $(h_n)_n$ of simple functions as follows:

$$g_n(x) = \begin{cases} m_k = \inf_{t\in[x_k,x_{k+1}[} f(t) & x_k \leq x < x_{k+1} \\ g_n(b) = f(b) \end{cases}$$

$$h_n(x) = \begin{cases} M_k = \sup_{t\in[x_k,x_{k+1}[} f(t) & x_k \leq x < x_{k+1} \\ h_n(b) = f(b) \end{cases}$$

The sequence $(g_n)_n$ is increasing and the sequence $(h_n)_n$ is decreasing. Let $g = \lim_{n\to+\infty} g_n$ and $h = \lim_{n\to+\infty} h_n$. Then

$$U(\sigma_n, f) = \int_a^b h_n(x)dx = \int_{[a,b]} h_n(x)d\,\lambda(x).$$

.

$$L(\sigma_n, f) = \int_a^b g_n(x)dx = \int_{[a,b]} g_n(x)d\,\lambda(x).$$

Since the functions g and h are measurable, using the monotone convergence theorem, we get

$$\lim_{n\to+\infty} \int_a^b g_n(x)dx = L(f) = \int_{[a,b]} g(x)d\,\lambda(x) \tag{7.7}$$

$$\lim_{n\to+\infty} \int_a^b h_n(x)dx = U(f) = \int_{[a,b]} h(x)d\,\lambda(x). \tag{7.8}$$

De deduce from (7.7) and (7.8) that $\displaystyle\int_{[a,b]} h(x)d\lambda(x) = \int_{[a,b]} g(x)d\lambda(x)$.
Then $\displaystyle\int_{[a,b]} (h(x) - g(x))d\lambda(x) = 0$. and since the function $h - g$ is non negative

and integrable, then $h = g$ a.e. and $f = g$ a.e. So the function f is measurable and
$$\int_a^b f(x)dx = \int_{[a,b]} f(x)d\,\lambda(x).$$
$\square$

Theorem 4.2.
Let f be a bounded function on an interval $[a, b]$.

1. The function f is Riemann integral on $[a, b]$ if and only if the set of points where the function f is not continuous is a null set.

2. Inversely, if the set of points where the function f is not continuous is a null set, f is integrable and
$$\int_{[a,b]} f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

For the proof, we keep the same notations as in theorem (4.1) and we need the following lemma:

Lemma 4.3.
For every $x \in [a, b] \setminus \left(\bigcup_{n=1}^{+\infty} \sigma_n\right)$, $g(x) = h(x)$ if and only if the function f is continuous at x.

Proof .
Let $x \in [a, b] \setminus \left(\cup_{n=1}^{+\infty}\sigma_n\right)$ and $\delta_n = ||\sigma_n||$. If the function f is continuous at x, for each $\varepsilon > 0$, there exists $\eta > 0$ such that $|f(x) - f(t)| < \varepsilon$ for every $t \in [a, b]$ and $|t - x| < \eta$. Since the sequence $(\delta_n)_n$ converges to 0, there exists n_0 such that $\delta_{n_0} < \eta$ for every $n \geq n_0$.
For each partition σ_n, with $n > n_0$, there exists $k \in \{0, \dots, p_n - 1\}$ such that $x_k < x < x_{k+1}$.
Then $|f(x) - f(t)| < \varepsilon$ for every $t \in [x_k, x_{k+1}]$. Therefore $h_n(x) = M_k \leq f(x) + \varepsilon$, $g_n(x) = m_k \geq f(x) - \varepsilon$ and $h_n(x) - g_n(x) \leq \varepsilon$. and since this is for each $n \geq n_0$ then $h(x) - g(x) \leq \varepsilon$ for every $\varepsilon > 0$. Then $g(x) = h(x)$.
Inversely: let $x \notin (\bigcup_{n=1}^{\infty} \sigma_n)$, where $g(x) = h(x)$. as $g(x) \leq f(x) \leq h(x)$, then $f(x) = g(x) = h(x)$. So the two sequences $(g_n(x))_n$ and $(h_n(x))_n$ converge and have the same limit $f(x)$.
Let $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $0 \leq f(x) - g_n(x) < \varepsilon$ and $0 \leq h_n(x) - f(x) < \varepsilon$, for every $n \geq n_0$. Since σ_{n_0} is a partition of the interval $[a, b]$, there exists $k \in \{0, \dots, p_{n_0} - 1\}$ such that $x \in [x_k, x_{k+1}[$ and
$$h_{n_0}(x) - \varepsilon < f(x) < g_{n_0}(x) + \varepsilon.$$

On the other hend $h_{n_0}(x) = \sup_{t\in[x_k,x_{k+1}]} f(t)$ and $g_{n_0}(x) = \inf_{t\in[x_k,x_{k+1}]} f(t)$. Then $f(t)-\varepsilon < f(x) < f(t)+\varepsilon$ for every $t \in]x_k, x_{k+1}[$. So the function f is continuous at x. $\square$

Proof of Theorem (4.2).

1. The function f is Riemann integral if and only if $U(f) = L(f)$ and this is equivalent to $h = g$ a.e and we deduce the result from the previous lemma.

   The function f is Riemann integral if and only if $h = g$ a.e and this is equivalent to the set $\{x : \ h(x) \neq g(x)\} \cup (\bigcup_{n=1}^{\infty} \sigma_n)$ is a null set, which is equivalent to the function f is continuous a.e on the interval $[a, b]$.

2. If the set where f is not continuous is a null set, then $\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} h_n(x) = f(x)$ at each point of continuity of the function f. So the function f is measurable and we can deduce the result from the dominated convergence theorem.

$$\lim_{n\to+\infty} \int_{[a,b]} g_n(x)d\lambda(x) = \int_{[a,b]} f(x)d\lambda(x)$$

$$\lim_{n\to+\infty} \int_{[a,b]} h_n(x)d\lambda(x) = \int_{[a,b]} f(x)d\lambda(x).$$

   So the function f is Riemann integrable and

$$\int_{[a,b]} f(x)d\,\lambda(x) = \int_a^b f(x)dx.$$

$\square$

We now give another proof of the following theorem:

Theorem 4.4.
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The function f is Riemann integral if and only if f is continuous a.e. on the interval $[a, b]$.

Proof .

1. Assume that the function f is Riemann integral. For any $x \in [a, b]$, define the functions

$$g(x) = \sup_{\delta>0} \inf_{y\in[a,b],|y-x|\leq\delta} f(y) = \liminf_{y\to x} f(y),$$

$$h(x) = \inf_{\delta > 0} \sup_{y \in [a,b], |y-x| \leq \delta} f(y) = \limsup_{y \to x} f(y).$$

The function f is continuous at x if and only if $g(x) = h(x)$. We have $g \leq f \leq h$. If σ is a partition of interval $[a, b]$, then $U(\sigma, g) \leq U(\sigma, f) \leq U(\sigma, h)$ and $L(\sigma, g) \leq s(\sigma, f) \leq s(\ sigma, h)$. But $U(\sigma, f) = U(\sigma, h)$ and $L(\sigma, g) = s(\sigma, f)$. Because for every open interval $]c, d[\subset [a, b]$,

$$\inf_{x \in]c,d[} g(x) = \inf_{x \in]c,d[} f(x), \qquad \sup_{x \in]c,d[} f(x) = \sup_{x \in]c,d[} h(x).$$

Therefore

$$L(f) = L(g) \leq U(g) \leq U(f), \quad L(f) \leq L(h) \leq U(h) = U(f).$$

As the function f is Riemann integrable, the two functions g and h are Riemann integrable, and their integral is $\displaystyle\int_a^b f(x)dx$.
If the functions g and h are Lebesgue integrable and have the same integral. But $g \leq h$, therefore $g = h$ a.e. As the function f is continuous at every point where the two functions g and h are equal, the function f is continuous a.e.

2. Assume that the function f is continuous a.e.then for every $n \in \mathbb{N}$, let σ_n be the uniform partition of the interval $[a, b]$ and the number of points of σ_n is 2^n.
Let

$$h_n(x) = \sup_{y \in]c,d[} f(y), \quad g_n(x) = \inf_{y \in]c,d[} f(y).$$

If there is an open interval $]c, d[$ of the partition σ_n and contains the point x and $h_n(x) = g_n(x) = f(x)$ if $x \in \sigma_n$. So the sequences $(g_n)_n$ and $(h_n)_n$ are respectively increasing and decreasing and

$$L_(\sigma_n, f) = \int_a^b g_n(x)dx \qquad U(\sigma_n, f) = \int_a^b h_n(x)dx.$$

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} h_n(x) = f(x)$ at every point x where the function f is continuous, so

$$f = \lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n \quad a.e.$$

Using the dominated convergence theorem

$$\lim_{n\to\infty}\int_a^b g_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty}\int h_n(x)dx.$$

and this proves that $L(f) \geq \int_a^b f(x)dx \geq U(f)$. So the function f is Riemann integrable.

□

4.2 Improper Integral and Lebesgue Integral

Theorem 4.5.
Let $f\colon]a,b[\longrightarrow \mathbb{R}$ be a function Lebesgue integrable on every closed and bounded interval of $]a,b[$.
The function f is Lebesgue integrable on $]a,b[$ if and only if the iproper integral $\displaystyle\int_a^b |f(x)|dx$ is convergent. In this case, the Lebesgue and the Riemann integral of f are equal:

$$\int_a^b f(x)dx = \int_{]a,b[} f(x)d\,\lambda(x).$$

Proof .
Suppose that the integral $\displaystyle\int_a^b |f(x)|dx$ is convergent. Let $(a_n)_n$ and $(b_n)_n$ two sequences in $]a,b[$ so that the sequence $(a_n)_n$ is decreasing and tends to a and the sequence $(b_n)_n$ is increasing and tends to b. We define the sequence of functions $(F_n)_n$ as follows:

$$F_n(x) = |f(x)|\chi_{[a_n,b_n]}.$$

. The sequence $(F_n)_n$ is increasing, measurable. Its limit isthe function $|f|\chi_{]a,b[}$. So the function f is measurable and by using the dominate convergence theorem we get:

$$\lim_{n\to+\infty}\int_{\mathbb{R}} F_n(x)d\,\lambda(x) = \int_{]a,b[} |f(x)|d\,\lambda(x).$$

On the other hand, using the previous theorem $\displaystyle\int_{\mathbb{R}} F_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx.$
Using the previous definition, we get:

$$\lim_{n\to+\infty}\int_{\mathbb{R}} F_n(x)d\,\lambda(x) = \int_a^b |f(x)|dx.$$

So the function f is Lebesgue integrable. To prove that the two integrals are equal, we define the sequence of functions $(g_n)_n$ as follows: $g_n = f\chi_{[a_n,b_n]}$. The sequence $(g_n)_n$ is convergent and its limit is the function $f\chi_{]a,b[}$. The functions g_n are integrable and $|g_n| \leq |f|\chi_{[a,b]}$. Using the dominate convergence theorem

$$\lim_{n\to+\infty} \int_{]a,b[} g_n(x)d\,\lambda(x) = \int_{]a,b[} f(x)\ d\,\lambda(x).$$

Inversely: If the function f is Lebesgue integrable on the interval $]a,b[$, the the function $|f|$ is also Lebesgue integrable on the interval $]a,b[$.
Let $(a_n)_n$ and $(b_n)_n$ two sequences in $]a,b[$ as previous. Using the dominate convergence theorem

$$\lim_{n\to+\infty} \int_{]a,b[} F_n(x)d\,\lambda(x) = \int_{]a,b[} |f(x)|d\,\lambda(x) < +\infty.$$

On the other hand $\displaystyle\int_{]a,b[} F_n(x)d\,\lambda(x) = \int_{a_n}^{b_n} |f(x)|dx$, So the limit $\displaystyle\lim_{n\to+\infty} \int_{a_n}^{b_n} |f(x)|dx$ in $\mathbb{R}$ and $\displaystyle\int_a^b |f(x)|dx < +\infty$. $\square$

4.3 Exercises

7-4-1 (a) Calculate the integral of the following functions on $[0, 1]$.

$$f(x) = \frac{1}{\sqrt{x}} + \chi_{\mathbb{Q}}(x) \qquad\qquad \begin{array}{l} g(x) = \sin x;\ \ x \in \mathbb{Q} \\ g(x) = \cos x;\ \ x \in \mathbb{R} \setminus \mathbb{Q} \end{array}$$

(b) Find whether the following functions are integrable on $]0, +\infty[$?

$$f(x) = \frac{\sin x}{x} \qquad\qquad h(x) = \frac{1}{x(1 + |\ln x|)^2}$$

$$g(x) = \frac{1}{(1 + x^2)\sqrt{|\sin x|}}$$

7-4-2 Calculate the following integrals:

(a) $\displaystyle\int_{[0,+\infty[} e^{-[x]} d\lambda(x)$, Where $[x]$ is the entire part of the real number x.

(b) $\displaystyle\int_{[0,\pi]} f(x) d\lambda(x)$, where $f(x) = \sin x$ if $x \in \mathbb{Q} \cap [0, \pi]$ and $f(x) = \cos x$ Otherwise.

(c) $\displaystyle\int_{[0,1]} \chi_{\mathbb{Q}}(x) d\lambda(x)$.

Solutions of Exercises

4.4 Solutions of Exercises on Chapter 1

1-1-1

$$\lim_{n\to+\infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i)f'(x_i) = \int_a^b f(x)f'(x)dx = \frac{1}{2}(f^2(b) - f^2(a)).$$

1-1-2 (a) Let $f(x) = \dfrac{1}{1+x}$ on the interval $[0,1]$. The Riemann sum of f is $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} f(\frac{k}{n}) = \sum_{k=1}^{n} \frac{1}{n+k}$, then

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \ln 2.$$

(b) Let $f(x) = x^2$ on the interval $[0,1]$. The Riemann sum of f is $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \frac{k^2}{n^2}$, then

$$\lim_{n\longrightarrow+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k^2}{n^2} = \frac{1}{3}.$$

(c) Let $f(t) = \sin(xt)$ on the interval $[0,1]$. The Riemann sum of f is $\dfrac{1}{n}\displaystyle\sum_{k=1}^{n} \sin(\frac{kx}{n})$, then

$$\lim_{n\longrightarrow+\infty} \frac{1}{n} \sum_{k=1}^{n} \sin(\frac{kx}{n}) = \frac{1-\cos x}{x}.$$

(d) $$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}}{1+\frac{k^2}{n^2}} = \int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\ln(2).$$

(e) Let $f(t) = \dfrac{1}{1+x^2}$ on the interval $[0,1]$. The Riemann sum of f is $\dfrac{1}{n}\sum_{k=1}^{n}\dfrac{1}{1+\frac{k^2}{n^2}} = \dfrac{n}{n^2+k^2}$, then

$$\lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \frac{\pi}{4}.$$

(f) Let $f(t) = \dfrac{1}{\sqrt{1+x^2}}$ on the interval $[0,1]$. The Riemann sum of f is $\sum_{k=1}^{n}\dfrac{1}{\sqrt{n^2+k^2}}$, then

$$\lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k^2}} = \sinh^{-1}(1) = \ln(1+\sqrt{2}).$$

(g) Let $f(t) = x^2\sin(\pi x)$ on the interval $[0,1]$. The Riemann sum of f is $\dfrac{1}{n^3}\sum_{k=1}^{n}k^2\sin(\dfrac{k\pi}{n})$, then

$$\lim_{n\longrightarrow+\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2\sin(\frac{k\pi}{n}) = \int_0^1 x^2\sin(\pi x)dx = \frac{1}{\pi} - \frac{4}{\pi^3}.$$

(h) Let $S_n = \dfrac{1}{n^4}\prod_{k=1}^{2n}(n^2+k^2)^{\frac{1}{n}}$ and $T_n = \ln S_n = \dfrac{1}{n}\sum_{k=1}^{2n}\ln(1+4\dfrac{k^2}{4n^2})$, then

$$\lim_{n\longrightarrow+\infty}T_n = 2\int_0^1 \ln(1+4x^2)dx = 2\ln 5 - 4 + 2\tan^{-1}2.$$

(i) $\lim_{n\to+\infty}\sum_{k=1}^{n}\dfrac{1}{n}\cos\left(\dfrac{k\pi}{n}\right) = \displaystyle\int_0^1\cos(\pi x)dx = 0.$

(j) $\lim_{n\to+\infty}\sum_{k=1}^{2^n}\dfrac{k^3}{2^{4n}} = \lim\limits_{n\to+\infty}\dfrac{1}{2^n}\sum_{k=1}^{2^n}\dfrac{k^3}{2^{3n}} = \displaystyle\int_0^1 x^3dx = \dfrac{1}{4}.$

(k)

$$\sum_{j=1}^{(k-1)n}\frac{1}{n+j} = \sum_{j=1}^{k-1}(\frac{1}{n}\sum_{\ell=1}^{n}\frac{1}{j+\frac{\ell}{n}}).$$

Then $\lim\limits_{n\to+\infty}\sum_{j=1}^{(k-1)n}\dfrac{1}{n+j} = \lim\limits_{n\to+\infty}\sum_{j=1}^{k-1}\displaystyle\int_0^1\dfrac{1}{j+x}dx = \ln(\dfrac{j+1}{j}) =$ $\ln k$.

(l) $\frac{1}{n^2}\sum_{j=1}^{n-1}\sqrt{j(n-j)} = \frac{1}{n}\sum_{j=1}^{n-1}\sqrt{\frac{j}{n}(1-\frac{j}{n})}$. Then

$$\begin{aligned}\lim_{n\to+\infty}\frac{1}{n^2}\sum_{j=1}^{n-1}\sqrt{j(n-j)} &= \int_0^1\sqrt{x(1-x)}dx = \frac{1}{2}\int_0^1\sqrt{1-(2x-1)^2}dx \\ &\overset{t=2x-1}{=} \frac{1}{4}\int_{-1}^1\sqrt{1-t^2}dt = \frac{1}{4}.\end{aligned}$$

(m)

$$\begin{aligned}\lim_{n\to+\infty}\ln\left(\prod_{j=1}^{n}\left(1+\frac{j}{n}\right)^{\frac{1}{n}}\right) &= \lim_{n\to+\infty}\frac{1}{n}\sum_{j=1}^{n}\ln(1+\frac{j}{n}) \\ &= \int_0^1\ln(1+x)dx = 2\ln 2 - 1 = \ln(\frac{4}{e}).\end{aligned}$$

Then $\lim_{n\to+\infty}\prod_{j=1}^{n}\left(1+\frac{j}{n}\right)^{\frac{1}{n}} = \frac{4}{e}$.

1-1-3 (a) i. The function $x \longmapsto \frac{1}{x}$ is decreasing, then on the interval $[j, j+1]$,

$$\frac{1}{j+1} \le \frac{1}{x} \le \frac{1}{j} \quad \text{and} \quad \frac{1}{j+1} \le \int_j^{j+1}\frac{dx}{x} \le \frac{1}{j}.$$

ii. We deduce that

$$\sum_{j=n+1}^{kn}\int_j^{j+1}\frac{dx}{x} = \int_{n+1}^{kn+1}\frac{dx}{x} \le \sum_{j=n+1}^{kn}\frac{1}{j} = S_n(k)$$

and

$$\sum_{j=n}^{kn-1}\frac{1}{j+1} = \sum_{j=n+1}^{kn}\frac{1}{j} = S_n(k) \le \sum_{j=n}^{kn-1}\int_j^{j+1}\frac{dx}{x} = \int_n^{kn}\frac{dx}{x}.$$

Then

$$\int_{n+1}^{kn+1}\frac{dx}{x} \le S_n(k) \le \int_n^{kn}\frac{dx}{x}.$$

(b) $\lim_{n\to+\infty} S_n(k) = \ln k$.

(c) The sequence $(T_n)_n$ is not convergent, because it is not a Cauchy sequence. $\lim\limits_{n\to+\infty} T_{2n} - T_n = \ln 2$.

1-1-4 The function $\varphi\colon [0,+\infty[$ defined by $\varphi(x) = \dfrac{1}{1+x}$ is continuous, then $\varphi\circ f = \frac{1}{1+f}$ is Riemann-Integrable.
$\displaystyle\int_a^b \frac{dx}{1+f(x)} - \int_a^b dx = \int_a^b \frac{f(x)}{1+f(x)} dx \leq \int_a^b f(x)\,dx = 0$. Then $\displaystyle\int_a^b \frac{dx}{1+f(x)} = \int_a^b dx = (b-a)$.

1-1-5 1) $\displaystyle\int_0^x \sin^3 t \cos t dt \overset{u=\sin t}{=} \int_0^{\sin x} u^3 du = \frac{1}{4}\sin^4 x$

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2) $\displaystyle\int_1^x t^2\sqrt{1+t^3}dt \overset{u^2=1+t^3}{=} \frac{2}{9}(1+x^2)^{\frac{3}{2}} - \frac{4\sqrt{2}}{9}.$

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3) $\displaystyle\int_0^x \frac{tdt}{1+\sqrt{t}} \overset{t=(u-1)^2}{=} \int_1^{1+\sqrt{x}} \frac{2(u-1)^3}{u} du = \frac{2}{3}(1+\sqrt{x})^3 - \frac{3}{2}(1+\sqrt{x})^2 + 3(1+\sqrt{x}) - \ln(1+\sqrt{x}) - \frac{13}{6}.$

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4)

$$\begin{aligned}
\int_0^x \frac{t^3+t^2+t+1}{1+t+t^2}dt &= \int_0^x t + \frac{1}{1+t+t^2}dt \\
&= \int_0^x t + \frac{4}{3(1+(\frac{2t+1}{\sqrt{3}})^2)}dt \\
&= x + \frac{2}{\sqrt{3}}(\tan^{-1}(\frac{2x+1}{\sqrt{3}}) - \tan^{-1}(\frac{1}{\sqrt{3}})) \\
&= x + \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) - \frac{\pi}{3\sqrt{3}}.
\end{aligned}$$

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5) $\displaystyle\int_0^\pi \frac{dx}{1+\sin x} \overset{t=\tan(\frac{x}{2})}{=} \int_0^{+\infty} \frac{2dt}{(1+t)^2} = 2.$

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6)

$$\begin{aligned}
\int_0^\pi \frac{dx}{1+\sin^2 x} &= 2\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin^2 x} \\
&\overset{t=\tan(x)}{=} 2\int_0^{+\infty} \frac{dt}{1+2t^2} = \frac{\pi\sqrt{2}}{2}.
\end{aligned}$$

7)

$$\int_0^{\frac{\pi}{4}} \frac{\sin^3 x \cos x}{1+\cos^2(2x)} dx = \frac{1}{4}\int_0^{\frac{\pi}{4}} \frac{\sin(2x)(1-\cos(2x))}{1+\cos^2(2x)} dx \overset{t=\cos(2x)}{=} \frac{1}{8}\int_0^1 \frac{(1-t)}{1+t^2} dt = \frac{1}{16}(\pi - \ln 2).$$

8)

$$\int_0^{\pi} \frac{dx}{3+\cos(2x)} = 2\int_0^{\frac{\pi}{2}} \frac{dx}{3+\cos(2x)} \overset{t=\tan(x)}{=} \int_0^{+\infty} \frac{dt}{2+t^2} = \frac{\pi}{2\sqrt{2}}.$$

9)

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx &= \int_0^{\frac{\pi}{4}} \ln(\sqrt{2}\cos(\frac{\pi}{4}-x))dx - \int_0^{\frac{\pi}{4}} \ln(\cos x)dx \\ &\overset{u=\frac{\pi}{4}-x}{=} \int_0^{\frac{\pi}{4}} \ln(\sqrt{2}\cos(u))du - \int_0^{\frac{\pi}{4}} \ln(\cos x)dx \\ &= \frac{\pi \ln 2}{8}.\end{aligned}$$

10) $\int_1^2 \frac{e^x dx}{(3+e^x)\sqrt{e^x-1}} \overset{e^x=1+u^2}{=} \int_{\sqrt{e-1}}^{\sqrt{e^2-1}} \frac{2du}{4+u^2} = \tan^{-1}(\frac{\sqrt{e^2-1}}{2}) - \tan^{-1}(\frac{\sqrt{e-1}}{2}).$

11) $\cosh(3x) - \cosh x = \cosh x \cosh(2x) + 2\sinh^2 x \cosh x - \cosh x = 4\cosh x \sinh^2 x.$

$$\begin{aligned}\int \frac{dx}{\cosh(3x)-\cosh x} &= \int \frac{dx}{4\cosh x \sinh^2 x} \\ &= \frac{1}{4}\int \frac{\cosh x}{\sinh^2 x(1+\sinh^2 x)} dx \\ &= \frac{1}{4}\int \frac{\cosh x}{\sinh^2 x} - \frac{\cosh x}{1+\sinh^2 x} dx \\ &= -\frac{1}{4\sinh x} + \frac{1}{4}\tan^{-1}(\sinh x) + C.\end{aligned}$$

12)

$$\begin{aligned}
\int \frac{dx}{\sinh^3 x + \cosh^3 x - 1} &\overset{t=e^x}{=} \int \frac{4dt}{(t-1)^2((t+1)^2+2)} \\
&= \int (-\frac{4}{9(t-1)} + \frac{2}{3(t-1)^2} + \frac{2}{9}\frac{2t+3}{t^2+2t+3})dt \\
&= -\frac{4}{9}\ln|t-1| - \frac{2}{3(t-1)} + \frac{2}{9}\ln(t^2+2t+3) \\
&\quad + \frac{1}{\sqrt{2}}\tan^{-1}(\frac{t+1}{\sqrt{2}}) + C \\
&= -\frac{4}{9}\ln|e^x-1| - \frac{2}{3(e^x-1)} + \frac{2}{9}\ln(e^{2x}+2e^x+3) \\
&\quad + \frac{1}{\sqrt{2}}\tan^{-1}(\frac{e^x+1}{\sqrt{2}}) + C.
\end{aligned}$$

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13)

$$\begin{aligned}
\int \frac{dx}{5\cosh x + 3\sinh x + 4} &\overset{t=e^x}{=} \int \frac{dt}{(2t+1)^2} \\
&= -\frac{1}{2(2e^x+1)} + C.
\end{aligned}$$

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14)

$$\begin{aligned}
\int (1-x^2)\sqrt{1-x^2}dx &\overset{x=\sin\theta}{=} \int \cos^4\theta d\theta \\
&= \frac{1}{4}\int \cos^2(2\theta) + 1 + 2\cos(2\theta)d\theta \\
&= \frac{1}{8}\int \cos(4\theta) + 3 + 4\cos(2\theta)d\theta \\
&= \frac{1}{32}\sin(4\theta) + \frac{3}{8}\theta + \frac{1}{4}\sin(2\theta) + C \\
&= \frac{x\sqrt{1-x^2}}{8}(5-2x^2) + \frac{3}{8}\sin^{-1}x + C.
\end{aligned}$$

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15)

$$\begin{aligned}
\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} &\overset{x=t^6}{=} \int \frac{6t^5}{t^3+t^2}dt = \int 6t^2 - 6t + 6 - \frac{6}{1+t}dt \\
&= 2t^3 - 3t^2 + 6t - 6\ln(1+t) + C \\
&= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(1+\sqrt[6]{x}) + C.
\end{aligned}$$

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16)

$$\int \sqrt{\frac{1-x}{1+x}} \frac{dx}{(1+x)^2} \overset{\frac{1-x}{1+x}=t^2}{=} -\int t^2 dt = -\frac{t^3}{3} + C = -\frac{1}{3}(\frac{1-x}{1+x})^{\frac{3}{2}}.$$

1-1-6 (a) $I_0 = \frac{\pi}{2}$, $I_1 = 1$ and for $n \geq 2$, by integration by parts $I_n = \int_0^{\frac{\pi}{2}} \cos^n(x)dx$ we have $nI_n = (n-1)I_{n-2}$. Then

$$I_{2n} = \frac{2n!}{(2^n n!)^2}\frac{\pi}{2}, \qquad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}.$$

$J_n = I_n$ by change of variable $x = \frac{\pi}{2} - t$.
By the change of variable $x = \sin t$,

$$\begin{aligned} K_n &= \int_{-1}^{+1} (x^2-1)^n dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-1)^n \cos^{2n+1} t dt \\ &= 2(-1)^n I_{2n+1} = 2(-1)^n \frac{(2^n n!)^2}{(2n+1)!}. \end{aligned}$$

(b) i.

$$\begin{aligned} L_{n-1} &= \int_0^x \frac{dt}{(1+t^2)^{n-1}} \\ &\overset{\text{by parts}}{=} \frac{x}{(1+x^2)^{n-1}} + 2(n-1)\int_0^x \frac{t^2}{(1+t^2)^n} \\ &= \frac{x}{(1+x^2)^{n-1}} + 2(n-1)L_{n-1} - 2(n-1)L_n \end{aligned}$$

$$L_n = \frac{2n-1}{2(n-1)}L_{n-1} + \frac{x}{2(n-1)(1+x^2)^{n-1}}.$$

ii.

$$\begin{aligned} I = \int_0^x \frac{1+t}{(t^2+1)^3}dt &= I_3 - \frac{1}{4(x^2+1)^2} = \frac{5}{4}I_2 + \frac{x}{4(1+x^2)^2} \\ &= \frac{15}{8}\tan^{-1} x + \frac{5x}{8(1+x^2)} + \frac{x}{4(1+x^2)^2} \end{aligned}$$

1-1-7 The decomposition of the rational function $\dfrac{(x-a)(x-b)}{(x-c)^2(x-d)^2}$ is

$$\frac{(x-a)(x-b)}{(x-c)^2(x-d)^2} = \frac{\alpha}{x-c} + \frac{\beta}{(x-c)^2} + \frac{\gamma}{x-d} + \frac{\delta}{(x-d)^2},$$

where $\alpha = \dfrac{(c-d)(2c-a-b) - 2(c-a)(c-b)}{(c-d)^3} = -\gamma$.

Then the primitives of $x \longmapsto \dfrac{(x-a)(x-b)}{(x-c)^2(x-d)^2}$ are rational functions if and only if $(c-d)(2c-a-b) - 2(c-a)(c-b) = 0$.

1-1-8 (a) $\displaystyle\int e^{-2x}\cos(2x)dx = \frac{1}{2}e^{-2x}(-\cos(2x)+\sin(2x))$. Then $\displaystyle K = \int_0^{\frac{\pi}{4}} e^{-2x}\cos(2x)dx = \frac{1}{2}(e^{-\frac{\pi}{2}}+1)$.

(b) $\displaystyle I + J = \int_0^{\frac{\pi}{4}} e^{-2x}dx = \frac{1-e^{-\frac{\pi}{2}}}{2}$.

$\displaystyle I - J = \int_0^{\frac{\pi}{4}} e^{-2x}\cos(2x)dx = \frac{1}{2}(e^{-\frac{\pi}{2}}+1)$. Then $I = \dfrac{1}{2}$ and $J = -\dfrac{e^{-\frac{\pi}{2}}}{2}$.

1-1-9 $\displaystyle\int_0^1 xf(x)dx \overset{t=1-x}{=} -\int_1^0 (1-t)f(1-t)dt = \int_0^1 (1-x)f(x)dx$. Then

$$\int_1^0 (1-t)f(1-t)dt = \frac{1}{2}\left(\int_0^1 xf(x)dx + \int_0^1 xf(x)dx\right) = \frac{1}{2}\int_0^1 f(x)dx.$$

1-1-10 If $x \le 0$, $\displaystyle F(x) = \int_0^{\pi} (t-x)\sin t dt = \pi - 2x$.

If $0 \le x \le \pi$,

$\displaystyle F(x) = \int_0^x (x-t)\sin t dt + \int_x^{\pi} (t-x)\sin t dt = x - \sin x + \pi - x - \sin x = \pi - 2\sin x$.

If $x \ge \pi$, $\displaystyle F(x) = \int_0^{\pi} (x-t)\sin t dt = 2x - \pi$.

F is continuous.

1-1-11 (a) The function $x \longmapsto \int_0^{ax} f(t)dt$ is continuous, then f is continuous. Assume that f is $\mathcal{C}^k$. The function $x \longmapsto \int_0^{ax} f(t)dt$ is $\mathcal{C}^{k+1}$, which yields that f is C^∞.

$f'(x) = af(ax)$. Assume that $f^{(n)}(x) = a^{\frac{n(n+1)}{2}} f(a^n x)$.

$f^{(n+1)}(x) = a^{\frac{n(n+1)}{2}} a^n f'(a^n x) = a^{\frac{n(n+1)}{2}} a^n a f(a^{n+1}x) = a^{\frac{(n+1)(n+2)}{2}} f(a^{n+1}x)$.

(b) $f(0) = 0$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. By the Taylor formula at order n at 0, there exists c_n between 0 and x such that

$$f(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c_n x).$$

Then $|f(x)| \leq \frac{|x|^{n+1}}{(n+1)!} a^{\frac{(n+1)(n+2)}{2}} |f(a^{n+1} c_n x)|$.

Since $|a| < 1$, $\lim\limits_{n \to +\infty} \frac{|x|^{n+1}}{(n+1)!} a^{\frac{(n+1)(n+2)}{2}} |f(a^{n+1} c_n x)| = 0$ which proves that $f(x) \equiv 0$. ($\lim\limits_{n \to +\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ since the series $\sum_{n \geq 0} \frac{|x|^{n+1}}{(n+1)!}$ converges.)

1-1-12

$$F(x) = \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt + \int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt.$$

(a) The functions $t \longmapsto \cos^{-1} \sqrt{t}$ and $t \longmapsto \sin^{-1} \sqrt{t}$ are continuous on $[0, 1]$ and the functions $x \longmapsto \cos^2 x$ and $x \longmapsto \sin^2 x$ are C^∞ with values on the interval $[0, 1]$, then F is differentiable on $\mathbb{R}$ and $F'(x) = -x \sin(2x) + x \sin(2x) = 0$ for $x \in [0, \frac{\pi}{2}]$.

(b) i. Since the functions $x \longmapsto \cos^2 x$ and $x \longmapsto \sin^2 x$ are even and π-periodic, then F is even π-periodic on $\mathbb{R}$. Moreover F is constant on $[0, \frac{\pi}{2}]$, then F is constant on $\mathbb{R}$.

ii. $\frac{d}{du}(\cos^{-1} u + \sin^{-1} u) = -\frac{1}{\sqrt{1-u^2}} + \frac{1}{\sqrt{1-u^2}} = 0$ on $]-1, 1[$, then $\forall u \in [-1, 1]$

$$\cos^{-1} u + \sin^{-1} u = \cos^{-1} 0 = \frac{\pi}{2}$$

iii. $F(x) = F(\frac{\pi}{4}) = \int_0^{\frac{1}{2}} \cos^{-1} \sqrt{t} \sin^{-1} \sqrt{t} dt = \frac{\pi}{4}.$

1-1-13 (a) $\int_0^a f(t)g(t)dt \overset{t=a-x}{=} \int_0^a f(a-x)g(a-x)dx = \int_0^a f(x)(k-g(x))dx = k\int_0^a f(x)dx - \int_0^a f(x)g(x)dx$. Then

$$\int_0^a f(t)g(t)dt = \frac{k}{2}\int_0^a f(t)dt.$$

(b) Let $f(x) = \dfrac{\sin x}{1+\cos^2 x}$ and $g(x) = x$. $f(\pi - x) = f(x)$ and $g(\pi - x) + g(x) = \pi$, then

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}(\tan^{-1}(\cos 0) - \tan^{-1}(\cos\pi)) = \frac{\pi^2}{4}.$$

1-1-14 $\int_0^1 (x-a)^2 f(x)dx = a^2 - 2a^2 + a^2 = 0$. Then $f = 0$ on $[0,1]$, which is impossible. Then there is no continuous function which fulfills theses conditions.

1-1-15 Let $f(t) = (1+\sin 2t)^{\frac{1}{t}} = e^{\frac{1}{t}\ln(1+\sin 2t)}$, for $t \in]-\frac{\pi}{4}, \frac{\pi}{4}[\setminus\{0\}$. f is continuous and $\lim_{t\to 0} f(t) = e^2$, then it can be extended to a continuous function on $]-\frac{\pi}{4}, \frac{\pi}{4}[$, with $f(0) = e^2$.

If $F(x) = \int_0^x (1+\sin 2t)^{\frac{1}{t}}dt$, then $\lim_{x\to 0}\frac{1}{x}\int_0^x (1+\sin 2t)^{\frac{1}{t}}dt = F'(0) = e^2$.

1-1-16 (a) $0 \le I_n \le \ln^n 2$, then $\lim_{n\longrightarrow+\infty} I_n = 0$.

(b) By integration by parts $u = \ln^{n+1}(x+1)$, $u' = \dfrac{(n+1)\ln^n(1+x)}{x+1}$, $v' = 1$, $v = x+1$.
$I_{n+1} = 2\ln^{n+1} 2 - (n+1)I_n$.

(c) $I_n = 2\ln^n 2 - nI_{n-1}$

$$I_n = 2\sum_{k=0}^{n-2}(-1)^k\frac{\ln^{n-k} 2n!}{(n-k)!} + (-1)^{n-2}n!(2\ln 2 - 1).$$

1-1-17 $U_n = \int_0^1 x^n.\dfrac{\sin 2x}{x^2-2}dx, \quad V_n = n.\int_0^1 x^n\dfrac{\sin 2x}{x^2-2}dx.$

(a) $|U_n| \le \int_0^1 x^n dx = \dfrac{1}{n+1}$, then $\lim_{n\longrightarrow+\infty} U_n = 0$.

(b) $\left|n\int_0^a x^n f(x)dx\right| \leq n\int_0^a x^n dx = \dfrac{na^{n+1}}{n+1}$.

Then $\lim\limits_{n\longrightarrow+\infty} n.\int_0^a x^n f(x)dx = 0$.

(c) It is obvious to prove that $V_n = n\int_0^1 x^n(f(x)-f(1))dx - \dfrac{nf(1)}{n+1}$.
Let $\varepsilon > 0$ and $a \in [0,1[$ such that $|f(x)-f(1)| \leq \varepsilon$ for all $x \in [a,1]$.
Thus
$V_n = n\int_0^a x^n(f(x)-f(1))dx + n\int_a^1 x^n(f(x)-f(1))dx + \dfrac{nf(1)}{n+1}$.

$$\begin{aligned}\left|V_n - \frac{nf(1)}{n+1}\right| &\leq \left|n\int_0^a x^n(f(x)-f(1))dx\right| + \left|n\int_a^1 x^n(f(x)-f(1))dx\right| \\ &\leq \left|n\int_0^a x^n(f(x)-f(1))dx\right| + \frac{n\varepsilon}{n+1}.\end{aligned}$$

It results that $\lim\limits_{n\longrightarrow+\infty} V_n = f(1)$.

1-1-18 (a) $\lim\limits_{n\longrightarrow+\infty}\left|\int_0^{\frac{\pi}{2}-\varepsilon}(-\sin x)^n dx\right| \leq \lim\limits_{n\longrightarrow+\infty}\dfrac{\pi}{2}(\sin(\dfrac{\pi}{2}-\varepsilon))^n = 0$.

$$\begin{aligned}\lim_{n\longrightarrow+\infty}\left|\int_0^{\frac{\pi}{2}}(-\sin x)^n dx\right| &\leq \lim_{n\longrightarrow+\infty}\int_0^{\frac{\pi}{2}-\varepsilon}|\sin^n x|dx \\ &\quad + \lim_{n\longrightarrow+\infty}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}|\sin^n x|dx.\end{aligned}$$

Then $\lim\limits_{n\longrightarrow+\infty}\int_0^{\frac{\pi}{2}}(-\sin x)^n dx = 0$.

(b) i. By the Mean Value Theorem $\sin x = x\cos c$, where $c \in]0,\frac{\pi}{2}]$. Then $\sin x \leq x$. Moreover on the interval $[0,\frac{\pi}{2}]$ the function $x \longmapsto g(x) = \sin x$ is concave and $g(0) = 1$ and $g(\frac{\pi}{2}) = 1$, then the graph of g is aver the chord $h(x) = \frac{2}{\pi}x$. (We can also study the variation of the function $x \longmapsto \sin x - \frac{2}{\pi}x$.

ii. $\left|\int_0^{\frac{\pi}{2}}(\sin rx).e^{-r\sin x}dx\right| \leq \int_0^{\frac{\pi}{2}} e^{-\frac{2rx}{\pi}}dx = \pi\dfrac{1-e^{-r}}{2r}$.

Then $\lim\limits_{r\longrightarrow+\infty}\int_0^{\frac{\pi}{2}}(\sin rx).e^{-r\sin x}dx = 0$.

(c) i. By the Taylor Formula $\cos x = 1 - \frac{x^2}{2}\cos c$, where $c \in \mathbb{R}$, then $\forall x \geq 0$, $\cos x \geq 1 - \frac{x^2}{2}$.

ii. $\displaystyle\int_x^{2x} (\frac{1}{t} - \frac{t}{2})dt \leq h(x) = \int_x^{2x} \frac{\cos t}{t} dt \leq \int_x^{2x} \frac{dt}{t} = \ln 2.$ $\displaystyle\lim_{\substack{x \longrightarrow 0 \\ x>0}} h(x) = \ln 2.$

1-1-19 (a) i. Since f is continuous, F is $\mathcal{C}^1$ on $\mathbb{R}$.

ii. $F^{'}(x) = xf(x)$. $\displaystyle\lim_{x\to 0} \frac{F^{'}(x)}{x} = \lim_{x\to 0} f(x) = f(0)$. Then $F^{''}(0) = f(0)$.

(b) If f is even, (resp odd), F is even (resp odd).

(c) $\displaystyle\int_0^x t\tan^{-1} t dt \overset{\text{by parts}}{=} \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x.$

$\displaystyle\int_0^x t^2 \tan^{-1} t dt \overset{\text{by parts}}{=} \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2).$

$\displaystyle\int_0^x t\frac{\tan^{-1} t}{(1+t^2)^2} dt \overset{\text{by parts}}{=} -\frac{\tan^{-1} x}{2(1+x^2)} - \frac{1}{4}\tan^{-1} x + \frac{x}{4(1+x^2)}.$

$\displaystyle\int_0^x \frac{t}{\cosh^2 t} dt \overset{\text{by parts}}{=} x\tanh x - \ln(\cosh(x)).$

1-1-20 The integration by parts $u = f$, $u' = f'$, $v' = 1$, $v = \frac{1}{2}(x-a) + \frac{1}{2}(x-b)$ gets

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)) - \int_a^b \left(\frac{1}{2}(x-a) + \frac{1}{2}(x-b)\right) f'(x)dx.$$

The integration by parts $u = f'$, $u' = f^{''}$, $v' = \frac{1}{2}(x-a) + \frac{1}{2}(x-b)$, $v = \frac{1}{2}(x-a)(x-b)$ gets

$$-\int_a^b \left(\frac{1}{2}(x-a) + \frac{1}{2}(x-b)\right) f'(x)dx = \frac{1}{2}\int_a^b (x-a)(x-b)f^{''}(x)dx.$$

Then

$$\int_a^b f(x)dx = \frac{b-a}{2}(f(a)+f(b)) + \frac{1}{2}\int_a^b (x-a)(x-b)f^{''}(x)dx.$$

1-1-21 $I_n = \int_0^c x^n \ln(1+x^2)dx \leq c^{n+1}\ln 2$ and $J_n = \int_0^1 x^n \ln(1+x^2)dx \leq \ln 2 \frac{1}{n+1}$. Then $\lim_{n\longrightarrow+\infty} I_n = \lim_{n\longrightarrow+\infty} J_n = 0$.

Moreover $J_n = I_n + \int_c^1 x^n \ln(1+x^2)dx \geq I_n + (\ln(1+c^2)\frac{1-c^{n+1}}{n+1})$.

$$\frac{I_n}{J_n} \leq \frac{(n+1)c^{n+1}}{\ln(1+c^2)(1-c^{n+1})}.$$

Then $\lim_{n\longrightarrow+\infty} \frac{I_n}{J_n} = 0$.

1-1-22

$$\begin{aligned}
\varphi(x) &= \int_a^b f(x-t)(1+t^2+\sin t)dt \\
&\overset{x-t=u}{=} \int_{x-b}^{x-a} f(u)(1+(x-u)^2+\sin(x-u))du \\
&= \int_{x-b}^{x-a} f(u)(1+x^2-2xu+u^2+\sin x\cos u-\cos x\sin u)du \\
&= \int_{x-b}^{x-a} f(u)du + x^2\int_{x-b}^{x-a} f(u)du - 2x\int_{x-b}^{x-a} uf(u)du \\
&\quad + \int_{x-b}^{x-a} u^2 f(u)du + \sin x\int_{x-b}^{x-a} f(u)\cos u du \\
&\quad - \cos x\int_{x-b}^{x-a} f(u)\sin u du.
\end{aligned}$$

Then φ is differentiable and

$$\begin{aligned}
\varphi'(x) &= f(x-a) - f(x-b) + 2x\int_{x-b}^{x-a} f(u)du + x^2(f(x-a)-f(x-b)) \\
&\quad -2\int_{x-b}^{x-a} uf(u)du - 2x((x-a)f(x-a)-(x-b)f(x-b)) \\
&\quad +2((x-a)^2 f(x-a)-(x-b)^2 f(x-b)) + \cos x\int_{x-b}^{x-a} f(u)\cos u du \\
&\quad +\sin x(f(x-a)\cos(x-a) - f(x-b)\cos(x-b)).
\end{aligned}$$

$\boxed{\text{1-1-23}}$ (a) $\int_0^{\frac{\pi}{2}} \cos x \sin^n x dx \overset{u=\sin t}{=} \frac{1}{n+1}$, then $\lim_{n\longrightarrow+\infty} n \int_0^{\frac{\pi}{2}} \cos x \sin^n x dx =$ 1.

(b) i. $\lim_{n\longrightarrow+\infty} n \sin^n a = \lim_{n\longrightarrow+\infty} n e^{n \ln(\sin a)} = 0.$

ii. Since g is bounded set $|g| \leq M$,

$$\begin{aligned}
\lim_{n\longrightarrow+\infty} n \left| \int_0^a \cos x \sin^n x g(x) dx \right| &\leq M \lim_{n\longrightarrow+\infty} n \int_0^a \sin^n x dx \\
&\leq M \lim_{n\longrightarrow+\infty} n \sin^n a = 0.
\end{aligned}$$

(c) Let $\ell = \lim_{x\longrightarrow(\frac{\pi}{2})^-} g(x)$. For $\varepsilon > 0$, there exists $a \in [0, \frac{\pi}{2}[$, such that $|g(x) - \ell| \leq \varepsilon$, for all $x \in [a, \frac{\pi}{2}]$.

$$\begin{aligned}
n \int_0^{\frac{\pi}{2}} \cos x (\sin x)^n g(x) dx - \frac{n\ell}{n+1} &= n \int_0^a \cos x (\sin x)^n (g(x) - \ell) dx \\
&\quad + n \int_a^{\frac{\pi}{2}} \cos x (\sin x)^n (g(x) - \ell) dx
\end{aligned}$$

$\lim_{n\longrightarrow+\infty} n \int_0^a \cos x (\sin x)^n (g(x) - \ell) dx = 0$ and
$\left| n \int_a^{\frac{\pi}{2}} \cos x (\sin x)^n (g(x) - \ell) dx \right| \leq \frac{n\varepsilon}{n+1}$. Then

$$\lim_{n\longrightarrow+\infty} n \int_0^{\frac{\pi}{2}} \cos x (\sin x)^n g(x) dx = \ell.$$

$\boxed{\text{1-1-24}}$ (a) Assume that $\int_a^b g(x) dx = 0$

Since g is non negative and continuous, the function $G(x) = \int_a^x g(t) dt$ is increasing and $G(a) = G(b) = 0$. Then $G = 0$ and $G' = g = 0$.

It is obvious that if $g = 0$, $\int_a^b g(x) dx = 0$.

(b) The polynomial

$$P(\lambda) = \int_a^b (f(x) - \lambda g(x))^2 dx = \lambda^2 \int_a^b g^2(x) dx - 2\lambda \int_a^b f(x) g(x) dx + \int_a^b f^2(x) dx$$

is non negative and of degree 2. Then its discriminant is non negative. Thus

$$|\int_a^b f(x)g(x)\ dx| \leq \left(\int_a^b |f(x)|^2\ dx\right)^{\frac{1}{2}}\left(\int_a^b |g(x)|^2\ dx\right)^{\frac{1}{2}}.$$

If we have the equality, the polynomial P has a root, then there exists $\lambda \in \mathbb{R}$ such that $\displaystyle\int_a^b (f(x)-\lambda g(x))^2 dx = 0$, then $f = \lambda g$.

(c) Let $f\colon [a,b] \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^1$ such that $f(a) = 0$.

i. $f(x) = \displaystyle\int_a^x f'(t)dt$, by the previous question (The Cauchy Schwarz inequality)

$$|f(x)|^2 \leq (x-a)\int_a^x |f'(t)|^2 dt \leq (x-a)\int_a^b |f'(t)|^2 dt.$$

ii. We integrate the previous inequality, we have

$$\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{2}\int_a^b |f'(x)|^2 dx.$$

iii. It is obvious that if f s constant, we have the equality.

If we have the equality, then $\displaystyle\int_a^b |f'(t)|dt = (x-a)\int_a^x |f'(t)|^2 dt$. Then by the previous question $f' = \lambda$ is constant. But $\displaystyle\int_a^b |f(x)|^2 dx \leq \frac{(b-a)^2}{2}\int_a^b |f'(x)|^2 dx$ yields that $\lambda = 0$ and f is constant.

1-2-1 In a neighborhood of 1, the function $\dfrac{1}{t^{n+1}\sqrt{t-1}}$ is equivalent to $\dfrac{1}{\sqrt{t-1}}$ which is integrable and in a neighborhood of $+\infty$, $\dfrac{1}{t^{n+1}\sqrt{t-1}} \leq \dfrac{1}{t^{n+1}}$ which is integrable.

By the change of variable $t-1=x^2$, $I_n = 2\displaystyle\int_0^{+\infty} \frac{dx}{(1+x^2)^{n+1}}$. By integration by parts, $I_n = \dfrac{2n+1}{n} I_{n-1}$ with $I_0 = \frac{\pi}{2}$. Then $I_n = \pi \dfrac{2^{2n}(n!)^2}{(2n+1)!}$.

1-2-2 Let $x \in [-1,1[$. In a neighborhood of 1, $\dfrac{1}{\sqrt{t(t-1)(t-x)}} \approx \dfrac{1}{\sqrt{(t-1)(1-x)}}$, which is integrable and in a neighborhood of $+\infty$, $\frac{1}{\sqrt{t(t-1)(t-x)}} \approx t^{-\frac{3}{2}}$.

Then the integral $f(x) = \displaystyle\int_1^{+\infty} \frac{dt}{\sqrt{t(t-1)(t-x)}}$ converges.

If $x=1$, In a neighborhood of 1, $\dfrac{1}{\sqrt{t(t-1)^2}} \approx \dfrac{1}{t-1}$, which is not integrable.

The function $x \longmapsto \dfrac{1}{\sqrt{t(t-1)(t-x)}}$ is continuous on the interval $[-1,1[$. Let $a \in [-1,1[$ and $\alpha > 0$ such that $a+\alpha < 1$. For $t \geq 1$ and $x \in [-1, a+\alpha]$, $\dfrac{1}{\sqrt{t(t-1)(t-x)}} \leq \dfrac{1}{\sqrt{t(t-1)(t-a-\alpha)}}$, which is integrable, then by the Dominate Convergence Theorem, f is continuous at a and then continuous on the interval $[-1,1[$.

1-2-3 (a) If g is decreasing, $x^{\alpha+1}g(2x) \leq \displaystyle\int_x^{2x} t^\alpha g(t)dt \leq x(2x)^\alpha g(x)$. Moreover since the integral $\displaystyle\int_0^1 x^\alpha g(x)dx$ converges, $\displaystyle\lim_{x\to 0^+}\int_x^{2x} t^\alpha g(t)dt = 0$, then $\displaystyle\lim_{x\to 0^+} x^{\alpha+1}g(x) = 0$.
The same result if g is increasing.

(b) By the change of variable $x = \dfrac{1}{t}$, the integral $\displaystyle\int_0^1 x^{-2-\alpha}h(\frac{1}{x})dx$ converges. Then $\displaystyle\lim_{x\to 0^+} x^{-\alpha-1}h(\frac{1}{x}) = 0$, which is equivalent to $\displaystyle\lim_{x\to+\infty} x^{\alpha+1}h(x) = 0$.

1-2-4 The function $f(x) = \sin(x^2)$ is continuous on $[1,+\infty[$ and the integral $\displaystyle\int_1^{+\infty} f(x)dx$ converges, but $\displaystyle\lim_{x\to+\infty} f(x)$ does not exists. (We prove that the integral converges by integration by parts).

1-2-5 (a) The integral $\displaystyle\int_1^{+\infty} \frac{dx}{x^\alpha(1+\lambda x)}$ is convergent the function $x \longmapsto \frac{1}{x^\alpha(1+\lambda x)}$ is continuous and in a neighborhood of $+\infty$, $\displaystyle\frac{1}{x^\alpha(1+\lambda x)} \approx \frac{1}{\lambda x^{\alpha+1}}$, which is integrable.
The same result for $J(\lambda)$.

(b) $\displaystyle I(\lambda)-\lambda^{\alpha-1}\int_0^{+\infty} \frac{dx}{x^\alpha(1+x)} = I(\lambda)-\int_0^{+\infty} \frac{dx}{x^\alpha(1+\lambda x)} = -\int_0^1 \frac{dx}{x^\alpha(1+\lambda x)}$.
Since $0 < \alpha < 1$, the function $\displaystyle\lambda \longmapsto \int_0^1 \frac{dx}{x^\alpha(1+\lambda x)}$ is continuous on $[0, +\infty[$ and then $\displaystyle\lim_{\lambda\to+} I(\lambda) - \lambda^{\alpha-1}\int_0^{+\infty} \frac{dx}{x^\alpha(1+x)} = \int_0^1 \frac{dx}{x^\alpha} = \frac{1}{1-\alpha}$.

(c) In a neighborhood of $+\infty$, $\displaystyle\frac{1}{x^\beta(1+x^\alpha)(1+\lambda x)} \approx \frac{1}{\lambda x^{\alpha+\beta}}$, which is integrable. Then the integral $\displaystyle\int_1^{+\infty} \frac{dx}{x^\beta(1+x^\alpha)(1+\lambda x)}$ is convergent. By the Monotone Convergence Theorem $\displaystyle\lim_{\lambda\to 0^+} K(\lambda) = \int_1^{+\infty} \frac{dx}{x^\beta(1+x^\alpha)}$.

1-2-6 By Abel Theorem the integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}}dx$ is convergent. Then the nature of the integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}+\sin x}dx$ is the same as the integral integral $\displaystyle\int_0^{+\infty} (\frac{\sin x}{\sqrt{x}+\sin x} - \frac{\sin x}{\sqrt{x}})dx$.

$$\frac{\sin x}{\sqrt{x}}-\frac{\sin x}{\sqrt{x}+\sin x} = \frac{\sin^2 x}{\sqrt{x}(\sqrt{x}+\sin x)} = \frac{1}{2\sqrt{x}(\sqrt{x}+\sin x)}-\frac{\cos(2x)}{2\sqrt{x}(\sqrt{x}+\sin x)}.$$

The last integral is convergent by Abel Theorem and the first is divergent since it is equivalent in a neighborhood of $+\infty$ to $\displaystyle\frac{1}{2x}$. Then the improper integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}+\sin x}dx$ diverges.

1-2-7 (a) Since $\sin(\pi - t) = \sin t$, $\displaystyle\int_0^\pi xf(\sin x)dx = \int_0^\pi (\pi - x)f(\sin x)dx$. Then

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

(b) By the change of variable $x=\pi-t$, $\int_0^\pi \ln(\sin x)dx = 2\int_0^{\frac{\pi}{2}} \ln(\sin t)dt$ and by the change of variable $x=\frac{\pi}{2}-t$, we have

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln(\sin x)dx &= \int_0^{\frac{\pi}{2}} \ln(\cos x)dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin x\cos x)dx \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin(2x))dx - \frac{\pi}{4}\ln 2 \\
&= \frac{1}{4}\int_0^{\pi} \ln(\sin x)dx - \frac{\pi}{4}\ln 2 \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin x)dx - \frac{\pi}{4}\ln 2.
\end{aligned}$$

Then $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx = -\frac{\pi}{2}\ln 2$ and $\int_0^\pi t\ln(\sin x)dx = -\frac{\pi^2}{2}\ln 2.$

1-2-8 (a) $\int_a^{2a} \frac{2x^3}{\sqrt{x^4-a^4}}dx = \sqrt{8a^4-a^4} = \sqrt{7}a^2.$

$$\begin{aligned}
\int_a^{2a} \frac{a^2}{\sqrt{x^4-a^4}}dx &\overset{x=a\sec t}{=} a\int_0^{\frac{\pi}{3}} \frac{\sec t}{\sqrt{1+\sec^2 t}}dt \\
&= a\int_0^{\frac{\pi}{3}} \frac{dt}{\sqrt{1+\cos^2 t}} \\
&\overset{u=\tan(\frac{t}{2})}{=} a\sqrt{2}\int_0^{\frac{1}{\sqrt{3}}} \frac{dt}{\sqrt{1+u^2}} \\
&= a\sqrt{2}\cosh^{-1}(\frac{1}{\sqrt{3}}) = \frac{a\ln 3}{\sqrt{2}}.
\end{aligned}$$

$$\int_a^{2a} \frac{a^2+2x^3}{\sqrt{x^4-a^4}}dx = \sqrt{7}a^2 + \frac{a\ln 3}{\sqrt{2}}.$$

(b) By integration by parts $\int_0^1 \frac{x\ln x}{(1-x^2)^{3/2}}dx \overset{x^2=1-t^2}{=} \frac{1}{2}\int_0^1 \frac{\ln(1-t^2)}{t^2}dt = -\ln 2$.

(c) $\int_0^1 \frac{\ln x}{(1-x)^{\frac{3}{2}}}dx \overset{x=1-t^2}{=} 2\int_0^1 \frac{\ln(1-t^2)}{t^2}dt = 2-2\ln 2$.

(d) By integration by parts $\int_0^{\frac{\pi}{2}} \sin 2x \ln(\tan x)dx = \sin^2 \ln(\tan x) + \ln(\cos x)\Big]_0^{\frac{\pi}{2}} = 0$.

(e) $\int_0^{\frac{\pi}{2}} \cos x\ln(\tan x)dx = \int_0^{\frac{\pi}{4}} \cos x\ln(\tan x)dx - \int_0^{\frac{\pi}{4}} \sin x\ln(\tan x)dx$.
Integration by parts yields
$\int_0^{\frac{\pi}{4}} \cos x\ln(\tan x)dx = -\int_0^{\frac{\pi}{4}} \frac{dx}{\cos x} = -\ln(1+\sqrt{2})$.

$$\begin{aligned} -\int_0^{\frac{\pi}{4}} \sin x\ln(\tan x)dx &= [(\cos x-1)\ln(\sin x)+\ln(1+\cos x)]_0^{\frac{\pi}{4}} \\ &= -\frac{\sqrt{2}}{4}\ln 2 - \ln 2 + \ln(1+\sqrt{2}). \end{aligned}$$

Then

$$\int_0^{\frac{\pi}{2}} \cos x\ln(\tan x)dx = -\frac{\sqrt{2}}{4}\ln 2 - \ln 2.$$

(f) $\int_1^{+\infty} \frac{x^4+1}{x^3(x+1)(1+x^2)}dx = \int_1^{+\infty} (\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x+1} + 2\frac{x+1}{1+x^2})dx = \frac{\pi}{2} - \frac{1}{2}$.

(g) $\int_1^{+\infty} \frac{dx}{x\sqrt[4]{1+x^2}} \overset{t^4=1+x^2}{=} \int_{2^{\frac{1}{4}}}^{+\infty} \frac{2t^2dt}{(t^4-1)} = \int_{2^{\frac{1}{4}}}^{+\infty} (\frac{1}{2(t-1)} - \frac{1}{2(t+1)} + \frac{1}{1+t^2})dt = \frac{1}{2}\ln(\frac{2^{\frac{1}{4}}+1}{2^{\frac{1}{4}}-1}) + \frac{\pi}{2} - \tan^{-1}(2^{\frac{1}{4}})$.

(h) $\int_0^{+\infty} x^n e^{-x}dx = n!$.

1-2-9 (a) In a neighborhood of 0, $\dfrac{\ln x}{1-x^2} \approx \ln x$, which is integrable and in a neighborhood of 1, $\dfrac{\ln x}{1-x^2} \approx -\dfrac{1}{1+x}$, which is integrable. Then $\displaystyle\int_0^1 \frac{\ln x}{1-x^2} dt$ converges.

(b) In a neighborhood of 0, $\dfrac{\ln^2 t}{1+x^2} \approx \ln^2 x$, which is integrable. Then $\displaystyle\int_0^1 \frac{\ln^2 x}{1+x^2} dx$ converges.

(c) In a neighborhood of 0, $\dfrac{1}{\sqrt{x}(1+|\ln x|)} \leq \dfrac{1}{\sqrt{x}}$, which is integrable and $\displaystyle\lim_{t\to+\infty} x\frac{1}{\sqrt{x}|\ln x|} = +\infty$, then $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(1+|\ln x|)}$ diverges.

(d) In a neighborhood of 0, $\dfrac{1}{\sqrt{x}(1+x^2)} \approx \dfrac{1}{\sqrt{x}}$, which is integrable and $\displaystyle\lim_{x\to+\infty} x^2\frac{1}{\sqrt{x}(1+x^2)} = 0$, then $\displaystyle\int_0^{+\infty} \frac{dx}{\sqrt{x}(1+x^2)}$ converges.

(e) $\left|\dfrac{x\sin x}{(1+x^2)}\right| \leq \dfrac{x}{(1+x^2)}$, then $\displaystyle\int_0^{+\infty} \frac{x\sin x}{(1+x^2)} dx$ converges.

(f) If $\alpha \leq 0$, $\displaystyle\lim_{x\to+\infty}\left|\frac{\cos x}{(1+x^\alpha)}\right| = 1$, then the integral $\displaystyle\int_0^{+\infty} \frac{\cos x}{(1+x^\alpha)} dx$ diverges.
If $\alpha > 0$, the function $x \longmapsto \frac{1}{(1+x^\alpha)}$ decreases to 0 when $x \to +\infty$, then by Abel Theorem, the integral $\displaystyle\int_0^{+\infty} \frac{\cos x}{(1+x^\alpha)} dx$ converges.

(g) $\left|\dfrac{\cos(\alpha x)}{1+e^x}\right| \leq \dfrac{1}{1+e^x}$, the the integral $\displaystyle\int_0^{+\infty} \frac{\cos(\alpha x)}{1+e^x} dx$ converges.

(h) By Abel Theorem, the integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x}} dx$ converges. Moreover
$\dfrac{\sin x}{\sqrt{x}} - \dfrac{\sin x}{\sqrt{x+\cos x}} = \dfrac{\sin x \cos x}{\sqrt{x}(\sqrt{x+\cos x})(\sqrt{x}+\sqrt{x+\cos x})}$.
Since $\left|\dfrac{\sin x\cos x}{\sqrt{x}(\sqrt{x+\cos x})(\sqrt{x}+\sqrt{x+\cos x})}\right| \leq \dfrac{1}{\sqrt{x}(\sqrt{x-1})(\sqrt{x}+\sqrt{x-1})}$

for $x \geq 2$, the integral $\displaystyle\int_0^{+\infty} \frac{\sin x}{\sqrt{x+\cos x}}dx$ converges.

(i) $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(1) = \frac{\pi}{2}$.

(j) In a neighborhood of 0, $\dfrac{\ln x}{(1-x)^\alpha} \approx \ln x$ which is integrable, and in a neighborhood of 1, $\dfrac{\ln x}{(1-x)^\alpha} \approx -(1-x)^{1-\alpha}$ which is integrable if and only if $\alpha < 2$. Then the integral $\displaystyle\int_0^1 \frac{\ln x}{(1-x)^\alpha}dx$ converges if and only if $\alpha < 2$.

(k) In a neighborhood of 0, $\dfrac{x^{\alpha-1}}{1+x} \approx x^{\alpha-1}$ which is integrable if and only if $\alpha > 0$. In a neighborhood of $+\infty$, $\dfrac{x^{\alpha-1}}{1+x} \approx x^{\alpha-2}$ which is integrable if and only if $\alpha < 1$. Then the integral $\displaystyle\int_0^{+\infty} \frac{x^{\alpha-1}}{1+x}dx$ converges if and only if $0 < \alpha < 1$.

(l) In a neighborhood of 0, $x^\alpha e^{-x} \approx x^\alpha$ which is integrable if and only if $\alpha > -1$ and since $\displaystyle\lim_{x\to+\infty} x^{\alpha+2}e^{-x} = 0$, the integral $\displaystyle\int_0^{+\infty} x^\alpha e^{-x}dx$ converges if and only if $\alpha > -1$.

(m) The function $x \longmapsto \frac{x}{1+x^2}$ is decreasing and tends to 0 when $x \to +\infty$ and since $|\int_a^b \sin x dx| \leq 2$, then by the Abel Theorem the integral $\displaystyle\int_0^{+\infty} \frac{x\sin x}{1+x^2}dx$ converges.

(n) $\displaystyle\int_0^{+\infty} \sin(x^2)dx = \int_0^{+\infty} \frac{\sin x}{2\sqrt{x}}dx$. The integral converges by the Abel Theorem.

(o) $\displaystyle\int_0^{+\infty} \frac{2\tan^{-1} x - \pi}{\sqrt{x}}dx = -\int_0^{+\infty} \frac{2\tan^{-1}(\frac{1}{x})}{\sqrt{x}}dx$. In a neighborhood of 0, $\dfrac{2\tan^{-1}(\frac{1}{x})}{\sqrt{x}} \approx \dfrac{\pi}{\sqrt{x}}$ which is integrable, and in a neighborhood of $+\infty$, $\dfrac{2\tan^{-1}(\frac{1}{x})}{\sqrt{x}} \approx \dfrac{2}{x\sqrt{x}}$ which is integrable. Then the integral $\displaystyle\int_0^{+\infty} \frac{2\tan^{-1} x - \pi}{\sqrt{x}} dx$ is convergent.

(p) In a neighborhood of 0, the function $\dfrac{\tan^{-1} x}{x} - \dfrac{\pi}{2(1+x)}$ is bounded and in a neighborhood of $+\infty$, $\dfrac{\tan^{-1} x}{x} - \dfrac{\pi}{2(1+x)} \approx \dfrac{\pi}{2x(1+x)}$ which is integrable. Then the integral $\displaystyle\int_0^{+\infty} (\frac{\tan^{-1} x}{x} - \frac{\pi}{2(1+x)}) dx$ is convergent.

(q) In a neighborhood of 0, the function $\sin(x)\sin(\dfrac{1}{x})$ is bounded and in a neighborhood of $+\infty$, the function $\sin(\frac{1}{x})$ is non negative decreasing and tends to 0, the function $\sin x$ fulfills $|\displaystyle\int_a^b \sin x dx| \leq 2$, then by the Abel Lemma, the integral $\displaystyle\int_0^{+\infty} \sin(x)\sin(\frac{1}{x}) dx$ is convergent.

(r) In a neighborhood of 0, the function $\dfrac{1}{\sqrt{x}}|\sin\left(x + \dfrac{1}{x}\right)| \leq \dfrac{1}{\sqrt{x}}$ which is integrable.

Since $0 \leq \dfrac{1}{\sqrt{x}}\sin(\dfrac{1}{x}) \leq \dfrac{1}{x\sqrt{x}}$ for $x \geq 1$, the integral $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x}}\sin(\frac{1}{x})$ converges. Moreover $\dfrac{1}{\sqrt{x}}\left|\sin\left(x + \dfrac{1}{x}\right) - \dfrac{1}{\sqrt{x}}\sin(\dfrac{1}{x})\right| = 2\dfrac{1}{\sqrt{x}}|\cos(\dfrac{1}{x} + \dfrac{x}{2})|\sin(\dfrac{1}{2x}) \leq 2\dfrac{1}{\sqrt{x}}\sin(\dfrac{1}{2x})$ for $x \geq 1$. Then the integral $\displaystyle\int_0^{+\infty} \sin\left(x + \frac{1}{x}\right)\frac{dx}{\sqrt{x}}$ converges.

(s) For $0 < \alpha < \frac{\pi}{2}$, $\displaystyle\int_0^{\alpha} \tan x dx = \ln\sec(\alpha) \underset{\alpha\to\frac{\pi}{2}}{\longrightarrow} +\infty$. Then the integral $\displaystyle\int_0^{\frac{\pi}{2}} \tan x dx$ diverges.

(t) In a neighborhood of $\dfrac{\pi}{2}$, $\sqrt{\tan x} \approx \frac{1}{\sqrt{\cos x}} \approx \frac{1}{\sqrt{\frac{\pi}{2} - x}}$. Then the integral $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\tan x} dx$ is convergent.

(u) $\dfrac{e^{\sin x}}{x} \geq \dfrac{1}{ex}$, then the integral $\displaystyle\int_1^{+\infty} \frac{e^{\sin x}}{x} dx$ diverges.

(v) The function $\dfrac{1}{x}\left(e^{\frac{1}{x}} - \cos\dfrac{1}{x}\right)$ is non negative and equivalent in a neighborhood of $+\infty$ to $\dfrac{1}{x^2}$, then the integral $\displaystyle\int_1^{+\infty} \frac{1}{x}\left(e^{\frac{1}{x}} - \cos\frac{1}{x}\right) dx$ converges. ($e^{\frac{1}{x}} \approx 1 + \dfrac{1}{x}$ and $\cos\dfrac{1}{x} \approx 1 - \dfrac{1}{2x^2}$ in a neighborhood of $+\infty$).

(w) $\displaystyle\lim_{\alpha\to 1}\int_0^{\alpha} \frac{dx}{\cos^{-1} x} \overset{x=\cos t}{=} \lim_{\alpha\to 1}\int_{\cos^{-1}\alpha}^{\frac{\pi}{2}} \frac{\sin t}{t} dt = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} dt$ and this last integral converges, then the integral $\displaystyle\int_0^1 \frac{dx}{\cos^{-1} x}$ converges.

1-2-10 (a) i. For $s < 0$, $t^{s-1}|\cos x| \leq t^{s-1}$, which is integrable. Then the integral $\displaystyle\int_1^{+\infty} t^{s-1}\cos x dx$ is absolutely convergent for $s < 0$.

ii. By integration by parts, $\displaystyle\int_1^{+\infty} t^s \sin t dt = \cos 1 + s\int_1^{+\infty} t^{s-1}\cos x dx$.

(b) With the change of variable $x = t^2$, $\displaystyle\int_1^{+\infty} \sqrt{t}\sin(t^2)dt = \frac{1}{2}\int_1^{+\infty} x^{-\frac{1}{4}}\sin(x)dx$. This integral is well defined.

1-2-11 (a) If $\ell \neq 0$ say $\ell > 0$, there exists $A > 0$ such that $f(x) \geq \frac{\ell}{2}$ for $x \geq A$, then $\displaystyle\int_A^{+\infty} f(x)dx \geq +\infty$. Thus $\ell = 0$.

(b) i. The integral $\displaystyle\int_0^{+\infty} \sin(x^2)dx$ exists, but $\displaystyle\lim_{x\longmapsto+\infty} g(x)$ does not exists.

ii. The same result if g is non negative. Take $g(x) = \frac{1}{n^3}$ on the interval $[n - \frac{1}{n}, n + \frac{1}{n}]$ for $n \geq 2$ and $g = 0$ elsewhere. $\displaystyle\int_0^{+\infty} g(x)dx$ converges but $\displaystyle\lim_{x\longmapsto+\infty} g(x)$ does not exists.

1-2-12 (a) In a neighborhood of 0, the function $\dfrac{\sin x}{x}$ is bounded and continuous.

For $x \geq 1$, the function $x \longmapsto \frac{1}{x}$ is decreasing and tends to 0 when $x \longrightarrow +\infty$ and $|\int_a^b \sin x dx| \leq 2$ for all $a, b \in \mathbb{R}$, then by the Abel Theorem, the integral $\int_0^{+\infty} \frac{\sin x}{x} dx$ is convergent.

(b) For $x \geq 1$, $\frac{|\sin x|}{x} \geq \frac{\sin^2 x}{x} = \frac{1 - \cos 2x}{2x} = \frac{1}{2x} - \frac{\cos 2x}{2x}$. The Riemann integral of $\frac{\cos 2x}{2x}$ is convergent. Then for all $a, b \in \mathbb{R}$, $\int_a^b \frac{\cos 2x}{2x} dx$ is bounded, but $\int_a^b \frac{dx}{2x}$ is not bounded. Then the integral $\int_1^{+\infty} \frac{|\sin t|}{t} dt$ is not convergent.

1-2-13 (a) In a neighborhood of a the function $\frac{1}{\sqrt{(x-a)(b-x)}}$ is equivalent to $\frac{1}{\sqrt{(x-a)(b-a)}}$ which is integrable. The same result in a neighborhood of b.

(b) $J = \int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \int_0^{\frac{\pi}{2}} \frac{2(b-a)\sin(2t)}{(b^2-a^2)\sin(2t)} dt = \frac{\pi}{b+a}$.

1-2-14 (a) i. In a neighborhood of 0, the function $t^{x-1}e^{-t}$ is equivalent to t^{x-1} which is integrable for $x > 0$.

ii. By integration by parts, $\Gamma(x+1) = x\Gamma(x)$.

(b) i. In use of the Hölder inequality, we prove $\Gamma(\frac{x}{p} + \frac{y}{q}) \leq \Gamma(x)^{1/p}\Gamma(y)^{1/q}$ for $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

ii. $\ln(\Gamma)(\frac{x}{p} + \frac{y}{q}) \leq \ln \Gamma(x)^{1/p} + \ln \Gamma(y)^{1/q}$, then $\ln \Gamma$ is a convex function.

1-2-15 (a) In a neighborhood of 0, $\frac{\ln(1+t^\alpha)}{t^\beta} \approx t^{\alpha-\beta}$, which is integrable if and only if $\alpha - \beta > -1$.

In a neighborhood of $+\infty$, $\frac{\ln(1+t^\alpha)}{t^\beta}$ is integrable if and only if $\beta > 1$. Then the integral $\int_0^{+\infty} \frac{\ln(1+t^\alpha)}{t^\beta} dt$ is convergent if and only if $1 < \beta < \alpha + 1$.

(b) In a neighborhood of 0, $\frac{\sin^2 t}{t^\alpha} \approx \frac{1}{t^{\alpha-2}}$, which is integrable if and only if $\alpha < 3$.

In a neighborhood of $+\infty$, the integral $\displaystyle\int_1^{+\infty} \frac{\sin^2 t}{t^\alpha} dt$ is convergent if and only if $\alpha > 1$. Then the integral $\displaystyle\int_0^{+\infty} \frac{\sin^2 t}{t^\alpha} dt$ is convergent if and only if $1 < \alpha < 3$.

(c) i. The integral $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dt}{1 + \cos\alpha \cos t}$ is convergent if and only if $\alpha \neq \pi + 2k\pi$, $k \in \mathbb{Z}$.

ii. Set $x = \tan(\frac{t}{2})$, $\displaystyle I(\alpha) = \frac{2}{1 + \cos\alpha} \int_0^1 \frac{dt}{1 + x^2 \tan^2(\frac{\alpha}{2})} = \frac{\pi\alpha}{2\sin\alpha}$.

1-2-16 (a) Let $f(u) = \ln 2 + \ln u$, f is differentiable and $f'(u) = \dfrac{1}{u} \geq 0$, then f is increasing and $f(\frac{1}{2} = 0$, then $\forall u \in [\frac{1}{2}, 1]$, $-\ln u \leq \ln 2$.

(b) In a neighborhood of 1, $u < 1$, $\displaystyle\frac{\ln u}{\sqrt{1 - u^2}} \approx \frac{u - 1}{\sqrt{1 - u^2}} = -\sqrt{\frac{1 - u}{1 + u}}$ which is integrable, then the integral $\displaystyle\int_0^1 \frac{\ln u}{\sqrt{1 - u^2}} du$ is convergent.

(c) With the change of variable $u = \sin\theta$, $\displaystyle\int_0^1 \frac{\ln u}{\sqrt{1 - u^2}} du = \int_0^{\frac{\pi}{2}} \ln(\sin\theta) d\theta$ and with the change of variable $u = \cos\theta$, $\displaystyle\int_0^1 \frac{\ln u}{\sqrt{1 - u^2}} du = \int_0^{\frac{\pi}{2}} \ln(\cos\theta) d\theta$. Then the integrals $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin\theta) d\theta$ and $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\cos\theta) d\theta$ are convergent and have the same value.

(d) The integral $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\sin\theta) d\theta$ is convergent and by the change of variable $t = \pi - \theta$, the integral $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \ln(\sin\theta) d\theta$ is also convergent and $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \ln(\sin\theta) d\theta = \int_0^{\frac{\pi}{2}} \ln(\sin t) dt$. Then

$$\int_0^{\pi} \ln(\sin\theta) d\theta = 2 \int_0^{\frac{\pi}{2}} \ln(\sin\theta) d\theta.$$

(e)

$$\begin{aligned}
I &= \int_0^1 \frac{\ln u}{\sqrt{1-u^2}}du \\
&= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln(\sin\theta)d\theta + \int_0^{\frac{\pi}{2}} \ln(\cos\theta)d\theta\right) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin\theta\cos\theta)d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin(2\theta)) - \ln 2 d\theta \\
&= \frac{1}{4}\int_0^{\pi} \ln(\sin\theta)d\theta - \frac{\pi}{4}\ln 2 \\
&= \frac{1}{2}I - \frac{\pi}{4}\ln 2.
\end{aligned}$$

Then $I = -\frac{\pi}{2}\ln 2$.

1-2-17 (a) $\lim_{t\to 1^-} \frac{\ln t}{\sqrt{1-t}} = 0$, then the integral $\int_0^1 \frac{\ln t}{\sqrt{1-t}}dt$ is convergent.

With the change of variable $1-t=s^2$, $\int_0^1 \frac{\ln t}{\sqrt{1-t}}dt = 2\int_0^1 \ln(1-s^2)ds$ and by integration by parts; $\int_0^1 \frac{\ln t}{\sqrt{1-t}}dt = -2\int_0^1 \frac{s}{1-s}ds = 2\ln 2 - 2$.

(b) i. In a neighborhood of 0, $\frac{\sin^2 t}{t^2} \approx 1$ and for $t \geq 1$, $\frac{\sin^2 t}{t^2} \leq \frac{1}{t^2}$.

Then the integral $\int_0^{+\infty} \frac{\sin^2 t}{t^2}dt$ is convergent.

ii. By l'Hopital rule

$$\lim_{\varepsilon\to 0} \frac{1-\cos\varepsilon}{\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to +\infty} \frac{1-\cos\varepsilon}{\varepsilon} = 0.$$

iii. By integration by parts

$$\begin{aligned}
\int_a^b \frac{\sin t}{t}dt &= \frac{1-\cos b}{b} - \frac{1-\cos a}{a} - \int_a^b \frac{1-\cos t}{t^2}dt \\
&= \frac{1-\cos b}{b} - \frac{1-\cos a}{a} - \int_a^b \frac{2\sin^2\frac{t}{2}}{t^2}dt \\
&= \frac{1-\cos b}{b} - \frac{1-\cos a}{a} - \int_{\frac{a}{2}}^{\frac{a}{2}} \frac{\sin^2 t}{t^2}dt.
\end{aligned}$$

Then the integral $\displaystyle\int_0^{+\infty} \frac{\sin t}{t} dt$ is convergent and

$$\int_0^{+\infty} \frac{\sin t}{t} dt = \int_0^{+\infty} \frac{\sin^2 t}{t^2} dt.$$

$\boxed{1\text{-}2\text{-}18}$ The function $\frac{\ln t}{1+t^2}$ is equivalent to $\ln t$ in a neighborhood of 0 and $\ln t$ is integrable. For t large $\dfrac{\ln t}{1+t^2} \leq \dfrac{\sqrt{t}}{1+t^2}$ which is integrable on $[A, +\infty[$. Then $\displaystyle\int_0^{+\infty} \frac{\ln t}{1+t^2} dt$ converges.

$\displaystyle\int_1^{+\infty} \frac{\ln t}{1+t^2} dt \overset{s=\frac{1}{t}}{=} -\int_0^1 \frac{\ln t}{1+t^2} dt$. Then $\displaystyle\int_0^{+\infty} \frac{\ln t}{1+t^2} dt = 0$

Problem-1-1 (a) i. $I_n - I_{n-1} = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin((2n+1)x) - \sin((2n-1)x)}{\sin x} dx = \int_0^{\frac{\pi}{2}} 2\cos(2nx)dx = 0$, then the sequence $(I_n)_n$ is constant and $I_n = I_0 = \frac{\pi}{2}$.

ii. $I_n - J_n = \displaystyle\int_0^{\frac{\pi}{2}} 2\frac{\cos((2n+\frac{1}{2})x)\sin(\frac{1}{2})x}{\sin x} dx \overset{x=2t}{=} \int_0^{\frac{\pi}{4}} 2\frac{\cos((4n+1)t)}{\cos t} dt = 2K_n$. Since $\displaystyle\lim_{n\longrightarrow+\infty} K_n = 0$, then $\displaystyle\lim_{n\longrightarrow+\infty} J_n = \frac{\pi}{2}$.

(b) i. $J_n - J_{n-1} = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2nx - \sin 2(n-1)x}{\sin x} dx = 2\int_0^{\frac{\pi}{2}} \cos(2n-1)xdx = \frac{2(-1)^{n-1}}{2n-1}$.

ii. $J_n = \displaystyle\sum_{k=1}^{n} \frac{2(-1)^{k-1}}{2k-1}$.

iii. $\displaystyle\lim_{n\longrightarrow+\infty}(1 - \frac{1}{3} + \frac{1}{5} + \ldots + \frac{(-1)^{n-1}}{2n-1}) = \lim_{n\longrightarrow+\infty} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}$.

Problem-1-2 (a) i. Let $\varepsilon > 0$, there exists $A > 0$ such that $|f(x) - \ell| \leq \varepsilon$ for all $x \geq A$.

$F(x) - \ell = \displaystyle\frac{1}{x}\int_0^A (f(t)-\ell)dt + \frac{1}{x}\int_A^x (f(t)-\ell)dt$.

$\displaystyle\lim_{x\longrightarrow+\infty} \frac{1}{x}\int_0^A (f(t)-\ell)dt = 0$ and $\displaystyle\frac{1}{x}\left|\int_A^x (f(t)-\ell)dt\right| \leq \varepsilon$. Then $\displaystyle\lim_{x\longrightarrow+\infty} F(x) = \ell$.

ii. $F(x) = \displaystyle\frac{1}{x}\int_0^x \cos t dt = \frac{\sin x}{x}$. $\displaystyle\lim_{x\longrightarrow+\infty} F(x) = 0$, but $\displaystyle\lim_{x\longrightarrow+\infty} \cos x$ does not exist.

iii. i) Since $\displaystyle\lim_{n\longrightarrow+\infty} u_n = \ell$, then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|u_n - \ell| \leq \varepsilon$. Thus for $x \geq N$, there exists $n \geq N$ such that $x \in [n, n+1]$, $f(x) = u_n$ and $|f(x) - \ell| \leq \varepsilon$. Which proves that $\displaystyle\lim_{x\longrightarrow+\infty} f(x) = \ell$.

ii) $\displaystyle\int_0^n f(t)dt = \sum_{k=0}^{n-1}\int_k^{k+1} f(t)dt = \sum_{k=0}^{n-1} u_k$. Since $\displaystyle\lim_{x\longrightarrow+\infty} F(x) = \ell$, then

$$\lim_{n\longrightarrow+\infty} \frac{u_0 + u_1 + \ldots + u_n}{n} = \ell.$$

and

$$\lim_{n\longrightarrow+\infty} \frac{u_0 + u_1 + \ldots + u_{n-1}}{n} = \ell.$$

(b) Since f is continuous at 0, for $\varepsilon > 0$, there exists $\alpha > 0$ such that for $x < \alpha$, $|f(x) - f(0)| \leq \varepsilon$.
For $x < \alpha$, $|f(x) - f(0)| \leq \frac{1}{x}\int_0^x |f(t) - f(0)|dt \leq \varepsilon$. Then $\lim_{x \to 0} F(x) = f(0)$ and F can be extended as a continuous function on $[0, 1]$.

(c) Since f is continuous, F is differentiable on $]0, 1]$ and $F'(x) = -\dfrac{F(x)}{x} + \dfrac{f(x)}{x}$.

(d) Assume that $F = \lambda f$.

  i. Since F is differentiable on $]0, 1]$, then f is differentiable on $]0, 1]$ and $(\lambda - 1)f(x) + \lambda x f'(x) = 0$.

  ii. The solution of the differential equation $(\lambda - 1)y + \lambda x y' = 0$ fulfills $\ln|y| = \frac{1-\lambda}{\lambda}\ln|x|$. Then f is a polynomial if and only if: $\dfrac{1-\lambda}{\lambda} \in \mathbb{N}$.

(e) Let $t \in]0, 1[$ such that $F(t) = \sup_{x \in [0,1]} f(x) = \ell$, then $\dfrac{1}{t}\displaystyle\int_0^t (f(x) - \ell)dx = 0$. But $f(x) - \ell \leq 0$ for all $x \in [0, t]$, then f is constant on $[0, t]$.

(f) If f is differentiable at 0, then $f(t) = f(0) + tf'(0) + t\varepsilon(t)$, where $\lim_{t \to 0} \varepsilon(t) = 0$. Then

$$F(t) = f(0) + \frac{t}{2}f'(0) + \frac{1}{t}\int_0^t x\varepsilon(x)dx.$$

It results that F is differentiable at 0 and $F'(0) = \dfrac{t}{2}f'(0)$.

Problem-1-3

(a) Let $a \in \mathbb{R}$ and $r > 0$, for $x \in [a - r, a + r]$; $|F(x) - F(a)| = |\displaystyle\int_a^x f(t)dt| \leq |x - a| \sup_{t \in [a-r, a+r]} |f(t)|$, which proves that F is continuous at a.

(b)

$$\begin{aligned}
G(y) &= F(x_0+y) - F(y-x_0) \\
&= \int_0^{x_0+y} f(t)dt - \int_0^{y-x_0} f(t)dt \\
&= \int_0^{y} f(t)dt + \int_y^{x_0+y} f(t)dt - \int_0^{y} f(t)dt - \int_y^{y-x_0} f(t)dt \\
&= \int_y^{y+x_0} f(t)dt - \int_y^{y-x_0} f(s)ds \\
&\overset{u=t-y,v=y-s}{=} \int_0^{x_0} f(u+y)du + \int_0^{x_0} f(y-v)dv \\
&= 2f(y)\int_0^{x_0} f(u)du = 2f(y)F(x_0).
\end{aligned}$$

(c) Since F is continuous and $G(y) = F(x_0+y)-F(y-x_0) = 2f(y)F(x_0)$, f is continuous, which yields that F is $\mathcal{C}^1$ and then f is $\mathcal{C}^1$. The result is obtained by induction.

(d) For x fixed in $\mathbb{R}$, we set $H(y) = f(x+y) + f(x-y) = 2f(x)f(y)$.

i. $H'(0) = f'(x) - f'(x) = 0 = 2f(x)f'(0)$. If we take x such that $f(x) \neq 0$, we deduce that $f'(0) = 0$.

ii. $H^{''}(0) = 2f(x)f^{''}(0) = 2f^{''}(x)$, which yields that $f^{''}(x) = f^{''}(0)f(x)$, $\forall\, x \in \mathbb{R}$.

(e) i. Using i) with $x = y = 0$ we have $f(0) = f^2(0)$, then $f(0) = 0$ or $f(0) = 1$.

ii. If $f(0) = 0$, then $f^{''} = 0$. Then $f(x) = ax$. The relation i) gives that $f = 0$.

iii. Since F is not the 0 function, then $f(0) = 1$.

Problem-1-4

$$f_n(x) = \int_0^x \frac{dt}{\cosh^n t}$$

(a) Since $\cosh t$ is C^∞ and positive and even, f_n is $\mathcal{C}^\infty$ and odd. $f_n'(x) = \dfrac{1}{\cosh^n x}$ and $f_n^{''}(x) = -\frac{n \sinh x}{\cosh^{n+1} x}$.

(b) $f_1(x) = \displaystyle\int_0^x \frac{2}{e^t + e^{-t}} dt = 2\tan^{-1}(e^x)$, $f_2(x) \displaystyle\int_0^x \frac{1}{\cosh^2 t} dt = \tanh(x)$, $\lambda_1 = \lim\limits_{x\to+\infty} f_1(x) = \pi$ and $\lambda_2 = \lim\limits_{x\to+\infty} f_2(x) = 1$.

(c) $f_n(x) = \displaystyle\int_0^x \frac{dt}{\cosh^n t} \leq f_1(x) \leq \pi$ for $x \in [0, +\infty[$.

(d) f_n is bounded and increasing on the interval $[0,+\infty[$, then $\lim_{x\to+\infty} f_n(x) = \lambda_n$ exists in $\mathbb{R}$.

(e) By integration by parts ($u = \dfrac{1}{\cosh^{n-2} t}$ and $v' = \dfrac{1}{\cosh^2 t}$), we find

$$f_n(x) = \frac{\sinh x}{\cosh^{n-1} x} + (n-2)\int_0^x \frac{\sinh^2 t}{\cosh^n t}dt = \frac{\sinh x}{\cosh^{n-1} x} + (n-2)f_{n-2}(x) - (n-2)f_n(x).$$

Then

$$(n-1)f_n(x) = (n-2)f_{n-2}(x) + \frac{\sinh x}{(\cosh x)^{n-1}}, \quad \forall\, n \geq 3.$$

(f) From the previous formula, we have $(n-1)\lambda_n = (n-2)\lambda_{n-2}$. $\lambda_{2n+1} = \dfrac{2n!}{2^{2n}n!^2}\pi$, $\lambda_{2n} = \dfrac{2^{2n-2}(n-1)!^2}{(2n-1)!}$.

Problem-1-5

(a) By integration by parts, we have $W_{n+2} = \dfrac{n+1}{n+2}W_n$.

(b) $W_0 = \frac{\pi}{2}$, $W_1 = 1$, then $W_{2n} = \dfrac{(2n)!}{2^{2n}(n!)^2}\dfrac{\pi}{2}$ and $W_{2n+1} = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$.

(c) $nW_nW_{n-1} = n\frac{n-1}{n}W_{n-2}\frac{n-2}{n-1}W_{n-3} = (n-2)W_{n-2}W_{n-3}$. Since $2W_2W_1 = W_1W_0 = \frac{\pi}{2}$, then the sequence $(nW_nW_{n-1})_n$ is constant equal to $\frac{\pi}{2}$.

(d) Since $0 \leq \cos x \leq 1$, then $W_{n+1} \leq W_n \leq W_{n-1}$. Moreover $\dfrac{W_{n+1}}{W_{n-1}} = \dfrac{n}{n+1} \leq \dfrac{W_n}{W_{n-1}} \leq 1$. Then $\lim_{n\to+\infty} \dfrac{W_n}{W_{n-1}} = 1$.

$nW_nW_{n-1} = nW_n^2\dfrac{W_{n-1}}{W_n} = \dfrac{\pi}{2}$. Then $\lim_{n\to+\infty} \sqrt{n}W_n = \sqrt{\dfrac{\pi}{2}}$.

(e) i. $B_n = \displaystyle\int_0^{\sqrt{n}} (1-\frac{t^2}{n})^n dt \overset{t=\sqrt{n}u}{=} \sqrt{n}\int_0^1 (1-u^2)^n du.$

ii. For the change of variable $u = \sin v$, $B_n = \sqrt{n}\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n+1} v dv.$

(f) For $x > 0$, define $A_n(x) = \displaystyle\int_0^x (1+\frac{t^2}{n})^{-n}dt$, for $n \in \mathbb{N}$.

i. $A_n(x) \overset{t=\sqrt{n}u}{=} \displaystyle\int_0^{\frac{x}{\sqrt{n}}} (1+u^2)^{-n}du.$

and

$$A_n(x) \overset{u=\tan v}{=} \sqrt{n}\int_0^{\tan^{-1}\frac{x}{n}} \cos^{2n-2} v dv.$$

ii. It is obvious that $\lim_{x\to+\infty} A_n(x) = \sqrt{n}W_{2n-2}$, then $\lim_{n\to+\infty}\lim_{x\to+\infty} A_n(x) = \lim_{n\to+\infty} \sqrt{n}W_{2n-2} = \frac{\sqrt{\pi}}{2}$.

(g) i. By the Taylor Formula, $e^y = 1 + y + \frac{y^2}{2}e^c$, where c between 0 and y, then $\forall y \in \mathbb{R}$, $e^y \geq 1 + y$ and $\ln(1 + y) \leq y$. $(1 + \frac{x^2}{n})^n = e^{n\ln(1+\frac{x^2}{n})} \leq e^{x^2}$.

In the same way for $0 < x < 1$, $\ln(1-x) \leq -x$, then $(1-\frac{x^2}{n})^n = e^{n\ln(1-\frac{x^2}{n})} \leq e^{-x^2}$, $\forall x \in [0, \sqrt{n}]$.

ii. $B_n = \int_0^{\sqrt{n}} (1-\frac{t^2}{n})^n dt \leq \int_0^{\sqrt{n}} e^{-t^2} dt \leq \int_0^{\sqrt{n}} (1+\frac{t^2}{n})^{-n} dt$, then

$$\lim_{n\to+\infty} B_n \leq \lim_{n\to+\infty} \int_0^{\sqrt{n}} e^{-x^2} dx \leq \lim_{n\to+\infty}\lim_{x\to+\infty} A_n(x).$$

Then

$$\frac{\sqrt{\pi}}{2} = \lim_{n\to+\infty} \int_0^{\sqrt{n}} e^{-x^2} dx = \lim_{n\to+\infty}\lim_{x\to+\infty} A_n(x).$$

Problem-1-6 (a) i. For $x \geq 0$, $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}e^c$. For $x = 1$ and since $\frac{x^4}{24}e^c \geq 0$, we have $\frac{8}{3} < e$.

ii. $\int_0^1 \ln(1 + t)dt = 2\ln 2 - 1 = \ln\frac{4}{e}$ and $\ln(\int_0^1 (1 + t)dt) = \ln\frac{3}{2}$.

Since $\frac{8}{3} < e$, then $\int_0^1 \ln(1 + t)dt < \ln(\int_0^1 (1 + t)dt)$.

iii. If $f(x) = \ln(1 + x) - x$, $f'(x) = \frac{-x}{1 + x} \leq 0$. f is decreasing and $f(0) = 0$, then $\ln(1 + v) \leq v$, for $v > -1$.

(b) i. $\int_0^1 \ln u(t)dt \leq \int_0^1 u(t) - 1dt = 0$. The equality done if and only if $u = 1$.

ii. Let $u(t) = \frac{f(t)}{\int_0^1 f(t)dt}$. $\int_0^1 u(t)dt = 1$ and then $\int_0^1 \ln u(t)dt \leq 0$, which yields that

$$\int_0^1 \ln f(t)dt \leq \ln(\int_0^1 f(t)dt). \tag{7.9}$$

(c) Since the function ln is concave, $\ln(\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n})) \geq \frac{1}{n}\sum_{k=1}^{n}\ln(f(\frac{k}{n}))$ for all $n \in \mathbb{N}$. f and $\ln \circ f$ are continuous, then

$$\lim_{n\to+\infty} \ln(\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n})) = \ln(\int_0^1 f(t)dt)$$

and

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\ln(f(\frac{k}{n})) = \int_0^1 \ln f(t)dt,$$

which gives the proof.

Problem-1-7 (a) The function $t \longmapsto f(t) = \frac{1}{(n+t)^2}$ is C^∞ on $[0,+\infty[$, $f^{(k)}(t) = \dfrac{(-1)^k(k+1)!}{(t+n)^{k+2}}$, $|f^{(k)}(t)| \leq \dfrac{(k+1)!}{n^{k+2}}$ and the series $\sum_{n\geq 1}\dfrac{1}{(n)^{k+2}}$ is convergent, then ψ is C^∞ on $[0,+\infty[$ and $\lim_{t\to+\infty}\psi(t) = 0$.

(b) i. As the sequence $((n+t)(n+t+1))_n$ is decreasing, then for any $t \geq 0$ and $n \geq 1$

$$\frac{1}{(n+t)(n+t+1)} \leq \frac{1}{(n+t)^2} \leq \frac{1}{(n+t)(n+t-1)}.$$

ii. $\dfrac{1}{(n+t)(n+t+1)} = \dfrac{1}{n+t} - \dfrac{1}{n+t+1}$ and $\dfrac{1}{(n+t)(n+t-1)} = \dfrac{1}{n+t-1} - \dfrac{1}{n+t}$, then $\dfrac{1}{t+1} \leq \psi(t) \leq \dfrac{1}{t}$, for any $t > 0$, and that $\psi(t) \approx \frac{1}{t}$; $(t \to +\infty)$.

iii. $\dfrac{1}{(n+t)^2} = \dfrac{1}{(n+t)^2(n+t+1)} + (\dfrac{1}{n+t} - \dfrac{1}{n+t+1})$, since $\sum_{n=1}^{+\infty}(\dfrac{1}{n+t} - \dfrac{1}{n+t+1}) = \dfrac{1}{1+t}$, then

$$\psi(t) - \frac{1}{t+1} = \sum_{n=1}^{+\infty}\frac{1}{(n+t)^2(n+t+1)}.$$

(c) i. Since the function $x \longmapsto \dfrac{1}{(x+t)^3}$ is decreasing, $\dfrac{1}{(n+t+1)^3} \leq \displaystyle\int_n^{n+1}\frac{dx}{(t+x)^3} \leq \frac{1}{(n+t)^3}$. Then for any $t > 0$

$$\begin{aligned}
\frac{2}{(2+t)^2} &= \int_2^{+\infty} \frac{dx}{(t+x)^3} \leq \sum_{n=1}^{+\infty} \frac{1}{(n+t+1)^3} \\
&\leq \psi(t) - \frac{1}{t+1} \leq \sum_{n=1}^{+\infty} \frac{1}{(n+t)^3} \\
&\leq \int_0^{+\infty} \frac{dx}{(t+x)^3} = \frac{2}{t^2}.
\end{aligned}$$

Then

$$\psi(t) - \frac{1}{t+1} = \frac{\eta(t)}{t^2},$$

ii. Since $\psi(t) - \dfrac{1}{1+t} \geq 0$ and the function $t \longmapsto \frac{1}{t^2}$ is integrable, then $\displaystyle\int_0^{+\infty} (\psi(t) - \frac{1}{1+t})dt$ is convergent.

(d) Since the series $\displaystyle\sum_{n\geq 1} \frac{1}{(n+t)^2(n+t+1)}$ converges uniformly on $[0, +\infty[$,

$$\int_0^{+\infty} (\psi(t) - \frac{1}{t+1})dt = \sum_{n=1}^{+\infty} (\frac{1}{n} - \ln(\frac{n+1}{n})).$$

Problem-1-1 (a) i. $I_n - I_{n-1} = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin((2n+1)x) - \sin((2n-1)x)}{\sin x}dx = \int_0^{\frac{\pi}{2}} 2\cos(2nx)dx = 0$, then the sequence $(I_n)_n$ is constant and $I_n = I_0 = \frac{\pi}{2}$.

ii. $I_n - J_n = \displaystyle\int_0^{\frac{\pi}{2}} 2\frac{\cos((2n+\frac{1}{2})x)\sin(\frac{1}{2})x}{\sin x}dx \overset{x=2t}{=} \int_0^{\frac{\pi}{4}} 2\frac{\cos((4n+1)t)}{\cos t}dt = 2K_n$. Since $\displaystyle\lim_{n\longrightarrow+\infty} K_n = 0$, then $\displaystyle\lim_{n\longrightarrow+\infty} J_n = \frac{\pi}{2}$.

(b) i. $J_n - J_{n-1} = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin 2nx - \sin 2(n-1)x}{\sin x}dx = 2\int_0^{\frac{\pi}{2}} \cos(2n-1)xdx = \frac{2(-1)^{n-1}}{2n-1}$.

ii. $J_n = \displaystyle\sum_{k=1}^{n} \frac{2(-1)^{k-1}}{2k-1}$.

iii. $\displaystyle\lim_{n\longrightarrow+\infty}(1 - \frac{1}{3} + \frac{1}{5} + \ldots + \frac{(-1)^{n-1}}{2n-1}) = \lim_{n\longrightarrow+\infty}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}$.

Problem-1-2 (a) i. Let $\varepsilon > 0$, there exists $A > 0$ such that $|f(x) - \ell| \le \varepsilon$ for all $x \ge A$.

$F(x) - \ell = \displaystyle\frac{1}{x}\int_0^A (f(t) - \ell)dt + \frac{1}{x}\int_A^x (f(t) - \ell)dt$.

$\displaystyle\lim_{x\longrightarrow+\infty} \frac{1}{x}\int_0^A (f(t) - \ell)dt = 0$ and $\displaystyle\frac{1}{x}\left|\int_A^x (f(t) - \ell)dt\right| \le \varepsilon$. Then $\displaystyle\lim_{x\longrightarrow+\infty} F(x) = \ell$.

ii. $F(x) = \displaystyle\frac{1}{x}\int_0^x \cos t dt = \frac{\sin x}{x}$. $\displaystyle\lim_{x\longrightarrow+\infty} F(x) = 0$, but $\displaystyle\lim_{x\longrightarrow+\infty}\cos x$ does not exist.

iii. i) Since $\displaystyle\lim_{n\longrightarrow+\infty} u_n = \ell$, then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \ge N$, $|u_n - \ell| \le \varepsilon$. Thus for $x \ge N$, there exists $n \ge N$ such that $x \in [n, n+1]$, $f(x) = u_n$ and $|f(x) - \ell| \le \varepsilon$. Which proves that $\displaystyle\lim_{x\longrightarrow+\infty} f(x) = \ell$.

ii) $\displaystyle\int_0^n f(t)dt = \sum_{k=0}^{n-1}\int_k^{k+1} f(t)dt = \sum_{k=0}^{n-1} u_k$. Since $\displaystyle\lim_{x\longrightarrow+\infty} F(x) = \ell$, then

$$\lim_{n\longrightarrow+\infty} \frac{u_0 + u_1 + \ldots + u_n}{n} = \ell.$$

and

$$\lim_{n\longrightarrow+\infty} \frac{u_0 + u_1 + \ldots + u_{n-1}}{n} = \ell.$$

(b) Since f is continuous at 0, for $\varepsilon > 0$, there exists $\alpha > 0$ such that for $x < \alpha$, $|f(x) - f(0)| \leq \varepsilon$.
For $x < \alpha$, $|f(x)-f(0)| \leq \frac{1}{x}\int_0^x |f(t)-f(0)|dt \leq \varepsilon$. Then $\lim\limits_{x\to 0} F(x) = f(0)$ and F can be extended as a continuous function on $[0,1]$.

(c) Since f is continuous, F is differentiable on $]0,1]$ and $F'(x) = -\dfrac{F(x)}{x} + \dfrac{f(x)}{x}$.

(d) Assume that $F = \lambda f$.

i. Since F is differentiable on $]0,1]$, then f is differentiable on $]0,1]$ and $(\lambda-1)f(x) + \lambda x f'(x) = 0$.

ii. The solution of the differential equation $(\lambda-1)y + \lambda x y' = 0$ fulfills $\ln|y| = \frac{1-\lambda}{\lambda}\ln|x|$. Then f is a polynomial if and only if: $\dfrac{1-\lambda}{\lambda} \in \mathbb{N}$.

(e) Let $t \in]0,1[$ such that $F(t) = \sup\limits_{x\in[0,1]} f(x) = \ell$, then $\dfrac{1}{t}\displaystyle\int_0^t (f(x) - \ell)dx = 0$. But $f(x) - \ell \leq 0$ for all $x \in [0,t]$, then f is constant on $[0,t]$.

(f) If f is differentiable at 0, then $f(t) = f(0) + tf'(0) + t\varepsilon(t)$, where $\lim_{t\to 0}\varepsilon(t) = 0$. Then

$$F(t) = f(0) + \frac{t}{2}f'(0) + \frac{1}{t}\int_0^t x\varepsilon(x)dx.$$

It results that F is differentiable at 0 and $F'(0) = \dfrac{t}{2}f'(0)$.

Problem-1-3

(a) Let $a \in \mathbb{R}$ and $r > 0$, for $x \in [a-r, a+r]$; $|F(x) - F(a)| = |\displaystyle\int_a^x f(t)dt| \leq |x-a| \sup\limits_{t\in[a-r,a+r]} |f(t)|$, which proves that F is continuous at a.

(b)

$$\begin{aligned}
G(y) &= F(x_0+y) - F(y-x_0) \\
&= \int_0^{x_0+y} f(t)dt - \int_0^{y-x_0} f(t)dt \\
&= \int_0^{y} f(t)dt + \int_y^{x_0+y} f(t)dt - \int_0^{y} f(t)dt - \int_y^{y-x_0} f(t)dt \\
&= \int_y^{y+x_0} f(t)dt - \int_y^{y-x_0} f(s)ds \\
&\overset{u=t-y, v=y-s}{=} \int_0^{x_0} f(u+y)du + \int_0^{x_0} f(y-v)dv \\
&= 2f(y)\int_0^{x_0} f(u)du = 2f(y)F(x_0).
\end{aligned}$$

(c) Since F is continuous and $G(y) = F(x_0+y) - F(y-x_0) = 2f(y)F(x_0)$, f is continuous, which yields that F is $\mathcal{C}^1$ and then f is $\mathcal{C}^1$. The result is obtained by induction.

(d) For x fixed in $\mathbb{R}$, we set $H(y) = f(x+y) + f(x-y) = 2f(x)f(y)$.

  i. $H'(0) = f'(x) - f'(x) = 0 = 2f(x)f'(0)$. If we take x such that $f(x) \neq 0$, we deduce that $f'(0) = 0$.

  ii. $H^{''}(0) = 2f(x)f^{''}(0) = 2f^{''}(x)$, which yields that $f^{''}(x) = f^{''}(0)f(x)$, $\forall\, x \in \mathbb{R}$.

(e) i. Using i) with $x = y = 0$ we have $f(0) = f^2(0)$, then $f(0) = 0$ or $f(0) = 1$.

  ii. If $f(0) = 0$, then $f^{''} = 0$. Then $f(x) = ax$. The relation i) gives that $f = 0$.

  iii. Since F is not the 0 function, then $f(0) = 1$.

Problem-1-4

$$f_n(x) = \int_0^x \frac{dt}{\cosh^n t}$$

(a) Since $\cosh t$ is C^∞ and positive and even, f_n is $\mathcal{C}^\infty$ and odd. $f_n'(x) = \dfrac{1}{\cosh^n x}$ and $f_n^{''}(x) = -\frac{n\sinh x}{\cosh^{n+1} x}$.

(b) $f_1(x) = \displaystyle\int_0^x \frac{2}{e^t + e^{-t}}dt = 2\tan^{-1}(e^x)$, $f_2(x) \displaystyle\int_0^x \frac{1}{\cosh^2 t}dt = \tanh(x)$, $\lambda_1 = \lim_{x\to+\infty} f_1(x) = \pi$ and $\lambda_2 = \lim_{x\to+\infty} f_2(x) = 1$.

(c) $f_n(x) = \displaystyle\int_0^x \frac{dt}{\cosh^n t} \leq f_1(x) \leq \pi$ for $x \in [0, +\infty[$.

(d) f_n is bounded and increasing on the interval $[0,+\infty[$, then $\lim_{x\to+\infty} f_n(x) = \lambda_n$ exists in $\mathbb{R}$.

(e) By integration by parts ($u = \dfrac{1}{\cosh^{n-2} t}$ and $v' = \dfrac{1}{\cosh^2 t}$), we find

$$f_n(x) = \frac{\sinh x}{\cosh^{n-1} x} + (n-2)\int_0^x \frac{\sinh^2 t}{\cosh^n t}dt = \frac{\sinh x}{\cosh^{n-1} x} + (n-2)f_{n-2}(x) - (n-2)f_n(x).$$

Then

$$(n-1)f_n(x) = (n-2)f_{n-2}(x) + \frac{\sinh x}{(\cosh x)^{n-1}}, \quad \forall\, n \geq 3.$$

(f) From the previous formula, we have $(n-1)\lambda_n = (n-2)\lambda_{n-2}$. $\lambda_{2n+1} = \dfrac{2n!}{2^{2n}n!^2}\pi$, $\lambda_{2n} = \dfrac{2^{2n-2}(n-1)!^2}{(2n-1)!}$.

Problem-1-5 (a) By integration by parts, we have $W_{n+2} = \dfrac{n+1}{n+2}W_n$.

(b) $W_0 = \frac{\pi}{2}$, $W_1 = 1$, then $W_{2n} = \dfrac{(2n)!}{2^{2n}(n!)^2}\dfrac{\pi}{2}$ and $W_{2n+1} = \dfrac{2^{2n}(n!)^2}{(2n+1)!}$.

(c) $nW_nW_{n-1} = n\frac{n-1}{n}W_{n-2}\frac{n-2}{n-1}W_{n-3} = (n-2)W_{n-2}W_{n-3}$. Since $2W_2W_1 = W_1W_0 = \frac{\pi}{2}$, then the sequence $(nW_nW_{n-1})_n$ is constant equal to $\frac{\pi}{2}$.

(d) Since $0 \leq \cos x \leq 1$, then $W_{n+1} \leq W_n \leq W_{n-1}$. Moreover $\dfrac{W_{n+1}}{W_{n-1}} = \dfrac{n}{n+1} \leq \dfrac{W_n}{W_{n-1}} \leq 1$. Then $\lim_{n\to+\infty} \dfrac{W_n}{W_{n-1}} = 1$.

$nW_nW_{n-1} = nW_n^2\dfrac{W_{n-1}}{W_n} = \dfrac{\pi}{2}$. Then $\lim_{n\to+\infty} \sqrt{n}W_n = \sqrt{\dfrac{\pi}{2}}$.

(e) i. $B_n = \displaystyle\int_0^{\sqrt{n}} (1-\frac{t^2}{n})^n dt \overset{t=\sqrt{n}u}{=} \sqrt{n}\int_0^1 (1-u^2)^n du.$

ii. For the change of variable $u = \sin v$, $B_n = \sqrt{n}\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n+1} v dv.$

(f) For $x > 0$, define $A_n(x) = \displaystyle\int_0^x (1+\frac{t^2}{n})^{-n}dt$, for $n \in \mathbb{N}$.

i. $A_n(x) \overset{t=\sqrt{n}u}{=} \displaystyle\int_0^{\frac{x}{\sqrt{n}}} (1+u^2)^{-n}du.$

and

$$A_n(x) \overset{u=\tan v}{=} \sqrt{n}\int_0^{\tan^{-1}\frac{x}{n}} \cos^{2n-2} v dv.$$

ii. It is obvious that $\lim\limits_{x\to+\infty} A_n(x) = \sqrt{n}W_{2n-2}$, then $\lim\limits_{n\to+\infty}\lim\limits_{x\to+\infty} A_n(x) = \lim\limits_{n\to+\infty}\sqrt{n}W_{2n-2} = \dfrac{\sqrt{\pi}}{2}$.

(g) i. By the Taylor Formula, $e^y = 1 + y + \frac{y^2}{2}e^c$, where c between 0 and y, then $\forall y \in \mathbb{R}$, $e^y \geq 1 + y$ and $\ln(1+y) \leq y$. $(1+\dfrac{x^2}{n})^n = e^{n\ln(1+\frac{x^2}{n})} \leq e^{x^2}$.

In the same way for $0 < x < 1$, $\ln(1-x) \leq -x$, then $(1-\dfrac{x^2}{n})^n = e^{n\ln(1-\frac{x^2}{n})} \leq e^{-x^2}$, $\forall x \in [0, \sqrt{n}]$.

ii. $B_n = \displaystyle\int_0^{\sqrt{n}} (1-\frac{t^2}{n})^n dt \leq \int_0^{\sqrt{n}} e^{-t^2} dt \leq \int_0^{\sqrt{n}} (1+\frac{t^2}{n})^{-n} dt$, then

$$\lim_{n\to+\infty} B_n \leq \lim_{n\to+\infty}\int_0^{\sqrt{n}} e^{-x^2}dx \leq \lim_{n\to+\infty}\lim_{x\to+\infty} A_n(x).$$

Then

$$\frac{\sqrt{\pi}}{2} = \lim_{n\to+\infty}\int_0^{\sqrt{n}} e^{-x^2}dx = \lim_{n\to+\infty}\lim_{x\to+\infty} A_n(x).$$

Problem-1-6 (a) i. For $x \geq 0$, $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24}e^c$. For $x = 1$ and since $\frac{x^4}{24}e^c \geq 0$, we have $\frac{8}{3} < e$.

ii. $\displaystyle\int_0^1 \ln(1+t)dt = 2\ln 2 - 1 = \ln\frac{4}{e}$ and $\displaystyle\ln(\int_0^1 (1+t)dt) = \ln\frac{3}{2}$.

Since $\frac{8}{3} < e$, then $\displaystyle\int_0^1 \ln(1+t)dt < \ln(\int_0^1 (1+t)dt)$.

iii. If $f(x) = \ln(1+x) - x$, $f'(x) = \dfrac{-x}{1+x} \leq 0$. f is decreasing and $f(0) = 0$, then $\ln(1+v) \leq v$, for $v > -1$.

(b) i. $\displaystyle\int_0^1 \ln u(t)dt \leq \int_0^1 u(t) - 1dt = 0$. The equality done if and only if $u = 1$.

ii. Let $u(t) = \dfrac{f(t)}{\int_0^1 f(t)dt}$. $\displaystyle\int_0^1 u(t)dt = 1$ and then $\displaystyle\int_0^1 \ln u(t)dt \leq 0$, which yields that

$$\int_0^1 \ln f(t)dt \leq \ln(\int_0^1 f(t)dt). \tag{7.10}$$

(c) Since the function ln is concave, $\ln(\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n})) \geq \frac{1}{n}\sum_{k=1}^{n}\ln(f(\frac{k}{n}))$ for all $n \in \mathbb{N}$. f and $\ln\circ f$ are continuous, then

$$\lim_{n\to+\infty} \ln(\frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n})) = \ln(\int_0^1 f(t)dt)$$

and

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{k=1}^{n}\ln(f(\frac{k}{n})) = \int_0^1 \ln f(t)dt,$$

which gives the proof.

Problem-1-7 (a) The function $t \longmapsto f(t) = \frac{1}{(n+t)^2}$ is C^∞ on $[0,+\infty[$, $f^{(k)}(t) = \dfrac{(-1)^k(k+1)!}{(t+n)^{k+2}}$, $|f^{(k)}(t)| \leq \dfrac{(k+1)!}{n^{k+2}}$ and the series $\sum_{n\geq 1}\dfrac{1}{(n)^{k+2}}$ is convergent, then ψ is C^∞ on $[0,+\infty[$ and $\lim_{t\to+\infty}\psi(t) = 0$.

(b) i. As the sequence $((n+t)(n+t+1))_n$ is decreasing, then for any $t \geq 0$ and $n \geq 1$

$$\frac{1}{(n+t)(n+t+1)} \leq \frac{1}{(n+t)^2} \leq \frac{1}{(n+t)(n+t-1)}.$$

ii. $\dfrac{1}{(n+t)(n+t+1)} = \dfrac{1}{n+t} - \dfrac{1}{n+t+1}$ and $\dfrac{1}{(n+t)(n+t-1)} = \dfrac{1}{n+t-1} - \dfrac{1}{n+t}$, then $\dfrac{1}{t+1} \leq \psi(t) \leq \dfrac{1}{t}$, for any $t > 0$, and that $\psi(t) \approx \frac{1}{t}$; $(t \to +\infty)$.

iii. $\dfrac{1}{(n+t)^2} = \dfrac{1}{(n+t)^2(n+t+1)} + (\dfrac{1}{n+t} - \dfrac{1}{n+t+1})$, since $\sum_{n=1}^{+\infty}(\dfrac{1}{n+t} - \dfrac{1}{n+t+1}) = \dfrac{1}{1+t}$, then

$$\psi(t) - \frac{1}{t+1} = \sum_{n=1}^{+\infty}\frac{1}{(n+t)^2(n+t+1)}.$$

(c) i. Since the function $x \longmapsto \dfrac{1}{(x+t)^3}$ is decreasing,

$\dfrac{1}{(n+t+1)^3} \leq \displaystyle\int_n^{n+1}\frac{dx}{(t+x)^3} \leq \dfrac{1}{(n+t)^3}$. Then for any $t > 0$

$$
\begin{aligned}
\frac{2}{(2+t)^2} &= \int_2^{+\infty} \frac{dx}{(t+x)^3} \leq \sum_{n=1}^{+\infty} \frac{1}{(n+t+1)^3} \\
&\leq \psi(t) - \frac{1}{t+1} \leq \sum_{n=1}^{+\infty} \frac{1}{(n+t)^3} \\
&\leq \int_0^{+\infty} \frac{dx}{(t+x)^3} = \frac{2}{t^2}.
\end{aligned}
$$

Then

$$
\psi(t) - \frac{1}{t+1} = \frac{\eta(t)}{t^2},
$$

ii. Since $\psi(t) - \dfrac{1}{1+t} \geq 0$ and the function $t \longmapsto \frac{1}{t^2}$ is integrable, then $\displaystyle\int_0^{+\infty} (\psi(t) - \frac{1}{1+t})dt$ is convergent.

(d) Since the series $\displaystyle\sum_{n\geq 1} \frac{1}{(n+t)^2(n+t+1)}$ converges uniformly on $[0, +\infty[$,

$$
\int_0^{+\infty} (\psi(t) - \frac{1}{t+1})dt = \sum_{n=1}^{+\infty} (\frac{1}{n} - \ln(\frac{n+1}{n})).
$$

4.5 Solutions of Exercises on Chapter 2

2-1-1 Consider the sequence $(v_n)_{n\geq 1}$, with $v_n = nu_n$. Remark that $\sum_{n\geq 1} u_n = \sum_{n\geq 1} \frac{1}{n} v_n$, the sequence $(\frac{1}{n})_n$ is decreasing and the series $\sum_{n\geq 1} v_n$ is convergent, then by the Abel's Theorem 1.13 the series $\sum_{n\geq 1} u_n$ is convergent.

2-1-2

1) Since the series $\sum_{n\geq 1} u_n$ is convergent and the sequence $(u_n)_{n\geq 1}$ is decreasing, then u_n is non negative and $\lim_{n\to+\infty} u_n = 0$. Moreover $\sum_{k=n+1}^{2n} u_k \geq nu_{2n}$. By Cauchy criterion (1.2), $\lim_{n\to+\infty} \sum_{k=n+1}^{2n} u_k = 0$, then $\lim_{n\to+\infty} nu_{2n} = 0$. Moreover $nu_{2n+1} \leq nu_{2n}$, then $\lim_{n\to+\infty} nu_{2n+1} = 0$.

2) Since $\lim_{n\to+\infty} nu_n = 0$, the series $\sum_{n\geq 1}(nu_n - (n+1)u_{n+1})$ converges. Moreover $n(u_n - u_{n+1}) = (nu_n - (n+1)u_{n+1}) + u_{n+1}$, then the series $\sum_{n\geq 1} n(u_n - u_{n+1})$ is convergent and

$$\sum_{n=1}^{+\infty} n(u_n - u_{n+1}) = \sum_{n=1}^{+\infty}(nu_n - (n+1)u_{n+1}) + \sum_{n=1}^{+\infty} u_{n+1} = u_1 + \sum_{n=1}^{+\infty} u_{n+1} = \sum_{n=1}^{+\infty} u_n.$$

3) $\sum_{n=1}^{+\infty} r^n = \frac{r}{1-r}$. The sequence $(r^n)_n$ is decreasing.

$$\sum_{n=1}^{+\infty} n(r^n - r^{n+1}) = \sum_{n=1}^{+\infty} nr^n(1-r) = \frac{r}{1-r}.$$

Then $\sum_{n=1}^{+\infty} nr^n = \frac{r}{(1-r)^2}$.

$$\sum_{n=1}^{+\infty} n(nr^n - (n+1)r^{n+1}) = \sum_{n=1}^{+\infty} n^2 r^n(1-r) - nr^{n+1} = \frac{r}{(1-r)^2}.$$

Then $\sum_{n=1}^{+\infty} n^2 r^n = \frac{r}{(1-r)^3} + \frac{r^2}{(1-r)^3} = \frac{r(1+r)}{(1-r)^3}$.

2-1-3 (a) The sequence $(\frac{1}{n+1})_n$ is decreasing to 0. Then the series $\displaystyle\sum_{n\geq 0}\frac{(-1)^n}{n+1}$ is convergent.

(b) For $t \geq 0$,

$$\sum_{k=0}^{n}(1)^k t^k - \frac{1}{1+t} = \frac{(-t)^{n+1}}{1+t}.$$

We integrate this identity on the interval $[0,1]$, we get

$$\sum_{k=0}^{n}\frac{(1)^k}{k+1} - \int_0^1 \frac{dt}{1+t} = \int_0^1 \frac{(-t)^{n+1}dt}{1+t}.$$

$\displaystyle\left|\int_0^1 \frac{(-t)^{n+1}dt}{1+t}\right| \leq \int_0^1 t^{n+1}dt = \frac{1}{n+2}$. Then

$$\left|\sum_{k=0}^{n}\frac{(-1)^k}{k+1} - \int_0^1 \frac{dt}{1+t}\right| \leq \frac{1}{n+2}.$$

(c) Since $\displaystyle\int_0^1 \frac{dt}{1+t} = \ln 2$, then $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = \ln 2$.

2-1-4 1) $\displaystyle\sum_{n=}^{+\infty}\frac{1}{\sqrt{n-1}} - \frac{2}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}}.$

2) $\displaystyle\frac{1}{n^2-1} = \frac{1}{2}.\frac{1}{n-1} - \frac{1}{2}.\frac{1}{n+1}$, then

$$\sum_{n=2}^{+\infty}\frac{1}{n^2-1} = \frac{1}{2}.$$

3) $\displaystyle\frac{1}{n(n+1)(n+2)} = \frac{1}{2}.\frac{1}{n} - \frac{1}{n+1} + \frac{1}{2}.\frac{1}{n+2}$, then

$$\sum_{n=1}^{+\infty}\frac{1}{n(n+1)(n+2)} = \frac{1}{4}.$$

4) $\displaystyle\sum_{n=1}^{+\infty}\frac{n^2}{n!} = \sum_{n=0}^{+\infty}\frac{n+1}{n!} = \sum_{n=1}^{+\infty}\frac{1}{(n-1)!} + e = 2e.$

5) $\displaystyle\sum_{n=0}^{+\infty} \frac{2n^3+1}{n!} = e+2\sum_{n=0}^{+\infty} \frac{(n+1)^2}{n!} = e+2\sum_{n=0}^{+\infty} \frac{n^2+2n+1}{n!} = e+4e+$ $4e+2e = 11e.$

6) $\displaystyle\sum_{n=2}^{m} \ln(1-\frac{1}{n^2}) = \ln \prod_{n=2}^{m}(\frac{n-1}{n}.\frac{n+1}{n}) = \ln \frac{m+1}{2m}.$

Then $\displaystyle\sum_{n=2}^{m} \ln(1-\frac{1}{n^2}) = \ln \frac{1}{2}.$

7)

$$\begin{aligned} 2^m \sin \frac{1}{2^m} \prod_{n=1}^{m} \cos \frac{1}{2^n} &= 2^m \sin \frac{1}{2^m} \cos \frac{1}{2^m} \prod_{n=1}^{m-1} \cos \frac{1}{2^n} \\ &= 2^{m-1} \sin \frac{1}{2^{m-1}} \prod_{n=1}^{m-1} \cos \frac{1}{2^n} = \sin 1. \end{aligned}$$

$\displaystyle\sum_{n=1}^{+\infty} \ln\cos \frac{1}{2^n} = \ln \sin 1.$

8) $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^x (\ln t)^n dt = \int_0^x t dt = \frac{x^2}{2}.$ (We use the Dominate Convergence Theorem $\displaystyle|\sum_{n=0}^{m} \frac{\ln^n t}{n!}| \leq \sum_{n=0}^{m} \frac{|\ln^n t|}{n!} \leq e^{|\ln t|}$.)

2-1-5

1) If $a_n = \dfrac{2^n n!}{n^n}$, $\dfrac{a_{n+1}}{a_n} = 2\left(\dfrac{n}{n+1}\right)^n = 2\left(1-\dfrac{1}{n+1}\right)^n \underset{n\to+\infty}{\longrightarrow} \dfrac{2}{e} < 1.$

Then the series $\displaystyle\sum_{n\geq 1} \frac{2^n n!}{n^n}$ is convergent.

---

2) If $a_n = \dfrac{3^n n!}{n^n}$, $\dfrac{a_{n+1}}{a_n} = 3\left(\dfrac{n}{n+1}\right)^n = 3\left(1-\dfrac{1}{n+1}\right)^n \underset{n\to+\infty}{\longrightarrow} \dfrac{3}{e} > 1.$

Then the series $\displaystyle\sum_{n\geq 1} \frac{3^n n!}{n^n}$ is divergent.

---

3) $u_n = \dfrac{n!}{n^n}$, $\displaystyle\lim_{n\to+\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to+\infty}(1+\frac{1}{n})^n = e > 1.$ Then $\displaystyle\sum_{n\geq 1} u_n$ converges.

---

4) The sequence $(\frac{\ln n}{n})_n$ is decreasing for $n \geq 3$ and its limit is 0, then the alternate series $\displaystyle\sum_{n\geq 2} (-1)^n \frac{\ln n}{n}$ is convergent.

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5) For $|x|<1$, $\ln(1+x)=x-\frac{x^2}{2(1+c(x))^2}$, where $|c(x)|\leq|x|$. Moreover for $|x|\leq\frac{1}{2}$, $\dfrac{2}{9}\leq\dfrac{1}{2(1+c(x))^2}\leq 2$.

The series $\sum_{n\geq 2}\dfrac{(-1)^n}{n^\alpha}$ is convergent and the series $\sum_{n\geq 2}\dfrac{1}{n^{2\alpha}}$ is convergent if and only if $\alpha>\frac{1}{2}$. Then $\sum_{n\geq 2}u_n$ is convergent if and only if $\alpha>\frac{1}{2}$, where $u_n=\ln\left(1+\frac{(-1)^n}{n^\alpha}\right)$.

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6) $u_n=\left(\frac{n}{n+1}\right)^{n^2}$, $\lim\limits_{n\to+\infty}u_n^{\frac{1}{n}}=\lim\limits_{n\to+\infty}(\dfrac{n}{n+1})^n=(1-\dfrac{1}{n})^n=\dfrac{1}{e}<1$, then the series $\sum_{n\geq 2}u_n$ is convergent.

---

7) $$\frac{\cos n}{\sqrt{n}+\cos n}-\frac{\cos n}{\sqrt{n}}=\frac{-\cos^2 n}{\sqrt{n}(\sqrt{n}+\cos n)}.$$

The sequence $(\frac{1}{\sqrt{n}})_n$ is decreasing to 0 and $|\sum\limits_{n=p}^{q}\cos n|\leq\dfrac{1}{\sin\frac{1}{2}}$. Then the series $\sum\limits_{n\geq 1}\dfrac{\cos n}{\sqrt{n}}$ is convergent.

$\dfrac{\cos^2 n}{\sqrt{n}(\sqrt{n}+\cos n)}=\dfrac{1}{2\sqrt{n}(\sqrt{n}+\cos n)}+\dfrac{\cos 2n}{2\sqrt{n}(\sqrt{n}+\cos n)}$. Then the series $\sum\limits_{n\geq 1}\dfrac{\cos n}{\sqrt{n}+\cos n}$ is divergent.

---

8) $u_n=\frac{1}{C_{2n}^n}=\frac{(n!)^2}{2n!}$, then $\lim\limits_{n\to+\infty}\dfrac{u_{n+1}}{u_n}=\lim\limits_{n\to+\infty}\dfrac{(n+1)^2}{(2n+1)(2n+2)}=\dfrac{1}{4}<1$, then the series $\sum\limits_{n\geq 1}u_n$ is convergent.

---

9) $u_n=\frac{(2n)!}{n^n(n-1)!}$, $\lim\limits_{n\to+\infty}\dfrac{u_{n+1}}{u_n}=\lim\limits_{n\to+\infty}\dfrac{2(2n+1)}{n}(\dfrac{n}{n+1})^n=\dfrac{4}{e}$, then the series $\sum\limits_{n\geq 1}u_n$ is divergent.

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10) $u_n=n\sin(\dfrac{1}{n})$, $\lim\limits_{n\to+\infty}u_n=1$, then the series $\sum\limits_{n\geq 1}u_n$ is divergent.

---

11) $u_n=e-(1+\frac{1}{n})^n=e-e^{n\ln(1+\frac{1}{n})}\geq e-e^{n(\frac{1}{n}-\frac{1}{4n^2})}=e-e^{1-\frac{1}{4n}}$, for $n\geq 1$. It follows that $u_n\geq e(1-e^{\frac{-1}{4n}})\geq e(1-(1-\frac{1}{4ne}))$. (It suffices to prove that $e^{-x}\leq 1-\frac{x}{e}$ for $x\leq 1$.) It follows that $u_n\geq\frac{1}{4n}$ and the series $\sum\limits_{n\geq 1}u_n$ is divergent.

---

12)

$$\begin{aligned}
u_n &= \cosh^\alpha n - \sinh^\alpha n = (\frac{e^n + e^{-n}}{2})^\alpha - (\frac{e^n - e^{-n}}{2})^\alpha \\
&= \frac{e^{n\alpha}}{2^\alpha}\left[(1+e^{-2n})^\alpha - (1-e^{-2n})^\alpha\right] \\
&= \frac{e^{n\alpha}}{2^\alpha}\left[\alpha e^{-2n} + \alpha e^{-2n} + e^{-2n} g(n)\right], \quad g(n) \underset{n\to+\infty}{\longrightarrow} 0 \\
&= \frac{e^{n(\alpha-2)}}{2^\alpha}\left[2\alpha + g(n)\right]
\end{aligned}$$

where $\lim_{n\to+\infty} g(n) = 0$. Then the series $\sum_{n\geq 1} u_n$ converges if and only if $\alpha < 2$.

---

13) $u_n = \cos^{-1}\left(\frac{n^3+1}{n^3+2}\right) = \cos^{-1}\left(1 - \frac{1}{n^3+2}\right) \sim \frac{1}{n^3+2}$, then the series $\sum_{n\geq 1} u_n$ converges.

---

14) $u_n = \ln\frac{(n^3+1)^2}{(n^2+1)^3} = \ln\frac{(1+n^{-3})^2}{(1+n^{-2})^3} = 2\ln(1+n^{-3}) - 3\ln(1+n^{-2}) = \frac{2}{n^3} - \frac{3}{n^2} + \frac{1}{n^2}g(n)$, where $g(n)$ bounded. Then the series $\sum_{n\geq 1} u_n$ converges.

---

15) $u_n = (\frac{1}{2})^{\sqrt{n}}$, The sequence $(u_n)_n$= is decreasing. Let $f(x) = \frac{1}{2\sqrt{x}}$.

$$\int_1^{+\infty} \frac{dx}{2\sqrt{x}} \overset{t=\sqrt{x}}{=} \int_1^{+\infty} 2t2^{-t}dt.$$

This integral converges, then the series $\sum_{n\geq 1} u_n$ converges.

---

16) $u_n = \sqrt{1 + \frac{(-1)^n}{\sqrt{n}}} - 1.$ $\sqrt{1 + \sqrt{1 + \frac{(-1)^n}{\sqrt{n}}}} = 1 + \frac{1}{2}.\sqrt{1 + \frac{(-1)^n}{\sqrt{n}}} - \frac{1}{8}\frac{1}{n} + \frac{1}{n^{\frac{3}{2}}}g(n)$, where $g(n)$ bounded. Then the series $\sum_{n\geq 1} u_n$ diverges.

---

17) $u_n = \frac{(\ln n)^n}{n^{\ln n}}$. $\lim_{n\to+\infty} u_n^{\frac{1}{n}} = \lim_{n\to+\infty} \frac{\ln n}{e^{\frac{\ln^2 n}{n}}} = +\infty$, the series $\sum_{n\geq 2} u_n$ diverges.

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18) $u_n = \frac{1}{(\ln n)^{\ln n}} = \frac{1}{e^{(\ln(\ln n))\ln n}} \leq \frac{1}{e^{2\ln n}} = \frac{1}{n^2}$, then the series $\sum_{n\geq 1} u_n$ converges.

---

19) $u_n = \sin\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{6n^3} + \frac{1}{n^4}g(n) - \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{n^2}h(n)$, where $\lim_{n\to+\infty} g(n) = \lim_{n\to+\infty} h(n) = 0$. Then the series $\sum_{n\geq 1} u_n$ converges.

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20) $u_n = \frac{(-1)^n}{n^\alpha + (-1)^n}$. $u_n - \frac{(-1)^n}{n^\alpha} = -\frac{1}{n^\alpha(n^\alpha + (-1)^n)}$. The series $\sum_{n\geq 2} \frac{(-1)^n}{n^\alpha}$ is convergent for $\alpha > 0$. Then the series $\sum_{n\geq 1} u_n$ converges if and only if $\alpha > \frac{1}{2}$.

---

21) $u_n = \frac{1}{n\ln n\big(\ln(\ln n)\big)^\alpha}$. We use the comparison with integral. Let $f(x) = \frac{1}{x\ln x\big(\ln(\ln x)\big)^\alpha}$.

$$\begin{aligned}
\int_4^{+\infty} f(x)dx &\overset{\ln x = t}{=} \int_{\ln 4}^{+\infty} \frac{dt}{t\ln^\alpha t} \\
&\overset{\ln t = u}{=} \int_{\ln\ln 4}^{+\infty} \frac{du}{u^\alpha}.
\end{aligned}$$

Then the series $\sum_{n\geq 1} u_n$ converges if and only if $\alpha > 1$.

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22) $\cos\frac{1}{\sqrt{n}} = 1 - \frac{1}{2n} + \frac{1}{4n^2} + \frac{g(n)}{n^3}$, where $\lim_{n\to+\infty} g(n) = 0$.

$\ln\cos\frac{1}{\sqrt{n}} = \ln\left(1 - \frac{1}{2n} + \frac{1}{4n^2} + \frac{g(n)}{n^3}\right) = -\frac{1}{2n} + \frac{1}{8n^2} + \frac{h(n)}{n^3}$, where $\lim_{n\to+\infty} g(n) = 0$.

$$\begin{aligned}
u_n &= \left(\cos\frac{1}{\sqrt{n}}\right)^n - \frac{1}{\sqrt{e}} \\
&= e^{-\frac{1}{2}+\frac{1}{8n}+\frac{h(n)}{n^2}} - e^{-\frac{1}{2}} \\
&= e^{-\frac{1}{2}}\left(e^{\frac{1}{8n}+\frac{h(n)}{n^2}} - 1\right) \sim \frac{e^{-\frac{1}{2}}}{8n}.
\end{aligned}$$

Then the series $\sum_{n\geq 2} u_n$ diverges.

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23) $u_n = \ln\frac{1}{\sqrt{n}} - \ln\left(\sin\frac{1}{\sqrt{n}}\right)$

$$\sin\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{n}} - \frac{1}{6n\sqrt{n}} + \frac{g(n)}{n^2},$$

where $\lim_{n\to+\infty} g(n) = 0$.

$$\begin{aligned} u_n &= \ln\frac{1}{\sqrt{n}} - \ln\left(\sin\frac{1}{\sqrt{n}}\right) \\ &= \ln\frac{1}{\sqrt{n}} - \ln\left(\frac{1}{\sqrt{n}} - \frac{1}{6n\sqrt{n}} + \frac{g(n)}{n^2}\right) \\ &= -\ln\left(1 - \frac{1}{6n} + \frac{g(n)}{n^{\frac{3}{2}}}\right) \sim \frac{1}{6n}. \end{aligned}$$

Then the series $\sum_{n\geq 1} u_n$ diverges.

$\boxed{\text{2-1-6}}$ (a) In a neighborhood of 0,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + x^5 g(x), \qquad (7.11)$$

where $\lim_{x\to 0} g(x) = 0$.

$$\begin{aligned} a\ln n + b\ln(n+1) + c\ln(n-1) &= \ln n\left(a+b+c+b\ln(1+\frac{1}{n}) + c\ln(1-\frac{1}{n})\right) \\ &= \ln n\left(a+b+c+\frac{b-c}{n} + \frac{h(n)}{n^2}\right), \end{aligned}$$

where $(h(n))_n$ bounded. Then the necessary condition of the convergent of the series $\sum_{n\geq 2} u_n$ is $b = c$ and $a+b+c = 0$.

(b) The series $\sum_{n\geq 2} u_n$ is absolutely convergent.

(c) If $a = -2$, $b = c = 1$, the conditions of the convergence of the series $\sum_{n\geq 2} u_n$ are satisfied. Its sum is $-\ln 2$.

$\boxed{\text{2-1-7}}$ (a) In use of (7.11), we have

$$\begin{aligned}
u_n = S_n - S_{n-1} &= \ln \frac{f(n)}{f(n-1)} \\
&= (n-1)\ln(1-\frac{1}{n}) + 1 - \frac{1}{2}\ln(1+\frac{1}{n}) \\
&= (n-1)(-\frac{1}{n} - \frac{1}{2n^2} + \frac{g(n)}{n^3}) + 1 - \frac{1}{2}(\frac{1}{n} - \frac{1}{2n^2} + \frac{g(n)}{n^3}) \\
&= \frac{3}{4n^2} + \frac{g(n)}{n^2},
\end{aligned}$$

where $(g(n))_n$ bounded. Then the series $\sum_{n\geq 2} u_n$ is convergent.

(b) $\sum_{n=2}^{m} u_n = S_m - S_1$. Then the sequence $(S_n)_n$ is convergent.

(c) $n! \underset{n\to+\infty}{\sim} e^{\ell} n^n e^{-n} \sqrt{n}$.

2-1-8 (a) Define the function $f(x) = 1 - e^{-x} - x$, $f'(x) = e^{-x} - 1$. f is increasing on the interval $]-\infty, 0]$ and decreasing on the interval $[0, +\infty[$ and $f(x) \leq 0$. We deduce that if the sequence $(u_n)_n$ is convergent its limit is 0.

- If $u_0 < 0$, the sequence $(u_n)_n$ is decreasing, then it diverges.
- If $u_0 > 0$, the sequence $(u_n)_n$ is also decreasing, then it converges because it is non negative.

(b) Assume $u_0 > 0$, then $\lim_{n\to+\infty} u_n = 0$. $e^{-x} = 1 - x + \frac{x^2}{2} + x^2\varepsilon(x)$, where $\lim_{x\to 0}\varepsilon(x) = 0$. Then $\lim_{n\to+\infty} \frac{u_{n+1} - u_n}{u_n^2} = -\frac{1}{2}$.

Since $\lim_{n\to+\infty} \frac{u_n - u_{n+1}}{u_n^2} = \frac{1}{2}$, then the convergence of the series $\sum_{n\geq 0} u_n^2$ is equivalent to the convergence of the series $\sum_{n\geq 0} u_n - u_{n+1}$, which is equivalent to the convergence to the sequence $(u_n)_n$. And since the sequence $(u_n)_n$ is convergent, then the series $\sum_{n\geq 0} u_n^2$ is convergent.

2-1-9 $\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n} = \frac{1}{n}(1 + (-1)^n\sqrt{n})$. For n even, $1 + (-1)^n\sqrt{n} \geq 0$ and for n odd, $1 + (-1)^n\sqrt{n} \leq 0$.

Then the series $\sum_{n\geq 1} u_n$ is alternate. The series $\sum_{n\geq 1} \frac{(-1)^n}{\sqrt{n}}$ is convergent, but the series $\sum_{n\geq 1} \frac{1}{n}$ is divergent. Then the series $\sum_{n\geq 1} u_n$ is divergent.

The sequence $(\frac{1}{\sqrt{n}} + \frac{(-1)^n}{n})_n$ is not decreasing.

$\boxed{\text{2-1-10}}$ (a) $\int_1^{+\infty} f(t)dt = \sum_{n=1}^{+\infty} \int_n^{n+1} f(t)dt$. But $\int_n^{n+1} f(t)dt = \int_{2n\pi}^{2(n+1)\pi} |\sin x|dx =$ 4. then the integral $\int_1^{+\infty} f(t)dt$ diverges. Since $f(n) = 0$, then the series $\sum_{n\geq 1} f(n)$ converges.

(b) Consider the function

$$g(x) = \begin{cases} n^2x + 1 - n^3 & \text{for} \quad x \in \left[n - \frac{1}{n^2}, n\right] \quad (n \geq 2) \\ -n^2x + 1 + n^3 & \text{for} \quad x \in \left[n, n + \frac{1}{n^2}\right] \quad (n \geq 2) \\ 0 & \text{for} \quad x \text{ does not in any of these intervals} \end{cases}.$$

Since $g(n) = 1$, then the series $\sum_{n\geq 1} g(n)$ diverges.

$\int_{n-\frac{1}{n^2}}^{n} g(t)dt = \frac{1}{2n^2}$ and $\int_n^{n+\frac{1}{n^2}} g(t)dt = \frac{1}{2n^2}$. Then the integral $\int_0^{+\infty} g(t)dt$ converges.

The function g is not decreasing.

$\boxed{\text{2-1-11}}$ Recall the Taylor formula with integral remainder

Let $f\colon I \longrightarrow \mathbb{R}$ be a function of class $\mathcal{C}^n$, then for $a, x \in I$:

$$f(x) = f(a) + \sum_{p=1}^{n-1} \frac{(x-a)^p}{p!} f^{(p)}(a) + \int_a^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt.$$

For $n < m \in \mathbb{N}$,

$f(n) = f(x) - \int_n^x f'(t)dt$. We integrate this relation on the interval $[n, n+1]$, we find $f(n) = \int_n^{n+1} f(x)dx - \int_n^{n+1} (\int_n^x f'(t)dt)dx$. By Fubini Theorem we get

$$f(n) = \int_n^{n+1} f(x)dx - \int_n^{n+1} (n+1-t)f'(t)dt,$$

Then

$$\left|\sum_{k=n}^{m} f(k)\right| \leq \left|\int_n^{m+1} f(x)dx\right| + \int_n^{m+1} |f'(t)|dt.$$

and the series $\sum_{n\geq 0} f(n)$ converges.

$$\begin{aligned}
\int_1^{+\infty} f(x)dx &= 2\int_1^{+\infty} \frac{\sin(\pi x)}{x}dx \\
&\overset{\text{by parts}}{=} -\frac{2}{\pi} - \frac{2}{\pi}\int_1^{+\infty} \frac{\cos(\pi x)}{x^2}dx.
\end{aligned}$$

Then $\int_1^{+\infty} f(x)dx$ is convergent.

$\int_1^{+\infty} |f'(x)|dx = \int_1^{+\infty} \left|\frac{\pi\cos(\pi\sqrt{x})}{2x\sqrt{x}} - \frac{\sin(\pi\sqrt{x})}{x^2}\right| dx$. It is evident that this integral is convergent. Which following that the series $\sum_{n=1}^{+\infty} \frac{\sin(\pi\sqrt{n})}{n}$ is convergent.

2-1-12 (a)

$$\begin{aligned}
\sin((2m+1)\theta) &= \operatorname{Im}(e^{\mathrm{i}\theta})^{2m+1} = \operatorname{Im}(\cos(\theta) + \mathrm{i}\sin(\theta))^{2m+1} \\
&= \sum_{k=0}^{m} (-1)^k C_{2m+1}^{2k+1} \cos^{2(m-k)}(\theta)\sin^{2k+1}(\theta) \\
&= \sin^{2m+1}(\theta)\sum_{k=0}^{m} (-1)^k C_{2m+1}^{2k+1} \cot^{2(m-k)}(\theta).
\end{aligned}$$

(b) Make the substitution $x = \cot^2(\theta)$, the roots of the polynomial P_m are given by: $\cot^2 \frac{k\pi}{2m+1}$, $k = 1, \ldots, m$. The sum of the roots is $\frac{C_{2m+1}3}{C_{2m+1}1} = \frac{m(2m-1)}{3}$. Then

$$\sum_{k=1}^{m} \cot^2\left(\frac{k\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

(c) We know that for $t \geq 0$, $|\sin t| \leq t$, then $1+\cot^2 t = \dfrac{1}{\sin^2 t} \geq \dfrac{1}{t^2}$. Moreover if $f(t) = \cot t - \dfrac{1}{t}$, $f'(t) = -\dfrac{1}{\sin^2 t} + \dfrac{1}{t^2} \leq 0$ and $\lim\limits_{t\to 0^+} f(t) = 0$, then $f(t) \leq 0$. Then $\forall t \in \left]0, \dfrac{\pi}{2}\right[\quad \cot^2 t \leq \dfrac{1}{t^2} \leq \cot^2 t + 1$.

(d) Apply this result for $t = \dfrac{k\pi}{2m+1}$, we have

$$\sum_{k=1}^{m} \left(\frac{2m+1}{k\pi}\right)^2 - m \leq \frac{m(2m-1)}{3} \leq \sum_{k=1}^{m} \left(\frac{2m+1}{k\pi}\right)^2,$$

Then

$$\frac{3(2m+1)^2}{\pi^2 m(2m-1)} \sum_{k=1}^{m} \frac{1}{k^2} - \frac{3}{(2m-1)\pi^2} \leq 1 \leq \frac{3(2m+1)^2}{\pi^2 m(2m-1)} \sum_{k=1}^{m} \frac{1}{k^2},$$

which yields

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

2-1-13 (a) There exists $N \in \mathbb{N}$ such that for $n \geq N$ $u_n \leq 1$, then $u_n^2 \leq u_n$ for all $n \geq N$, which yields that the series $\sum\limits_{n\geq 0} u_n^2$ converges. In the same way, $\sum\limits_{n\geq 0} v_n^2$ converges.

$u_n v_n \leq u_n^2 + v_n^2$, then the series $\sum\limits_{n\geq 0} \sqrt{u_n v_n}$ converges. (Cauchy Schwarz).

(b) If $\ell > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $nw_n \geq \frac{\ell}{2}$ and then $w_n \geq \frac{\ell}{2n}$ and the series $\sum\limits_{n\geq 0} w_n$ is divergent, which is absurd. Then $\ell = 0$.

2-1-14 (a) $u_{n+1} - u_n = -u_n^2 \leq 0$, then the sequence $(u_n)_n$ is decreasing.

(b) If $x \in]0,1[$, $0 \leq x - x^2 \leq 1$, then by induction $u_n \in]0,1[$.

(c) $(u_n)_n$ is decreasing and positive, then it converges. If ℓ is its limit, we find $\ell = \ell - \ell^2$. Then $\ell = 0$.

(d) $\sum\limits_{n=0}^{m} u_n^2 = \sum\limits_{n=0}^{m} u_n - u_{n+1} = u_0 - u_{m+1}$. Then the series $\sum\limits_{n\geq 0} u_n^2$ converges and its sum is u_0.

(e) $\displaystyle\sum_{n=0}^{m} \ln(\frac{u_{n+1}}{u_n}) = \ln \prod_{n=0}^{m} \frac{u_{n+1}}{u_n} = \ln \frac{u_{m+1}}{u_0}$. Then the series $\displaystyle\sum_{n\geq 0} \ln(\frac{u_{n+1}}{u_n})$ diverges and its sum is $-\infty$. Moreover $\ln(\frac{u_{n+1}}{u_n}) = \ln(1-u_n)$ and $\ln(1-u_n) \sim u_n$. Then the series $\displaystyle\sum_{n\geq 0} u_n$ diverges.

(f) i. $v_n = \dfrac{u_{n-1}-u_n}{u_n u_{n-1}} = \dfrac{1}{1-u_{n-1}}$. Since $\lim\limits_{+\infty} u_n = 0$, then $\lim\limits_{+\infty} v_n = 1$.

ii. For $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for $n \geq N$, $1-\varepsilon \leq v_n \leq 1+\varepsilon$, then for $n \geq N$,

$$(n-N)(1-\varepsilon) \leq \frac{1}{u_n} - \frac{1}{u_N} \leq (n-N)(1+\varepsilon),$$

which yields that $u_n \approx \frac{1}{n}$.

iii. Since $u_n^2 \in]0,1[$, the series $\displaystyle\sum_{n\geq 1} \sin(u_n^2)$ and the series $\displaystyle\sum_{n\geq 1} u_n^2$ converge or diverge together. Then the series $\displaystyle\sum_{n\geq 1} \sin(u_n^2)$ is convergent. Moreover since $u_n \approx \frac{1}{n}$, then the series $\displaystyle\sum_{n\geq 1} \frac{u_n}{\sqrt{n}}$ is convergent.

2-1-15 (a) i. For $|x| < 1$, $\ln(1+x) = xh(x)$, with $h \in C^\infty(]-1,1[)$ and $h(0) = 1$. In particular $|h(x)| \leq M$ for a suitable number M for $|x| \leq \frac{1}{2}$.

If the series $\displaystyle\sum_{n\geq 0} u_n$ is absolutely convergent, there exists $N \in \mathbb{N}$ such that for $n \geq N$; $|u_n| \leq \frac{1}{2}$, then $|\ln(1+u_n)| \leq M|u_n|$ and the series $\displaystyle\sum_{n\geq 0} \ln(1+u_n)$ is absolutely convergent.

In the other hand if the series $\displaystyle\sum_{n\geq 0} \ln(1+u_n)$ is absolutely convergent, then $\lim\limits_{n\to+\infty} u_n = 0$. Moreover $|x| \leq 2|\ln(1+x)|$, for $|x| \leq 1$. Then the series $\displaystyle\sum_{n\geq 0} |u_n|$ is convergent.

ii. Let $u_n = \dfrac{(-1)^n}{\sqrt{n}}$. The series $\displaystyle\sum_{n\geq 0} u_n$ is convergent, but the series $\displaystyle\sum_{n\geq 0} \ln(1+u_n)$ is not convergent.

iii. A. Assume that the series $\sum_{n\geq 0} u_n$ is absolutely convergent, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $|u_n| \leq \frac{1}{2}$, then $u_n^2 \leq |u_n|$ and $|\frac{u_n}{1+u_n}| \leq 2|u_n|$. Then the series $\sum_{n\geq 0} u_n^2$ and $\sum_{n\geq 0} \frac{u_n}{1+u_n}$ are absolutely convergent.

B. If $u_n = \frac{(-1)^n}{\sqrt{n}}$. The series $\sum_{n\geq 0} u_n$ is convergent but the series $\sum_{n\geq 0} u_n^2$ and $\sum_{n\geq 0} \frac{u_n}{1+u_n}$ are not convergent.

(b) Since $v_n = \frac{u_n}{1+u_n} \leq u_n$, if the series $\sum_{n\geq 0} u_n$ converges, the series $\sum_{n\geq 0} v_n$ is also convergent.

If the series $\sum_{n\geq 0} v_n$ converges, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $v_n \leq \frac{1}{2}$ which yields that $u_n = \frac{v_n}{1-v_n} \leq 2u_n$.

2-1-16 By the Cauchy criterion (1.2), for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n, m \geq N$; $|\sum_{k=n}^{m} u_k| \leq \varepsilon$ and $|\sum_{k=n}^{m} w_k| \leq \varepsilon$. Then

$$-\varepsilon \leq \sum_{k=n}^{m} u_k \leq \sum_{k=n}^{m} v_k \leq \sum_{k=n}^{m} w_k \leq \varepsilon.$$

Then the series $\sum_{n\geq 0} v_n$ converges.

2-1-17 (a) $u_n = \int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(x^2)dx = \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{2\sqrt{t}}dt$. It is well known that $\sin x \geq 0$, for $x \in [2n\pi, (2n+1)\pi]$ and $\sin x \leq 0$, for $x \in [(2n-1)\pi, 2n\pi]$. Then the series $\sum_{n\geq 1} u_n$ is an alternate series.

(b) From which previous $\sin x = (-1)^n |\sin x|$, for $x \in [n\pi, (n+1)\pi]$, then $|u_n| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin t|}{2\sqrt{t}}dt$.

$$|u_n| = \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{\sqrt{x}}dx = u_n = \int_0^{\pi} \frac{\sin x}{\sqrt{n\pi + x}}dx.$$

This proves that the sequence $(|u_n|)_n$ is decreasing, which yields that the series $\sum_{n\geq 1} u_n$ is convergent.

$$\begin{aligned} |u_n| &= \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{\sqrt{x}} dx \\ &\geq \int_{n\pi}^{(n+1)\pi} \frac{\sin^2 x}{\sqrt{x}} dx \\ &\geq \int_{n\pi}^{(n+1)\pi} \frac{1-\cos(2x)}{2\sqrt{x}} dx. \end{aligned}$$

In the same way we prove that the series $\sum_{n\geq 1} \int_{n\pi}^{(n+1)\pi} \frac{\cos(2x)}{\sqrt{x}} dx$ is convergent, but evidently the series $\sum_{n\geq 1} \int_{n\pi}^{(n+1)\pi} \frac{1}{\sqrt{x}} dx$ is divergent. Then the series $\sum_{n\geq 1} u_n$ is conditionally convergent.

2-1-18 $|u_n| \geq \dfrac{1}{n^{\frac{3}{4}}-1}$, the series $\sum_{n\geq 2} u_n$ is not absolutely convergent.

Let $v_n = u_n - \dfrac{(-1)^n}{n^{\frac{3}{4}}} = \dfrac{-(-1)^n \cos n}{n^{\frac{3}{4}}(n^{\frac{3}{4}} + \cos n)}$. The series $\sum_{n\geq 2} \dfrac{(-1)^n}{n^{\frac{3}{4}}}$ is convergent and the series $\sum_{n\geq 2} \dfrac{-(-1)^n \cos n}{n^{\frac{3}{4}}(n^{\frac{3}{4}} + \cos n)}$ is absolutely convergent. Then series $\sum_{n\geq 1} u_n$ convergent.

2-1-19 (a) The sequence $(u_n)_{n\geq 0}$ is increasing and $u_{n+1} = u_n + u_n^2 = u_n(1 + u_n) \geq (1+u_0)u_n \geq (1+u_0)^n u_0$. Then $\lim_{n\to+\infty} u_n = +\infty$.

(b) $v_{n+1} - v_n = 2^{-n-1} \ln \dfrac{u_{n+1}}{u_n^2} = 2^{-n-1} \ln(1 + \dfrac{1}{u_n})$. The sequence $\ln(1 + \dfrac{1}{u_n})$ is decreasing and its limit is 0 and the series $\sum_{n\geq 0} 2^{-n-1}$ is convergent, then the series $\sum_{n\geq 0}(v_{n+1} - v_n)$ is convergent, which yields that the sequence $(v_n)_n$ is convergent.

(c) If $\alpha = \lim_{n\to+\infty} v_n$, then $u_n \approx \alpha^{2^n}$.

2-1-20 (a) Since f is a continuous non negative function, then the integral $\int_0^{+\infty} f(x)dx$ is convergent if and only if the sequence $v_n = \int_0^{a_n} f(x)dx$ is convergent, with $(a_n)_n$ any sequence such that $\lim_{n\to+\infty} a_n = +\infty$.

Then the integral $\int_0^{+\infty} f(x)dx$ is convergent if and only if the series $\sum_{n\geq 0} u_n$ is convergent.

(b) On the interval $[n\pi, (n+1)\pi]$, $\cosh(x) \geq \frac{e^{n\pi}}{2}$, then

$$0 \leq u_n \leq \int_{n\pi}^{(n+1)\pi} \frac{1}{1+\frac{e^{n\pi}}{2}\sin^2 x}dx = \int_0^{\pi} \frac{1}{1+\frac{e^{n\pi}}{2}\sin^2 x}dx = 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{e^{n\pi}}{2}\sin^2 x}dx$$

(c) On the interval $[0, \frac{\pi}{2}]$, $\sin x \geq \frac{2}{\pi}x$.

$u_n \leq 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{e^{n\pi}}{2}\sin^2 x}dx \leq \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{4e^{n\pi}}{2\pi^2}x^2}dx \leq \int_0^{+\infty} \frac{1}{1+\frac{4e^{n\pi}}{2\pi^2}x^2}dx = \frac{\pi^2}{\sqrt{2}}e^{\frac{-n\pi}{2}}$.

This proves that the series $\sum_{n\geq 0} u_n$ is convergent and then the integral $\int_0^{+\infty} f(x)dx$ is convergent.

2-1-21 (a) The sequence $(R_n)_n$ is decreasing,

$$\frac{R_{n-1}^{1-\alpha} - R_n^{1-\alpha}}{1-\alpha} = \int_{R_n}^{R_{n-1}} \frac{dt}{t^\alpha} \geq \frac{a_n}{R_{n-1}^\alpha} = b_n.$$

Since $\lim_{n\to+\infty} R_n^{1-\alpha} = 0$, the series $\sum_{n\geq 1} \frac{R_{n-1}^{1-\alpha} - R_n^{1-\alpha}}{1-\alpha}$ is convergent and then the series $\sum_{n\geq 1} b_n$ is convergent.

(b) Since $\lim_{n\to+\infty} \ln R_n = -\infty$, the series $\sum_{n\geq 1} e_n$ is divergent. Moreover $0 \leq e_n \leq c_n$ and $c_n = \frac{d_n}{1-d_n} \geq d_n$, then the series $\sum_{n\geq 1} c_n$ and $\sum_{n\geq 1} d_n$ diverge.

(c) Define $v_n = \dfrac{1}{R_{n-1}^{\alpha}}$, $\lim\limits_{n \to +\infty} v_n = +\infty$ and the series $\sum\limits_{n \geq 0} u_n v_n$ is convergent.

2-2-1

4.6 Solutions of Exercises on Chapter 3

3-1-1 (a)

$$\tilde{f}(x) = \int_0^1 K(x,y)f(y)dy = (x-1)\int_0^x yf(y)dy + x\int_x^1 (y-1)f(y)dy.$$

and

$$\tilde{f}'(x) = \int_0^x yf(y)dy + \int_x^1 (y-1)f(y)dy.$$

Then $\tilde{f}'' = f$, $\tilde{f}$ is of class $\mathcal{C}^2$ and $\tilde{f}(1) = \tilde{f}(0) = 0$.

(b) Let f, g of E,

$$\begin{aligned}\int_0^1 \tilde{g}(x)f(x)dx &= \int_0^1 \left((x-1)g(x)\int_0^x yf(y)dy + xg(x)\int_x^1 (y-1)f(y)dy\right) d\\ &\overset{\text{by parts}}{=} \int_0^1 \left(xf(x)\int_x^1 (y-1)g(y)dy + (x-1)f(x)\int_0^x yg(y)dy\right) d\\ &= \int_0^1 \tilde{f}(x)g(x)dx.\end{aligned}$$

(For the first integration by parts, we set $u = \displaystyle\int_0^x yf(y)dy$, $v' = (x-1)g(x)$ and $v(x) = -\displaystyle\int_x^1 (y-1)g(y)dy$.)

3-1-2 (a) The integral $\displaystyle\int_1^{+\infty} \frac{t^{-(x+1)}}{\sqrt{t^2-1}}dt$ converges for $x \in I =]-1, +\infty[$.

(b) The function $f(x,t) = \dfrac{t^{-(x+1)}}{\sqrt{t^2-1}}$ is C^∞ on $I\times]1,+\infty[$. For $x \in [a,+\infty[$, with $a > 1$ and $n \geq 0$, $\left|\dfrac{\partial^n f}{\partial x^n}(x,t)\right| \leq |\ln^n t f(a,t)|$ which is integrable. Then F is C^∞ on I.

3-1-3

$$F(x) = \int_0^{+\infty} \frac{1-\cos(tx)}{t^2}e^{-t}dt; \quad x > 0$$

(a) The function $f(x,t) = \dfrac{1-\cos(tx)}{t^2}e^{-t} = \dfrac{2\sin^2(\frac{tx}{2})}{t^2}e^{-t}$ is C^∞ on $]0,+\infty[\times]0,+\infty[$. Moreover $|f(x,t)| \leq \dfrac{x^2}{2}e^{-t}$. Then the integral converges.

(b) From which previous f is continuous. $\left|\frac{\partial f}{\partial x}(x,t)\right| \leq |x|e^{-t}$ and for $n \geq 2$, $\left|\frac{\partial^n f}{\partial x^n}(x,t)\right| \leq t^{n-2}e^{-t}$.

Then F is C^∞ and $F^{(n)}(x) = \int_0^{+\infty} \frac{\partial^n f}{\partial x^n}(x,t)dt$. In particular $F''(x) = \int_0^{+\infty} \cos(xt)e^{-t}dt$ and by integration by parts, $F''(x) = \frac{1}{1+x^2}$.

(c) By integration by parts and since $|f(x,t)| \leq \frac{x^2}{2}e^{-t}$ and $\left|\frac{\partial f}{\partial x}(x,t)\right| \leq |x|e^{-t}$, then $F(x) = x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2)$.

3-1-4 For $x > 0$ define the functions $F(x) = \int_0^{+\infty} \frac{\sin t}{t+x}dt$ and $G(x) = \int_0^{+\infty} \frac{e^{-tx}}{1+t^2}dt$.

(a) Define the functions $f(x,t) = \frac{\sin t}{t+x}$ and $g(x,t) = \frac{e^{-tx}}{1+t^2}$ on $]0,+\infty[\times]0,+\infty[$. f and g are C^∞. The function $t \longmapsto \frac{1}{t+x}$ is decreasing and tends to 0 at ∞, $|\int_a^b \sin t dt| \leq 2$ for all $a,b \in \mathbb{R}$. Then by Abel Criterion F is well defined.

For the function G, $|g(x,t)| \leq \frac{1}{1+t^2}$ which is integrable. Then G is continuous.

Let $a > 0$ and $n \geq 1$, $\left|\frac{\partial^n f}{\partial x^n}(x,t)\right| = \left|\frac{n!\sin t}{(t+x)^{n+1}}\right| \leq \frac{n!}{(t+a)^{n+1}}$, which is integrable for $x \geq a$. The F is C^∞ and $F^{(n)}(x) = n!(-1)^n \int_0^{+\infty} \frac{\sin t}{(t+x)^{n+1}}dt$.

Moreover by integration by parts $F(x) = \frac{1}{x} - \int_0^{+\infty} \frac{\cos t}{(t+x)^2}dt$ and a second integration by parts yields that $F''(x) + F(x) = \frac{1}{x}$.

$\left|\frac{\partial^n g}{\partial x^n}(x,t)\right| = \left|\frac{(-t)^n e^{-xt}}{1+t^2}\right| \leq \frac{(t)^n e^{-at}}{(1+t^2)}$, which is integrable for $x \geq a$.

The G is C^∞ and $G^{(n)}(x) = (-1)^n \int_0^{+\infty} \frac{t^n e^{-xt}}{1+t^2}dt$. In particular $G''(x) = \int_0^{+\infty} \frac{t^2 e^{-xt}}{1+t^2}dt = -G(x) + \frac{1}{x}$.

(b) Since $F(x) = \frac{1}{x} - \int_0^{+\infty} \frac{\cos t}{(t+x)^2} dt$ and $|\frac{\cos t}{(t+x)^2}| \leq \frac{1}{(t+a)^2}$, for $x \geq a > 0$, then $\lim_{x \to +\infty} F(x) = 0$. The same result for the derivative of F since $F^{'}(x) = -\int_0^{+\infty} \frac{\sin t}{(t+x)^2} dt$ and $|\frac{\sin t}{(t+x)^2}| \leq \frac{1}{(t+a)^2}$, for $x \geq a > 0$.

$|g(x,t)| \leq \frac{e^{-ta}}{1+t^2}$ for $x \geq a$ and $\left|\frac{\partial g}{\partial x}(x,t)\right| \leq e^{-at}$, for $x \geq a > 0$. Then $\lim_{x \to +\infty} G(x) = \lim_{x \to +\infty} G'(x) = 0$. Then $F = G$.

(c) By the Monotone Convergence Theorem, $\lim_{x \to 0} G(x) = \frac{\pi}{2}$. Then $\lim_{x \to 0} F(x) = \frac{\pi}{2}$.

$F(x) - \int_0^{+\infty} \frac{\sin t}{t} dt = -x \int_0^{+\infty} \frac{\sin t}{t(t+x)} dt$. Since $|\frac{\sin t}{t(t+x)}| \leq \frac{1}{t(t+a)}$, which is integrable for $x \geq a$, then $\lim_{x \to 0} F(x) = \int_0^{+\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$.

3-1-5 (a) If $|f| \leq M$ and since $|f(t)e^{-xt}| \leq Me^{-xt}$ snd $|tf(t)e^{-xt}| \leq Mte^{-xt}$, which are integrable, the F and G are well defined for $x > 0$.

(b) The function $x \longmapsto f(x,t) = f(t)e^{-xt}$ is C^∞ on $]0, +\infty[$ and $|f(t)e^{-xt}| \leq Me^{-xt}$, then by the Dominate convergence Theorem or the Monotone convergence Theorem $\lim_{x \to +\infty} F(x) = 0$.

(c) $\left|\frac{\partial f}{\partial x}(x,t)\right| = tf(t)e^{-xt} \leq Mte^{-at}$ for all $x \geq a > 0$. Then F is differentiable and $F'(x) = G(x)$.

3-1-6 (a) $\varphi(x) = \int_{-\infty}^{+\infty} f(x-t)\psi(t)dt = \int_{-\infty}^{+\infty} f(t)\psi(x-t)dt$. The function $g(x,t) = f(t)\psi(x-t)$ is continuous on $\mathbb{R} \times \mathbb{R}$ and $|f(t)\psi(x-t)| \leq |\pi f(t)|$ which is integrable, then φ is continuous on $\mathbb{R}$.

(b) For $n \in \mathbb{N}$, $\left|\frac{\partial^n f}{\partial x^n}(x,t)\right| \leq |f(t)\psi^{(n)}(x-t)| \leq C_n|f(t)|$ which is integrable. Then φ is of class C^∞ on $\mathbb{R}$. (We can prove by induction that $\psi^{(n)}(t) = \frac{P_n(t)}{(1+t^2)^n}$, where P_n is a polynomial of degree $\leq n$, and this function is continuous and bounded since its limit at ∞ is 0.)

(c) The function $(x,t) \longmapsto f(x-t)\psi(t)$ is integrable on $\mathbb{R}^2$ the by the Fubini Theorem

$$\begin{aligned}\int_{-\infty}^{+\infty} \varphi(x)dx &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} f(x-t)\psi(t)dt\right)dx \\ &= \int_{-\infty}^{+\infty}\psi(t)\left(\int_{-\infty}^{+\infty} f(x-t)dx\right)dt \\ &= \int_{-\infty}^{+\infty} f(t)dt.\int_{-\infty}^{+\infty}\psi(t)dt.\end{aligned}$$

(d) Let $\tilde{\varphi}(x) = \int_{-\infty}^{+\infty} \frac{\cos(x-t)}{\pi(1+t^2)}dt$.

i. The function $x \ln g(x,t) = \frac{\cos(x-t)}{\pi(1+t^2)}$ is C^∞ on $\mathbb{R}$ and $\left|\frac{\partial^n g}{\partial x^n}(x,t)\right| = \left|\frac{\cos(x-t+n\frac{\pi}{2})}{\pi(1+t^2)}\right| \leq \frac{1}{\pi(1+t^2)}$, which is integrable, then $\tilde{\varphi}$ is C^∞ and $\tilde{\varphi}''(x) = -\tilde{\varphi}$.

ii. By the residue Theorem, $\tilde{\varphi}(0) = \int_{-\infty}^{+\infty} \frac{\cos(t)}{\pi(1+t^2)}dt = \frac{1}{e}$ $\tilde{\varphi} = \frac{\cos x}{e}$.

3-1-7 (a) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$.

i. By integration by parts $I_n = \frac{n-1}{n} I_{n-2}$, for $n \geq 2$. $I_0 = \frac{\pi}{2}$, $I_1 = 1$ and $I_2 = \frac{\pi}{4}$. Then $I_{2p} = \frac{(2p)!}{(2^p p!)^2}\frac{\pi}{2}$ and $I_{2p+1} = \frac{(2^p p!)^2}{(2p+1)!}$.

ii. Since $0 \leq \sin x \leq 1$, the sequence $(I_n)_n$ is decreasing, then $I_n I_{n+1} \leq I_n^2 \leq I_n I_{n-1}$ for all $n \in \mathbb{N}$.

$I_{2n}I_{2n+1} = \frac{\pi}{2(2n+1)}$ and $I_{2n-1}I_{2n} = \frac{\pi}{4n}$, then

$$I_n \sim_{+\infty} \sqrt{\frac{\pi}{2n}}.$$

(b) i. The function $(x,t) \longmapsto \psi(x,t) = \sin^x t$ is C^∞ on $]-1,+\infty[\times]0,\frac{\pi}{2}[$, $\frac{\partial^n \psi}{\partial x^n}(x,t) = \ln^n(\sin t)\psi(x,t)$ and dominated by $|\ln^n(\frac{2t}{\pi})|$ which is integrable. Then f is C^∞ on $]-1,+\infty[$.

ii. f is decreasing, then $f(E(x)+1) \leq f(x) \leq f(E(x))$. By the previous question $f(x) \sim_{+\infty} \sqrt{\frac{\pi}{2x}}$.

iii. By integration by parts, $f(x) = \dfrac{f(x+2)}{x+1} - f(x+2) = \dfrac{f(1)}{x+1} + (f'(1) - f(x+2)) + \dfrac{(x+1)}{2} f^{''}(1) + o(x-1)$.

3-1-8 (a) The function $f(x,t) = \dfrac{1}{\sqrt{(1+t^2)(x^2+t^2)}}$ is C^∞ on $\mathbb{R}^* \times [0,+\infty[$ and $\lim_{t\to+\infty} t^2 f(x,t) = 1$, then F is well defined. Moreover $0 \leq f(x,t) \leq f(a)$ for all $a > 0$, then F is continuous on $]0,+\infty[$.
$\left|\dfrac{\partial f}{\partial x}(x,t)\right| = \dfrac{x}{\sqrt{1+t^2}(x^2+t^2)^{\frac{3}{2}}} \leq \dfrac{b}{\sqrt{1+t^2}(a^2+t^2)^{\frac{3}{2}}}$, for all $x \in [a,b] \subset]0,+\infty[$. since this function in integrable, the F is of class C^1 on $]0,+\infty[$.

(b) By the change of variable $t = xu$, we find $F(x)\dfrac{1}{x}F(\dfrac{1}{x})$.

(c) Since the function $x \longmapsto f(x,t)$ is decreasing, integrable and $\lim_{x\to+\infty} f(x,t) = 0$, then $\lim_{x\to+\infty} F(x) = 0$.

(d) By the Monotone convergence Theorem, $\lim_{x\to 0}\displaystyle\int_0^1 \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}} = \int_0^1 \frac{dt}{t\sqrt{(1+t^2)}} = +\infty$.

(e) i. $F(x) = \displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}} + \int_{\sqrt{x}}^{+\infty} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$. In the second integral, we take the change of variable $t = \dfrac{x}{u}$ and we find $F(x) = \displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{(1+t^2)(x^2+t^2)}}$.

ii. Since $\dfrac{1}{\sqrt{1+t^2}} \sim_0 1$, then $F(x) \sim_0 2\displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{x^2+t^2}}$.

iii. $2\displaystyle\int_0^{\sqrt{x}} \frac{dt}{\sqrt{x^2+t^2}} = 2\ln(1+\sqrt{1+x}) - \ln x$. Then $F(x) \sim_0 -\ln x$ and $F(x) \sim_\infty \dfrac{\ln x}{x}$.

3-1-9 (a) In a neighborhood of 0, $\dfrac{t^x(1-t)}{\ln t} \sim \dfrac{t^x}{\ln t}$ which is integrable if and only if $x > -1$.

(b) The function $x \longmapsto f(x,t) = \frac{t^x(1-t)}{\ln t}$ is C^∞ on $]-1,+\infty[$ and $\dfrac{\partial f}{\partial x}(x,t) = t^x(1-t) \leq t^a$, for all $x > a > -1$. Then f is differentiable on $]-1,+\infty[$ and $f'(x) = \dfrac{1}{(x+1)(x+2)}$ for all $x > -1$.

(c) $\displaystyle\lim_{x\longrightarrow+\infty} \frac{t^x(1-t)}{\ln t} = 0$ for $t \neq 1$ and $\dfrac{t^x(1-t)}{\ln t} \leq \dfrac{1}{\ln t}$ for $t > 1$, which is integrable, then by the Dominate Convergence Theorem, $\displaystyle\lim_{x\longrightarrow+\infty} f(x) = 0$ and $f(x) = \ln(\frac{x+1}{x+2})$.

3-1-10 (a) By Fubini Theorem, $g(x) = \displaystyle\int_0^x f(t)\left(\int_t^x \frac{1}{\sqrt{(x-u)(u-t)}}du\right)dt$.

Then g is well defined.

(b) With the change of variables $u = t\cos^2\varphi + x\sin^2\varphi$,

$$\int_t^x \frac{du}{\sqrt{(x-u)(u-t)}} = \int_0^{\frac{\pi}{2}} 2d\varphi = \pi.$$

(c) From the first and second question $g(x) = \pi\displaystyle\int_0^x f(t)dt$ and $f(x) = \frac{g'(x)}{\pi}$.

3-1-11 (a) The function $f(x,t) = \dfrac{1}{\sqrt{1-x^2\sin^2 t}}$ is C^∞ on $]-1,1[\times[0,\frac{\pi}{2}]$. Then F is well defined on $]-1,1[$. For $x \notin]-1,1[$, $\dfrac{1}{x} \in]-1,1[$. Define $\alpha = |\sin^{-1}(\frac{1}{x})|$, then $f(x,t) = \dfrac{|x|}{\sqrt{\sin^2\alpha - \sin^2 t}}$, which its integral is not defined. (For $x \geq \alpha$, $\sin^2\alpha - \sin^2 t \leq 0$).

(b) The map $x \longmapsto f(x,t)$ is C^∞ on $]-1,1[$ and for $|x| \leq a < 1$, $f(x,t) \leq f(a,t)$, which is integrable, then f is continuous. Moreover, $\left|\dfrac{\partial f}{\partial x}\right| \leq \dfrac{a}{(1-a^2\sin^2 t)^{\frac{3}{2}}}$, which is integrable and $\left|\dfrac{\partial^2 f}{\partial x^2}\right| \leq \dfrac{1}{(1-a^2\sin^2 t)^{\frac{3}{2}}} + \dfrac{3a^2}{(1-a^2\sin^2 t)^{\frac{5}{2}}}$, which is integrable. Then F is of class $\mathcal{C}^2$ on $]-1,1[$, $F'(x) = \displaystyle\int_0^{\pi/2} \frac{x\sin^2 t}{(1-x^2\sin^2 t)^{\frac{3}{2}}}dt$. and $F'' \displaystyle\int_0^{\pi/2} \frac{\sin^2 t(1+2x^2\sin^2 t)}{(1-x^2\sin^2 t)^{\frac{5}{2}}}dt$.

(c) i. With the change of variables $u = x\sin t$, $F(x) = \displaystyle\int_0^x \frac{du}{\sqrt{(1-u^2)(x^2-u^2)}} \geq \int_0^x \frac{du}{1-u^2}$, for , $0 < x < 1$.

ii. $\displaystyle\int_0^x \frac{du}{1-u^2} = \frac{1}{2}\ln\frac{1+x}{1-x}$, then $\lim_{x\to 1^-} F(x) = +\infty$.

4.7 Solutions of Exercises on Chapter 4

4-1-1 (a) The sequence $(f_n(x))_n$ convergence if and only if $x = 0$ or $|1-x| < 1$, which is equivalent that $x \in [0, 2[$.

(b) $\displaystyle\lim_{n\to+\infty} \int_0^1 f_n(x)dx = \lim_{n\to+\infty} \frac{n^2}{(n+1)(n+2)} = 1$. But the sequence $(f_n)_n$ converges to 0 on $[0, 2[$. Then the sequence $(f_n)_n$ is not uniformly convergent on the interval $[0, 2[$.

(c) $\displaystyle\lim_{n\to+\infty} f_n(\frac{1}{n}) = \lim_{n\to+\infty} n(1 - \frac{1}{n})^n = +\infty$, then the sequence (f_n) is not uniformly convergent on the interval $[0, 2[$.

4-1-2 (a) $f_n(0) = 0$, and for $x \neq 0$, $\displaystyle\lim_{n\to+\infty} f_n(x) = 0$. But $f(\frac{1}{n}) = \frac{1}{2}$. Then the convergence is not uniform.

(b) $f_n(x) = \begin{cases} 2n^2x & \text{if} \quad x \in [0, \frac{1}{2n}] \\ 0 & \text{if} \quad x \in [\frac{1}{n}, 1] \\ 2n - 2n^2x & \text{if} \quad x \in [\frac{1}{2n}, \frac{1}{n}] \end{cases}$ on $[0, 1]$,

$f_n(0) = 0$, and for $x \in]0, 1]$, there exists $n \in \mathbb{N}$ such that for $x > \frac{1}{n}$, $f_n(x) = 0$. Then $\displaystyle\lim_{n\to+\infty} f_n(x) = 0$. But $f(\frac{1}{2n}) = n$. Then the convergence is not uniform on $[0, 1]$.

(c) $f_n(x) = \begin{cases} x^2 \sin(\frac{1}{nx}) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$ on $\mathbb{R}$,

$f_n(0) = 0$, and for $x \neq 0$, $\displaystyle\lim_{n\to+\infty} f_n(x) = 0$. But $\lim_{n\to+\infty} f(n) = n^2 \sin(\frac{1}{n^2}) = 1$, then the convergence is not uniform on $\mathbb{R}$.

(d) $f_n(x) = \begin{cases} \frac{\sin(x)}{x} e^{-nx} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$ on $\mathbb{R}_+$,

$f_n(0) = 1$, and for $x > 0$, $\displaystyle\lim_{n\to+\infty} f_n(x) = 0$. The convergence is not uniform on $[0, +\infty[$ because the limit is not continuous and f_n is continuous for all $n \in \mathbb{N}$. Moreover, the convergence is uniform on $[a, +\infty[$ for all $a > 0$ since $|f_n(x)| \leq e^{-na}$, for all $x \in [a, +\infty[$.

(e) $f_n(x) = n^\alpha x(1 - nx - |1 - nx|)$ on $\mathbb{R}_+$, $\alpha \in \mathbb{R}$.

$f_n(x) = 0$ if $x \leq \frac{1}{n}$ and $f_n(x) = 2n^\alpha x(1 - nx)$ if $n \geq \dfrac{1}{n}$.

- If $\alpha < -1$, $\displaystyle\lim_{n\to+\infty} f_n(x) = 0$, for all $x \in \mathbb{R}_+$, but $\displaystyle\lim_{n\to+\infty} f_n(n^{-\alpha}) = -\infty$, then the convergence is not uniform.

• If $\alpha = -1$, $\lim\limits_{n\to+\infty} f_n(x) = -2x^2$, for all $x \geq 0$. The convergence is not uniform.

• If $\alpha > -1$, $\lim\limits_{n\to+\infty} f_n(x) = -\infty$, for all $x > 0$.

(f) $f_n(x) = \begin{cases} n^\alpha x(1-nx) & \text{if} \quad 0 \leq x < \frac{1}{n} \\ 0 & \text{if} \quad \frac{1}{n} \leq x \leq 1 \end{cases}$, $\alpha \in \mathbb{R}$.

$\lim\limits_{n\to+\infty} f_n(x) = 0$, for all $x \in [0,1]$. Then the pointwise limit of the sequence is 0.

$f_n'(x) = n^\alpha(1-2nx)$, for $x \in [0, \frac{1}{n}]$. Then $\sup_{x\in[0,1]} f_n(x) = \frac{n^{\alpha-1}}{4}$. The convergence is uniform on $[0,1]$ if and only if $\alpha < 1$.

(g) $f_n(x) = \begin{cases} nx - \frac{1}{n} & \text{if} \quad x \in [0, \frac{1}{n}[\\ 1-x & \text{if} \quad x \in [\frac{1}{n}, 1] \end{cases}$ defined on $[0,1]$.

$\lim\limits_{n\to+\infty} f_n(x) = 1-x$ on $]0,1]$ and $\lim\limits_{n\to+\infty} f_n(0) = 0$. The functions f_n are continuous. Since the limit is not continuous, the convergence is not uniform. But the convergence is uniform on any interval $[a,1]$, for all $0 < a < 1$.

(h) $f_n(x) = \begin{cases} \dfrac{\sin nx}{n\sqrt{x}} & \text{if } x > 0 \\[2ex] 0 & \text{if } x = 0 \end{cases}$

For $x \neq 0$, $\lim\limits_{n\to+\infty} f_n(x) = 0$. The convergence is uniform on the interval $[0,+\infty[$, indeed, $\sup\limits_{x\geq 0} |f_n(x)| = \max\left(\sup\limits_{x\leq\frac{1}{n}} |f_n(x)|, \sup\limits_{x\geq\frac{1}{n}} |f_n(x)|\right)$.

$\sup\limits_{x\leq\frac{1}{n}} |f_n(x)| = \sup\limits_{x\leq\frac{1}{n}} \dfrac{\sin nx}{nx}\sqrt{x} \leq \dfrac{1}{\sqrt{n}}$, since $\dfrac{|\sin nx|}{nx} \leq 1$ and $\sup\limits_{x\geq\frac{1}{n}} |f_n(x)| = \sup\limits_{x\geq\frac{1}{n}} \dfrac{\sin nx}{nx}\sqrt{x} \leq \dfrac{1}{\sqrt{n}}$, since $|\sin nx| \leq 1$.

(i) $f_n(x) = \dfrac{x^n}{1+x^n}$ on each of the following interval, with $0 < a < 1$.

$[0, 1-a]$, $\quad [1-a, 1+a]$, $\quad [1+a, +\infty[$.

On the interval $[0, 1-a]$, the pointwise limit of f is 0. The convergence is uniform since $|f_n(x)| \leq (1-a)^n$.

On the interval $[1-a, 1+a]$, the pointwise limit of f is 0 if $|x| < 1$, 1 if $|x| > 1$, $\frac{1}{2}$ if $x = 1$. The convergence is not uniform.

On the interval $[1+a,+\infty[$, $\lim\limits_{n\to+\infty} f_n(x)=1$. Moreover $|f_n(x)-1| = |\dfrac{1}{1+x^n}| \leq \dfrac{1}{1+(1+a)^n}$. Then the convergence if uniform.

(j) $f_n(x)=\begin{cases} \dfrac{\sin^2(nx)}{nx} & \text{if} \quad x\notin \pi\mathbb{Z} \\ 0 & \text{if} \quad x\in\pi\mathbb{Z}\end{cases}$,

$\lim\limits_{n\to+\infty} f_n(x)=0$, and $f_n(\dfrac{1}{n})=\sin^2 1$. Then the convergence is not uniform on $\mathbb{R}$.

$\boxed{\text{4-1-3}}$ (a) i. The function $\varphi_n'\colon [0,n] \longrightarrow \mathbb{R}$ is continuous. It reaches its extremum on 0 or on n or on at an interior point of the interval $[0,n]$ where $\varphi_n^{''}$ vanishes. $\varphi_n'(0)=0$, $\varphi_n'(n)=-e^{-n}<0$ and let a such that $\varphi_n^{''}(a)=e^{-a}-\dfrac{n-1}{n}(1-\dfrac{a}{n})^{n-2}=0$. Then $e^{-a}=\frac{n-1}{n}(1-\frac{a}{n})^{n-2}$ and $\varphi_n'(a)=-\frac{n-1}{n}(1-\frac{a}{n})^{n-2}+(1-\frac{a}{n})^{n-1}=(1-\frac{a}{n})^{n-2}(\frac{1-a}{n})$. We remark that $\varphi_n'(1)>0$, then the extremum of φ_n' is positive. Then φ_n' has a unique zero on the interval $]0,n[$.

ii. Let b be the point in $]0,n[$ where $\varphi_n'(b)=0$. φ_n is increasing on the interval $[0,b]$ and decreasing on the interval $[b,n]$. $\varphi_n(b)=\dfrac{b}{n}(1-\dfrac{b}{n})^{n-1}$, $\varphi_n(0)=0$ and $\varphi_n(n)=e^{-n}$.

(b) The pointwise limit of the sequence $(f_n)_n$ is $f(x)=e^{-x}$. Since $\varphi_n = f-f_n$, then the sequence $(f_n)_n$ converges uniformly on $[0,+\infty[$.

$\boxed{\text{4-1-4}}$ (a) $f_n'(x)=-n\cos^{n-1}x\sin^2 x+\cos^{n+1}x=\cos^{n-1}x(\cos^2 x-n\sin^2 x)=\cos^{n-1}x(1-(n+1)\sin^2 x)$. The maximum of f_n is reached at α_n such that $\sin^2(\alpha_n)=\frac{1}{n+1}$. Since $f_n(\alpha_n)\leq\sin(\alpha_n)$, then the sequence $(f_n)_n$ converges uniformly on $[0,\frac{\pi}{2}]$.
$f_n\colon [0,\frac{\pi}{2}] \longrightarrow \mathbb{R}$ defined by: $\;f_n(x)=(\cos^n x)\sin x$.

(b) $\lim\limits_{n\to+\infty} g_n(x)=\lim\limits_{n\to+\infty}(1+\dfrac{x}{n})^n=e^x$. The sequence $(g_n)_n$ is increasing. $\sup\limits_{x\in]-\infty,a]}|e^x-g_n(x)|\leq \max(e^{-n},\sup\limits_{x\in[-n,a]}(e^x-g_n(x)))$.
For $x\in[-n,a]$, the function $f_n(x)=e^x-(1+\frac{x}{n})^n$ is differentiable and $f_n'(x)=0 \iff e^x=(1+\frac{x}{n})^{n-1}$. Let α_n the zero of f_n', then by the variation of f_n yields $\sup\limits_{x\in[-n,a]}(e^x-g_n(x))\leq \max(e^{-n},f_n(a),-\dfrac{\alpha_n}{n}e^{\alpha_n}$. Since $xe^x\leq\frac{1}{e}$ for $x\leq 0$, then

$$\sup_{x\in[-n,a]}(e^x-g_n(x))\leq\max(e^{-n},f_n(a),\frac{1}{en},$$

which proves that the sequence $(g_n)_n$ converges uniformly on $]-\infty, a]$, for all $a \in \mathbb{R}$.

$\boxed{\text{4-1-5}}$ • If $x = 0$, $f_n(0) = 0$.

• If $x > 0$, $\lim\limits_{n \to +\infty} f_n(x) = 1$.

• If $-2 < x < 0$, $x \neq -1$, $\lim\limits_{n \to +\infty} f_n(x) = -1$.

• If $x < -2$, $\lim\limits_{n \to +\infty} f_n(x) = 1$.

The pointwise limit of $(f_n)_n$ is the function f defined by:

$$f(x) = \begin{cases} 1 & \text{if } x \in]-\infty, -2[\cup]0, +\infty[\\ 0 & \text{if } x = 0 \\ -1 & \text{if } -2 < x < 0 \end{cases}$$

• If $[a, b]$ is an interval in $]0, +\infty[$, $|f_n(x) - f(x)| = \dfrac{2}{(x+1)^n + 1} \leq \dfrac{2}{(a+1)^n + 1}$. Then the convergence is uniform on $[a, b]$.

• If $[a, b]$ is an interval in $]-\infty, -2[$, $|f_n(x) - f(x)| \leq \dfrac{2}{|(x+1)^n + 1|} \leq \dfrac{2}{|(b+1)^n + 1|}$. Then the convergence is uniform on $[a, b]$.

• If $[a, b]$ is an interval in $]-2, 1[$, $|f_n(x) - f(x)| = |\dfrac{2(x+1)^n}{(x+1)^n + 1}| \leq \dfrac{2\max(|a+1|^n, |b+1|^n),}{|\min(|a+1|^n, |b+1|^n) + 1|}$. Then the convergence is uniform on $[a, b]$.

$\boxed{\text{4-1-6}}$ (a) $u_n(0) = 0$ and for $x \neq 0$, $\lim\limits_{n \to +\infty} u_n(x) = 0$.

(b) $\lim\limits_{n \longrightarrow +\infty} \displaystyle\int_0^1 u_n(x)dx = \int_0^{+\infty} te^{-t}dt = 1$.

(c) The convergence of the sequence $(u_n)_n$ on $[0, 1]$ is not uniform.

$\boxed{\text{4-1-7}}$ (a) $f_n(0) = 0$ and for $x \in]0, 1]$, $\lim\limits_{n \to +\infty} f_n(x) = 1$.

(b) The convergence of $(f_n)_n$ to f is not uniform on $[0, 1]$ since the functions f_n are continuous but the limit is not continuous.

The convergence of $(f_n)_n$ to f is uniform on $[1, +\infty[$? since $|f_n(x) - 1| = \dfrac{1}{1 + nx} \leq \dfrac{1}{1 + n}$.

The convergence of $(f_n)_n$ to f is not uniform on $[0, +\infty[$ since the convergence of $(f_n)_n$ is not uniform on $[0, 1]$.

(c) $F_n(x) = \displaystyle\int_0^x f_n(t)dt = x - \frac{1}{n}\ln(1+nx)$.

i. The pointwise limit of the sequence $(F_n)_n$ is F, with $F(x) = x$.

ii. The convergence of $(F_n)_n$ to F on $[0,1]$ is uniform since $|F_n(x) - F(x)| = \frac{1}{n}\ln(1+nx) \le \frac{1}{n}\ln(1+n) \underset{n\to+\infty}{\longrightarrow} 0$.

4-1-8 (a) $0 \le f_n \le \frac{1}{n}$, then the sequence $(f_n)_n$ converges uniformly on $\mathbb{R}$.

(b) For all n, the function f_n is differentiable on $\mathbb{R}$ and $f_n'(x) = x(x^2 + \dfrac{1}{n^2})^{-\frac{1}{2}}$.
The limit of the sequence $(f_n)_n$ is f defined by $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$ and $f(0) = 0$ which is not differentiable at 0.

4-1-9 (a) $D = \{x \in\in \mathbb{R}_+^*;\ |\ln x| < 1\} =]\frac{1}{e}, e[$.

(b) $\sup\limits_D |f_n(x)| = n$, then the sequence $(f_n)_n$ is not uniformly convergent D. Let K be any compact of D and $\alpha = \sup_K |\ln x|$, $\alpha < 1$, then the sequence $(f_n)_n$ converges uniformly on K.

4-1-10 (a) $f_n(0) = 0$ and for $x > 0$, $\lim\limits_{n\to+\infty} f_n(x) = e^{-x}(1+x^2)$. Since the limit is not continuous and the functions f_n are continuous, then the sequence $(f_n)_n$ is not uniformly convergent on $\mathbb{R}_+$.

(b) Let K be a closed and bounded interval of $]0,+\infty[$ and $a = \inf K$. $|f_n(x) - f(x)| = \dfrac{e^{-x}(1+x^2)}{1+nx^2}$. The map $x \longmapsto \dfrac{e^{-x}(1+x^2)}{1+nx^2}$ is decreasing, then $\sup\limits_{x\in K} |f_n(x) - f(x)| = |f_n(a) - f(a)| \underset{n\to+\infty}{\longrightarrow} 0$.

(c) The map $x \longmapsto \dfrac{e^{-x}(1+x^2)}{1+nx^2}$ is decreasing, then $\sup\limits_{x\in[0,1]} |f_n(x) - f(x)| = 1$

(d) For all $a > 0$, $\lim\limits_{n\to+\infty} \displaystyle\int_a^1 f_n(t)dt = \int_a^1 f(t)dt$ since the convergence is uniform on the interval $[a,1]$. Moreover $\left|\displaystyle\int_0^a f_n(t) - f(t)dt\right| \le a$.
Then $\lim\limits_{n\to+\infty} \displaystyle\int_0^1 f_n(t)dt = \int_0^1 f(t)dt$.

4-1-11 (a) $\mathbb{R}_+$ is the domain D of pointwise convergence of the sequence $(f_n)_n$ and $\lim\limits_{n\to+\infty} f_n(x) = 1$ if $x \in [0,1[$ and $\lim\limits_{n\to+\infty} f_n(x) = 0$ if $x \in [1,+\infty[$.

(b) $\sup\limits_{x\in[1,+\infty[} |f_n(x)| = e^{-n}$. Then the sequence $(f_n)_n$ converges uniformly on $[1,+\infty[$ to 0.

(c) The function $x \longmapsto 1 - f_n(x)$ is increasing on the interval $[0,1[$, then $\sup_{x\in[0,1[} |f_n(x)-1| = 1-e^{-n}$, which yields that the sequence $(f_n)_n$ is not uniformly convergent on $[0,1[$.

(d) Let K be a compact subset of $[0,1[$, there exists $a<1$ such that $K\subset[0,a]$. $\sup_{x\in[0,1[} |f_n(x)-1| = 1-f_n(a) \underset{n\to+\infty}{\longrightarrow} 0$. Then the sequence $(f_n)_n$ converges uniformly on K.

$g_n(x) = f_n'(x) = -n^2x^{n-1}f_n(x)$.

(e) $\mathbb{R}_+$ is the domain of pointwise convergence of the sequence $(g_n)_n$ and its limit is 0.

(f) $\lim_{n\to+\infty} g_n(\dfrac{n-1}{n^2})^{\frac{1}{n}} = \lim_{n\to+\infty} -n^2(\dfrac{n-1}{n^2})^{\frac{n-1}{n}}e^{-\frac{n-1}{n}} = -\infty.$

(g) Since $\lim_{n\to+\infty}\sup_{x\in[0,+\infty[} |g_n(x)| \leq \lim_{n\to+\infty} |g_n(\dfrac{n-1}{n^2})^{\frac{1}{n}}| = +\infty$, then the convergence of the sequence $(g_n)_n$ is not uniform on the interval $[0,+\infty[$. The same argument for the uniform convergence on the interval $[0,1[$.

$\sup_{x\in[1,+\infty[} f_n(x) = f_n(1)$. The sequence converges uniformly on the interval $[1,+\infty[$.

4-1-12 (a) $\ln(\frac{x^2+x+n}{n+x}) = \ln(1+\frac{x^2}{n+x})$, which proves that $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^{n^\beta+2}}{n}$.

(b) • If $\beta=0$, $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^3}{n}$. The sequence $(f_n)_n$ converges on $]0,+\infty[$ to 0.

• If $\beta<0$, $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^2}{n}$, the sequence converges on $]0,+\infty[$ to 0.

• If $\beta>0$, the sequence converges only on $[0,1]$ to 0.

(c) • If $\beta=0$, $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^3}{n}$. The sequence $(f_n)_n$ converges uniformly on any compact of $]0,+\infty[$.

• If $\beta<0$, $|f_n(x)| \underset{n\to+\infty}{\approx} \frac{x^2}{n}$, the sequence converges uniformly on any compact of $]0,+\infty[$.

• If $\beta>0$, the sequence converges uniformly and normally on $[0,1]$. (The function $x\longmapsto x^{n^\beta}\ln(\frac{x^2+x+n}{n+x})$ is increasing on $[0,1]$)

4-1-13 (a) By the Cauchy criterion (1.2), there is n_0 such that for $n,m\geq n_0$, $\sup_{x\in\mathbb{R}}|P_n(x)-P_m(x)|\leq 1$. Then $P_n-P_{n_0}$ is bounded on $\mathbb{R}$ for $n\geq n_0$.

(b) $P_n - P_{n_0}$ is bounded, then it is constant. There is a sequence $(c_n)_n$ such that $P_n = P_{n_0} + c_n$ and since $(P_n)_n$ converges to f then f is a polynomial.

4-1-14 (a) For $x \neq 0$, $1 + x\ln x \geq 0$, then $|f_n(x)| \leq \dfrac{1}{2ne}$. Then $(f_n)_n$ converges uniformly on $[0,1]$.

(b) The pointwise limit of $(f_n)_n$ is 0. $f_n'(x) = nx^{n-1}(1 + n\ln x)$. Then $\sup_{x\in[0,1]} f_n(x) = f(e^{-\frac{1}{n}}) = -\frac{1}{e}$. The convergence of the sequence $(f_n)_n$ is not uniform. The convergence is uniform on any interval $[a,b] \subset [0,1[$.

(c) The pointwise limit of $(f_n)_n$ is 0. The convergence of the sequence $(f_n)_n$ is not uniform because $f_n(\dfrac{1}{n}) = \dfrac{\sin^2 1}{n\sin\frac{1}{n}} \underset{n\to+\infty}{\longrightarrow} \sin^2 1$.

(d) The functions f_n are even and $f_n(x) = 4^n\left(x^{2^{n+1}} - x^{2^n}\right) = 4^n x^{2^n}\left(x^{2^n} - 1\right)$.

$f_n(1) = 0$, and the sequence $(f_n)_n$ converges only on the interval $[-1,1]$ and the limit is 0. $f_n'(x) = 0 \iff x = 2^{-\frac{1}{2^n}}$.

$$\sup_{x\in[0,1]} |f_n(x)| = |f_n(2^{-\frac{1}{2^n}})| = 4^{n-1}.$$

Then the convergence is not uniform on $[0,1]$.

(e) $f_n(0) = 0$ and $(f_n)_n$ converges to 0.

$\displaystyle\int_0^1 f_n(t)dt = \frac{\ln(n2^n+1)}{2n} \underset{n\to+\infty}{\longrightarrow} \frac{\ln 2}{2}$. Then the convergence is not uniform.

4-1-15 The functions f_n are odd, we study the sequence on $[0,+\infty[$.

(a) For $x \neq 0$, $f_n(x) \approx \frac{x}{n} \underset{n\to+\infty}{\longrightarrow} 0$. Moreover $|\sin x| \leq |x|$ yields that $|f_n(x)| \leq \dfrac{|x|}{n}$. On any interval $[a,b]$, the convergence is uniform.

(b) $f_n(n) = n^2 \sin\dfrac{1}{n^2} \underset{n\to+\infty}{\longrightarrow} 1$. Then the convergence is not uniform.

(c) $|f_n'(x)| = |2x\sin\dfrac{1}{nx} - \dfrac{1}{n}\cos\dfrac{1}{nx}| \leq \dfrac{2}{n} + \dfrac{1}{n} = \dfrac{3}{n}$. Then the sequence $(f_n')_n$ converges uniformly on $\mathbb{R}$.

4-1-16 For $x \in [0,1]$ and $n \in \mathbb{N}$, define $f_n(x) = \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k} - \ln(1+x)$

(a) $f_n'(x) = \sum_{k=0}^{n-1}(-1)^k x^k - \dfrac{1}{1+x} = -\dfrac{(-x)^n}{1+x}$. Then

$$f_n(x) = -\int_0^x \frac{(-t)^n}{1+t}dt.$$

$|f_n(x)| \leq \displaystyle\int_0^x t^n dt = \frac{1}{n+1}$ and the sequence $(f_n)_n$ converges uniformly to 0 on $[0,1]$.

(b) Sine the sequence $(f_n)_n$ converges uniformly to 0, then for all sequence $(x_n)_n$ in $[0,1]$, $\lim\limits_{n\to+\infty} f_n(x_n) = 0$. Moreover $\lim\limits_{n\to\infty} \displaystyle\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{n}{n+1}\right)^k = \lim\limits_{n\to\infty} \ln(1+\frac{n}{n+1}) = \ln 2$.

4-1-17 (a) The sequence $(f_n)_n$ converges to 0 but the convergence is not uniform since $f_n(\frac{1}{n}) = n$.

(b) $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n(x)dx = \lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \frac{1}{2}$ and $\displaystyle\int_0^1 \lim\limits_{n\to\infty} f_n(x)dx = 0$. Then the convergence is not uniform.

4-2-1 (a) $f_n(x) = \dfrac{\sin(n^2 x)}{n^2}$, with $x \in \mathbb{R}$,
$|f_n(x)| \le \frac{1}{n^2}$, then the series $\sum_{n\ge 1} f_n$ is t normally convergent on $\mathbb{R}$.

(b) $f_n(x) = \dfrac{1}{n}\tan^{-1}\dfrac{x}{n}$, with $x \in \mathbb{R}$.
Since f_n is odd, we study the series on the interval $[0, +\infty[$.
For $x \ge 0$, $\tan^{-1} x \le x$, then $f_n(x) \le \frac{x}{n^2}$. Then the series $\sum_{n\ge 1} f_n$ is absolutely convergent on $\mathbb{R}$ and normally convergent on any interval $[0, a]$ for all $a > 0$.
Since the functions $x \longmapsto f_k(x)$ is increasing on $[0, +\infty[$, for $m > n$,
$\sup\limits_{x\in[0,+\infty[} \sum\limits_{k=n}^{m} f_k(x) = \sum\limits_{k=n}^{m} \dfrac{\pi}{2k}$, then the series is not uniformly convergent on $\mathbb{R}$.

(c) $f_n(x) = x^{n^2}\sin(n\pi x)$, for $n \in \mathbb{N}$ and $x \in [0, a]$, with $0 < a < 1$.
$|x^{n^2}\sin(n\pi x)| \le a^{n^2} \le a^n$ and the series $\sum_{n\ge 1} a^n$ converges, then the series $\sum_{n\ge 1} f_n$ converges uniformly and normally on the interval $[0, a]$.

(d) $f_n(x) = \dfrac{x}{(1+x^2)^n}$, with $x \in \mathbb{R}$. $f_n'(x) = \dfrac{1-(2n-1)x^2}{(1+x^2)^{n+1}}$.
f_n is an odd function. The series $\sum_{n\ge 1} f_n$ is pointwise convergent on $\mathbb{R}$ since $f_n(0) = 0$ and $\lim\limits_{n\to+\infty} n^2 f_n(x) = 0$, for $x \ne 0$.
f_n is increasing on $[0, \frac{1}{\sqrt{2n-1}}]$ and decreasing on the interval $[\frac{1}{\sqrt{2n-1}}, +\infty[$.
$\sup\limits_{x\in[0,+\infty[} f_n(x) = f_n(\dfrac{1}{\sqrt{2n-1}}) \approx \dfrac{1}{\sqrt{e(2n-1)}}$. For $a > 0$ and n large,
$\sup\limits_{x\in[a,+\infty[} f_n(x) = f_n(a)$. Then the series $\sum_{n\ge 1} f_n$ converges uniformly and normally on any interval $[a, +\infty[$, with $a > 0$, but it is not uniformly convergent on $\mathbb{R}$.

(e) $f_n(x) = xe^{-nx^2}$, with $x \in \mathbb{R}$,
f_n is odd. The series $\sum_{n\ge 1} f_n$ is pointwise convergent on $\mathbb{R}$ since $f_n(0) = 0$ and $\lim\limits_{n\to+\infty} n^2 f_n(x) = 0$, for $x \ne 0$.
f_n is increasing on $[0, \frac{1}{\sqrt{2n}}]$ and decreasing on the interval $[\frac{1}{\sqrt{2n}}, +\infty[$.
$\sup\limits_{x\in[0,+\infty[} f_n(x) = f_n(\dfrac{1}{\sqrt{2n}}) \approx \dfrac{1}{\sqrt{2en}}$.

The series $\sum_{n\geq 1} f_n$ converges uniformly and normally on any interval $[a, +\infty[$, with $a > 0$, but it is not uniformly convergent on $\mathbb{R}$.

$\displaystyle\sum_{n=0}^{+\infty} f_n(x) = \frac{x}{1-e^{-x^2}}$, for $x \neq 0$. The function f defined by $f(x) = \displaystyle\sum_{n=0}^{+\infty} f_n(x)$ is not continuous on 0.

(f) $f_n(x) = x^2 e^{-x\sqrt{n}}$, with $x \in \mathbb{R}_+$,

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R}_+$ since $f_n(0) = 0$ and $\displaystyle\lim_{n\to+\infty} n^2 f_n(x) = 0$, for $x > 0$.

$\displaystyle\sup_{x\in[0,+\infty[} f_n(x) = f_n(\frac{2}{\sqrt{n}}) = \frac{4}{e^2 n}$.

The series $\sum_{n\geq 1} f_n$ converges uniformly and normally on any interval $[a, +\infty[$, with $a > 0$, but it is not uniformly convergent on $\mathbb{R}_+$.

(g) $f_n(x) = \dfrac{nx^2}{1+n^3x}$, with $x \in \mathbb{R}_+$,

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R}_+$ since $f_n(0) = 0$ and $\displaystyle\lim_{n\to+\infty} n^2 f_n(x) = x$, for $x > 0$.

For all $n \in \mathbb{N}$, the function f_n is increasing. $\displaystyle\sup_{x\in[0,+\infty[} f_n(x) = \lim_{x\to+\infty} f_n(x) = +\infty$. The series $\sum_{n\geq 1} f_n$ it is not uniformly convergent on $\mathbb{R}_+$.

Moreover since f_n is increasing, $\sup_{x\in[0,a]} f_n(x) = f_n(a)$ and since the series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R}_+$, then the series $\sum_{n\geq 1} f_n$ it is uniformly convergent on any interval $[0, a]$, $\forall a > 0$.

(h) $f_n(x) = \dfrac{(-1)^n}{n^x}$, with $x \in \mathbb{R}$,

The series $\sum_{n\geq 1} f_n$ is pointwise convergent if and only if $x > 0$.

The series $\sum_{n\geq 1} f_n$ is normally convergent on any interval $[a, +\infty[$, with $a > 1$ and not normally convergent on the interval $[1, +\infty[$.

$\displaystyle\sup_{x\in]0,+\infty[} |f_n(x)| = 1$, then the series $\sum_{n\geq 1} f_n$ it is not uniformly convergent on $]0, +\infty[$.

Since the series is alternate and the sequence $(\frac{1}{n^x})_n$ is decreasing and converges uniformly to 0 on the interval $[a, +\infty[$, $a > 0$, then the series $\sum_{n\geq 1} f_n$ is uniformly convergent on $[a, +\infty[$, for all $a > 0$.

(i) $f_n(x) = \dfrac{x^{2n}}{1+x^{2n}}$, with $x \in \mathbb{R}$,

$\lim\limits_{n\to+\infty} f_n(x) = 0$ if and only if $x \in]-1,1[$. Since f_n is even, we study the series on the interval $[0,1[$.

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $[0,1]$ since $f_n(x) \leq x^{2n}$.

f_n is increasing on the interval $[0,1[$ and $\sup_{x\in[0,1[} f_n(x) = f_n(1) = \frac{1}{2}$, then the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $[0,1[$.

The series $\sum_{n\geq 1} f_n$ is normally convergent on any interval $[0,a]$, with $a<1$.

(j) $f_n(x) = \dfrac{(-1)^n}{x^2+n}$, with $x \in \mathbb{R}$,

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $\mathbb{R}$ since the sequence $(\frac{1}{x^2+n})_n$ is decreasing and converges to 0.

The series $\sum_{n\geq 1} f_n$ is not normally convergent on any interval of $\mathbb{R}$ since $|f_n(x)| = \frac{1}{x^2+n} \approx \frac{1}{n}$.

The series $\sum_{n\geq 1} f_n$ is uniformly convergent on $\mathbb{R}$ since the sequence $(\frac{1}{x^2+n})_n$ is decreasing and converges uniformly to 0. $(\frac{1}{x^2+n} \leq \frac{1}{n})$.

(k) $f_n(x) = \dfrac{x}{(1+nx^2)^n}$, with $x \in \mathbb{R}$. Since f_n is odd, we study the series on the interval $[0,+\infty[$.

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $[0,+\infty[$ since the sequence $f_n(0)=0$ and $\lim\limits_{n\to+\infty} n^2 f_n(x) = 0$, for $x>0$.

The function f_n is decreasing on any interval $[a,+\infty[$, $a>0$ for n large. Then the series $\sum_{n\geq 1} f_n$ is normally convergent on any interval $[a,+\infty[$, $\forall a>0$.

$f_n(\dfrac{1}{n}) = \dfrac{1}{n(1+\frac{1}{n})^n} \approx \dfrac{1}{en}$, then the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $[0,+\infty[$.

(l) $f_n(x) = \dfrac{(-1)^n x}{(1+x^2)^n}$, with $x \in \mathbb{R}$. Since f_n is odd, we study the series on the interval $[0,+\infty[$.

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $[0,+\infty[$ since, $f_n(0) = 0$ and the sequence $(\frac{x}{(1+x^2)^n})_n$ is decreasing and converges to 0 for $x>0$.

$f_n(\frac{1}{\sqrt{n}}) = \frac{1}{\sqrt{n}(1+\frac{1}{n})^n} \approx \frac{1}{e\sqrt{n}}$, then the series $\sum_{n\geq 1} f_n$ is not normally convergent on $[0,+\infty[$.

$\sum_{n=0}^{+\infty} f_n(x) = \frac{1+x^2}{x}$, for $x \neq 0$ and $\sum_{n=0}^{+\infty} f_n(0) = 0$. then the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $[0,+\infty[$.

(m) $f_n(x) = \frac{x}{n^\alpha(1+nx^2)}$, $\alpha > 0$. Since f_n is odd, we study the series on the interval $[0,+\infty[$.

The series $\sum_{n\geq 1} f_n$ is pointwise convergent on $[0,+\infty[$ since, $f_n(0) = 0$ and the sequence $n^{\alpha+1} f_n(x) = \frac{1}{x}$, for $x > 0$.

$\sup_{x\in[0,+\infty[} f_n(x) = f_n(\frac{1}{\sqrt{n}}) = \frac{1}{2n^{\alpha+\frac{1}{2}}}$. Then the series $\sum_{n\geq 1} f_n$ is normally convergent on $[0,+\infty[$ if and only if $\alpha > \frac{1}{2}$.

$$\sup_{x\in[0,+\infty[} \sum_{k=n+1}^{+\infty} nf_k(x) \geq \sum_{k=n+1}^{+\infty} nf_k(\frac{1}{\sqrt{n}}) \geq 2nf_{2n}(\frac{1}{\sqrt{n}}) = \frac{n}{3\sqrt{n}(2n)^\alpha}.$$

Then the series $\sum_{n\geq 1} f_n$ is uniformly convergent on $[0,+\infty[$ if and only if $\alpha > \frac{1}{2}$, $\forall\alpha > 0$.

The series $\sum_{n\geq 1} f_n$ is normally convergent on $[a,+\infty[$ for all $a > 0$.

4-2-2 (a) The series $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ is an alternate series. The sequence $(\ln\left(1+\frac{x}{n}\right))_n$ is decreasing and tends to 0, then the series $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ is pointwise convergent on $\mathbb{R}_+$.

(b) $\lim_{n\to+\infty} n|f_n(x)| = x$, then the series $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ is not absolutely convergent on any interval of $\mathbb{R}_+$.

$\sup_{x\geq 0} |f_n(x)| = +\infty$, then the series $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ is not uniformly convergent on $\mathbb{R}_+$.

$\sup_{x\in[0,a]} |f_n(x)| = |f_n(a)|$, then by Abel Theorem for the uniform convergence (2.5) of series, the series $\sum_{n\geq 1}(-1)^n \ln\left(1+\frac{x}{n}\right)$ is uniformly convergent on $[0,a]$.

4-2-3 The series $\sum_{n\geq 0}\frac{(-1)^n e^{-nx}}{n+1}$ converges if and only if $x \geq 0$. Moreover the series is alternate and $\frac{e^{-nx}}{n+1}$ is decreasing and converges uniformly to 0 on $[0,+\infty[$. Then f is continuous on $[0,+\infty[$. $(f(x) = e^x \ln(1+e^{-x}))$.

4-2-4 (a) The series $\sum_{n\geq 0}\frac{(-1)^n e^{-nx}}{n^2+1}$ converges if and only if $x \geq 0$. Then $\mathcal{D} = [0,+\infty[$.

(b) The series $\sum_{n\geq 0}\frac{(-1)^n e^{-nx}}{n^2+1}$ is pointwise convergent on $\mathcal{D}$. The series of derivatives $\sum_{n\geq 0}\frac{(-1)^{n+1} n e^{-nx}}{n^2+1}$ converges uniformly on $\mathcal{D}$ since the sequence $\frac{ne^{-nx}}{n^2+1}$ is decreasing and converges uniformly to 0. Moreover the functions $f_n(x) = \frac{(-1)^n e^{-nx}}{n^2+1}$ are C^∞ on D. Then g is of class C^1 on $\mathcal{D}$.

4-2-5 (a) The functions $f_n(x) = (-1)^n \frac{x^{2n+1}}{2n+1}$ are odd. We study the series on $[0,1]$.

The sequence $\frac{x^{2n+1}}{2n+1}$ is decreasing and tends to 0 uniformly on $[0,1]$, $f_n(x) \leq \frac{1}{2n+1}$. Then the series $\sum_{n\geq 0}(-1)^n\frac{x^{2n+1}}{2n+1}$ is uniformly convergent on $[-1,1]$.

We set $f(x) = \sum_{n=0}^{+\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$, for $x \in [-1,1]$.

(b) The function f_n are C^∞ on $\mathbb{R}$, $\sup_{x\in[0,a]}|f_n'(x)| = a^{2n}$ and $\lim_{n\to+\infty} a^{2n} = 0$, for all $0 \leq a < 1$. This proves that f is differentiable on $]-1,1[$ and

$$f'(x) = \sum_{n=0}^{+\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}.$$

(c) Since $f(0) = 0$, $f(x) = \tan^{-1}(x)$, for $-1 \leq x \leq 1$.

(d) By integration by parts, $\int_0^1 \tan^{-1} x dx = \frac{\pi}{4} - \frac{\ln 2}{2}$.

The series $\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{2n+1}$ is uniformly convergent on $[-1,1]$, then

$$\int_0^1 f(x)dx = \sum_{n=0}^{+\infty}(-1)^n \int_0^1 \frac{x^{2n+1}}{2n+1}dx = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+2)}.$$

then

$$\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)(2n+2)} = \frac{\pi}{4} - \frac{\ln 2}{2}.$$

4-2-6 (a) $f_n(1) = 0$, then the series $\sum_{n\geq 1} f_n(1)$ converges. For $x \in [0,1[$, $\sum_{n\geq 1} x^n$ is convergent and its sum is $\frac{x}{1-x}$. The series $\sum_{n\geq 1} x^{n-\frac{1}{2}}$ is convergent and its sum is $\frac{\sqrt{x}}{(1-x)}$.

$$f(x) = \begin{cases} \frac{\sqrt{x}(\sqrt{x}-1)}{1-x} = -\frac{\sqrt{x}}{1+\sqrt{x}} & \text{if } x \in [0,1[\\ 0 & \text{if } x = 1 \end{cases}$$

(b) $R_n(x) = \sum_{p=n+1}^{+\infty} u_p(x) = \sum_{p=n+1}^{+\infty} x^p - x^{p-\frac{1}{2}} = x^n f(x)$.

(c) $\sup_{x\in[0,1]} |R_n(x)| = \sup_{x\in[0,1]} x^n |f(x)| = \sup_{x\in[0,1[} \frac{x^n\sqrt{x}}{1+\sqrt{x}} = \frac{1}{2}$. Then the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $[0,1]$.

(d)

$$\left|\int_0^1 R_n(x)dx\right| = \int_0^1 \frac{x^n\sqrt{x}}{\sqrt{x}+1}dx \leq \frac{1}{n+1}.$$

(e) Since $\lim_{n\to+\infty} \int_0^1 R_n(x)dx = 0$, the series $\sum_{n\geq 1} g_n$ is convergent and its sum is $\int_0^1 f(x)dx$.

(f) $\int_0^1 u_n(x)dx = \frac{2}{2(n+1)} - \frac{2}{2n+1}$. Then $\int_0^1 f(x)dx = -\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}dx = -1 + 2\ln 2$. We deduce that $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} = -\ln 2$.

4-2-7 (a) $|f_n(x)| \leq \frac{1}{n^2}$, then the series $\sum_{n\geq 0} f_n$ converges uniformly on $\mathbb{R}$.

(b) $f'_n(x) = \cos n^2 x$, the series $\sum_{n\geq 0} f'_n$ is not convergent on any interval.

4-2-8 (a) f_n is C^∞, the sequence $(\frac{1}{\sqrt{n^2+x^2}})_n$ is decreasing dominated by $\frac{1}{n}$, then converges uniformly on $\mathbb{R}$. Then f is continuous on $\mathbb{R}$.

(b) $f'_n(x) = \frac{x(-1)^n}{(n^2+x^2)^{\frac{3}{2}}}$, $|f'_n(x)| \leq \frac{1}{n^2}$. (We study the variations of f'_n). Then f is of class $\mathcal{C}^1$.

4-2-9 (a) The series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^x}$ converges if and only if $x > 0$. Then $D =]0, +\infty[$. By Abel Theorem (2.5) the series $\sum_{n\geq 1} \frac{(-1)^{n+1}}{n^x}$ converges uniformly on any interval $[a, +\infty[$, for all $a > 0$. Then f is continuous on D.
$f_n^{(k)}(x) = \frac{(-1)^{n+k+1}\ln^k n}{n^x}$. The series $\sum_{n\geq 1} \frac{(-1)^{n+k+1}\ln^k n}{n^x}$ converges uniformly on any interval $[a, +\infty[$, for all $a > 0$, then f is $\mathcal{C}^\infty$ on D.

(b) For $x > 1$; define $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ and $\psi(x) = \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^x}$.

$$\zeta(x) = \frac{1}{2^x}\sum_{n=1}^{+\infty}\frac{1}{n^x} + \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^x} = \frac{1}{2^x}\zeta(x) + \psi(x).$$

$$f(x) = -\frac{1}{2^x}\sum_{n=1}^{+\infty}\frac{1}{n^x} + \sum_{n=0}^{+\infty}\frac{1}{(2n+1)^x} = -\frac{1}{2^x}\zeta(x) + \psi(x) = (1 - \frac{1}{2^x})\zeta(x).$$

4-2-10 (a) The sequence $(\frac{x}{2\sqrt{n}+\cos x})_n$ is decreasing and converges to 0 and $|\sum_{n=p}^{q} \sin nx| \leq \frac{1}{\sin(\frac{x}{2})}$, then the series $\sum_{n\geq 1} \frac{x \sin nx}{2\sqrt{n}+\cos x}$ is pointwise convergent on $]0, 2\pi[$

(b) if $x \in [\alpha, 2\pi - \alpha]$, $|\sum_{n=p}^{q} \sin nx| \leq \frac{1}{\sin(\frac{x}{2})} \leq \frac{1}{\sin(\frac{\alpha}{2})}$ and the sequence $(\frac{x}{2\sqrt{n} + \cos x})_n$ is decreasing and converges uniformly to 0 since $\frac{x}{2\sqrt{n} + \cos x} \leq \frac{2\pi}{2\sqrt{n} - 1}$. Then the series converges uniformly on the interval $[\alpha, 2\pi - \alpha]$ for all $0 < \alpha < 2\pi$.

4-2-11 (a) If $\alpha \leq 1$ and $x \neq 0$, $\lim_{n \to +\infty} n^{\alpha} f_n(x) = +\infty$, then the series $\sum_{n \geq 1} f_n(x)$ diverges.
If $\alpha > 1$, take β such that $\alpha > \beta > 1$, for $x \neq 0$, $\lim_{n \to +\infty} n^{\beta} f_n(x) = 0$, then the series $\sum_{n \geq 1} f_n(x)$ converges.

(b) Assume that $\alpha > 1$.

i. $\lim_{x \to +\infty} f_n(x) = +\infty$, then the series $\sum_{n \geq 1} f_n(x)$ is not uniformly convergent on $\mathbb{R}$?

ii. Let $K = [-a, a]$, for $a > 0$, the function f_n is continuous on $\mathbb{R}$, and $\sup_{x \in [-a,a]} f_n(x) = \frac{1}{n^{\alpha}} \ln(1 + n^{\alpha} a^2)$. Moreover for $\alpha > \beta > 1$, $\lim_{n \to +\infty} n^{\beta} \sup_{x \in [-a,a]} f_n(x) = 0$. Then the series $\sum_{n \geq 1} f_n(x)$ converges uniformly on $[-a, a]$ for all $a > 0$, the f is continuous on $\mathbb{R}$.

(c) The functions f_n are even. We study the differentiability on $[0, +\infty[$. Let $a > 0$ and $x \in [a, +\infty[$, $|f_n'(x)| = \left| \frac{2x}{1 + n^{\alpha} x^2} \right| \leq \frac{1}{n^{\alpha}}$. (The maximum of the function $g(x) = \frac{2x}{1 + n^{\alpha} x^2}$ is $g(\frac{1}{n^{\frac{\alpha}{2}}}) = \frac{1}{n^{\frac{\alpha}{2}}}$.)
Then the series $\sum_{n \geq 1} f_n'$ converges uniformly on $[a, +\infty[$, for all $a > 0$. then f is differentiable on $\mathbb{R}^*$.

(d) Assume $1 < \alpha \leq 2$.

i. The functions f_n are even. We study the differentiability on $[0, +\infty[$. Let $a > 0$ and $x \in [a, +\infty[$, $|f_n'(x)| = \left| \frac{2x}{1 + n^{\alpha} x^2} \right| \leq \frac{2a}{1 + n^{\alpha} a^2}$ for n large enough. Then the series $\sum_{n \geq 1} f_n'$ converges uniformly on $[a, +\infty[$, for all $a > 0$. then f is differentiable on $\mathbb{R}^*$.

ii. $f(n^{\frac{-\alpha}{2}}) = \sum_{k=1}^{+\infty} \frac{1}{k^\alpha} \ln(1+(\frac{k}{n})^\alpha) \geq \sum_{k=n}^{\infty} \frac{1}{k^\alpha} \ln(1+(\frac{k}{n})^\alpha) \geq \ln 2 . \sum_{k=n}^{+\infty} \frac{1}{k^\alpha}$.

We know that $\int_n^{n+1} \frac{dx}{x^\alpha} \leq \frac{1}{n^\alpha}$, then $\sum_{k=n}^{+\infty} \frac{1}{k^\alpha} \geq \int_n^{+\infty} \frac{dx}{x^\alpha} = \frac{n^{\alpha-1}}{1-\alpha}$.

Since $1 < \alpha \leq 2$, $n^{\frac{\alpha}{2}} \geq n^{\alpha-1}$ and

$$n^{\frac{\alpha}{2}} f(n^{\frac{-\alpha}{2}}) > \frac{\ln 2}{\alpha - 1}.$$

f is not differentiable at 0 because f is even and its derivative is odd. Then if f is differentiable, $f'(0) = 0$.

4-2-12 (a) The sequence f_n is a geometric sequence with common ratio $\frac{1}{1+x^2} < 1$ for $x \neq 0$. Then the series $\sum_{n\geq 0} f_n$ and $\sum_{n\geq 0} (-1)^n f_n$ converge and

$\sum_{n=0}^{+\infty} f_n(x) = \frac{1+x^2}{x}$ if $x \neq 0$ and $\sum_{n=0}^{+\infty} f_n(0) = 0$,

$\sum_{n=0}^{+\infty} (-1)^n f_n(x) = \frac{x(1+x^2)}{2+x^2}$.

(b) $\sup_{x\in[a,+\infty[} f_n(x) = f_n(a)$, for n large. Since the series $\sum_{n\geq 0} f_n(a)$ is convergent, then the series $\sum_{n\geq 0} f_n$ converges uniformly on $[a, +\infty[$.

(c) The series $\sum_{n\geq 0} (-1)^n f_n$ is alternate. The sequence $(f_n(x))_n$ is decreasing and converges uniformly on $[a, +\infty[$ to 0, then the series $\sum_{n\geq 0} (-1)^n f_n$ converges uniformly on $\mathbb{R}$.

4-2-13 (a) $f_n(x) \geq \frac{1}{nx^n}$ and the series $\sum_{n\geq 1} \frac{1}{nx^n}$ converges if and only if $x > 1$. Moreover for $x > 1$, $f_n(x) \leq \frac{1}{x^{n-1}}$ and the series $\sum_{n\geq 1} \frac{1}{x^{n-1}}$ converges. Then the domain of the pointwise convergence of the series $\sum_{n\geq 1} f_n(x)$ is $]1, +\infty[$.

(b) For all $n \geq 1$, the map $x \longmapsto f_n(x)$ is decreasing on the interval $]1, +\infty[$, then

$$\sup_{x\in]1,+\infty[}\sum_{n=p}^{q} f_n(x)=\sum_{n=p}^{q}\frac{\ln(1+n)}{n},$$

Then the series $\sum_{n\geq 1} f_n$ is not uniformly convergent on $]1,+\infty[$.

Moreover $\sup_{x\in[a,+\infty[} f_n(x)=f_n(a)$, which proves that the series $\sum_{n\geq 1} f_n$ converges normally on $[a,+\infty[$.

(c) The functions f_n are continuous on $]1,+\infty[$ and the series $\sum_{n\geq 1} f_n$ converges normally on $[a,+\infty[$, for all $a>1$, then f is continuous on $]1,+\infty[$.

The function f is increasing and $\lim_{x\to 1^+} f(x)\geq \lim_{m\to+\infty}\sup_{x\in]1,+\infty[}\sum_{n=1}^{m} f_n(x)=$ $\lim_{m\to+\infty}\sum_{n=1}^{m}\frac{\ln(1+n)}{n}=+\infty.$

4-2-14 (a) For $x<0$, $\lim_{n\to+\infty} f_n(x)=+\infty$ and for $x\geq 0$, $f_n(x)\leq\dfrac{1}{1+\sqrt{n^3}}$. Then the domain of definition of f is $[0,+\infty[$.

(b) The functions f_n are C^∞ on $\mathbb{R}$, the series $\sum_{n\geq 0} f_n$ converges normally on $[0,+\infty[$, then f is continuous on $\mathbb{R}_+=[0,+\infty[$.

(c) $f_n'(x)=-\sqrt{n}f_n(x)$ and for all $a>0$ and $x\in[a,+\infty[$, $|f_n'(x)|\leq\sqrt{n}f_n(a)$. Since $\lim_{n\to+\infty} n^2\sqrt{n}f_n(a)=0$, f is differentiable on $\mathbb{R}_+^*=]0,+\infty[$.

4-2-15 The sequence $(\frac{1}{n+x})_n$ is decreasing and converges uniformly to 0 on any interval $[a,+\infty[$, for all $a>-1$. By Abel Theorem (2.5) for the uniform convergence of alternate series, f is continuous on $]-1,+\infty[$.

Since the sequence converges uniformly on $[a,+\infty[$ and $\lim_{x\to+\infty} f_n(x)=0$, then $\lim_{x\to+\infty} f(x)=0$.

$f(x)=-\dfrac{1}{1+x}+\displaystyle\sum_{n=2}^{+\infty}\frac{(-1)^n}{n+x}$. The function g defined by $g(x)=\displaystyle\sum_{n=2}^{+\infty}\frac{(-1)^n}{n+x}$ is continuous on $]-2,+\infty[$, then $\lim_{x\to(-1)^+} f(x)=-\infty$.

4-2-16 Let $f_n(x) = \dfrac{e^{-nx}}{1+n^2}$. If $x < 0$, $\lim_{n\to+\infty} f_n(x) = +\infty$ and for $x \geq 0$, $f_n(x) \leq \dfrac{1}{1+n^2}$. the series $\sum\limits_{n\geq 0} f_n$ converges normally on $\mathbb{R}^+$.

$f_n'(x) = -nf_n(x)$. The series $\sum\limits_{n\geq 0} f_n'$ converges normally on $[a, +\infty[$ for all $a > 0$ and the functions f_n are C^∞, then f is of class $\mathcal{C}^1$ on $\mathbb{R}_+^*$.

4-2-17 The sequence $(\frac{1}{n})_n$ is decreasing and tends to 0. Moreover $\left|\sum\limits_{k=1}^{n} \cos k(k-1)x - \cos k(k+1)\right.$ $|1 - \cos n(n+1)x| \leq 2$. Then the series $\sum\limits_{n\geq 1} f_n(x)$ converges uniformly on $\mathbb{R}$.

4-2-18 For $x \neq 0$, $f_n(x) \geq \dfrac{1}{n^\alpha}$ for n large enough. Then the necessary condition for the convergence of the series $\sum\limits_{n\geq 1} f_n(x)$ for $x \neq 0$ is $\alpha > 1$. Moreover if $\alpha > 1$, let $1 < \gamma < \alpha$, then $\lim\limits_{n\to+\infty} n^\gamma f_n(x) = 0$. Then the series $\sum\limits_{n\geq 1} f_n'(x)$ is pointwise convergent on $\mathbb{R}$ if and only if $\alpha > 1$.

$f_n'(x) = \dfrac{2n^\beta x}{n^\alpha(1+n^\beta x^2)} \approx \dfrac{2}{n^\alpha x}$, for $x \neq 0$.

Then the series $\sum\limits_{n\geq 1} f_n(x)$ and $\sum\limits_{n\geq 1} f_n'(x)$ are pointwise convergent on $\mathbb{R}$ if and only if $\alpha > 1$.

4-2-19 (a) f_0 is a polynomial. Assume f_n is a polynomial. Since $f_n'(x) = f_{n-1}(x - x^2)$ is a polynomial, then f_n is a polynomial. Moreover if $g(x) = f_n(x) + f_n(1-x)$, $g'(x) = f_n'(x) - f_n'(1-x) = f_{n-1}(x(1-x)) - f_{n-1}(x(1-x)) = 0$. Then g is constant.

(b) Let $x \in [0,1]$, $f_1(x) - f_0(x) = \displaystyle\int_0^x dt = x$.

Assume that $0 \leq f_n(x) - f_{n-1}(x) \leq \dfrac{x^n}{n!}$.

$f_{n+1}(x) - f_n(x) = \displaystyle\int_0^x f_n(t(1-t)) - f_{n-1}(t(1-t))dt$ and

$$\begin{aligned}
0 \leq f_{n+1}(x) - f_n(x) &= \int_0^x f_n(t(1-t)) - f_{n-1}(t(1-t))dt \\
&\leq \int_0^x \frac{(t^n(1-t)^n)}{n!}dt \leq \int_0^x \frac{t^n}{n!}dt \\
&= \frac{x^{n+1}}{(n+1)!}.
\end{aligned}$$

(c) $\sup_{x\in[0,1]} |f_n(x) - f_m(x)| \leq \sup_{x\in[0,1]} \sum_{k=n}^{m-1} |f_k(x) - f_{k+1}(x)| \leq \sum_{k=n}^{m-1} \frac{1}{k!}$. Since the series $\sum_{k\geq 0} \frac{1}{k!}$ converges, the sequence $(f_n)_n$ converges uniformly on $[0,1]$ to a continuous function f. Since the convergence is uniform, the function f fulfills

$$f(x) = 1 + \int_0^x f(t(1-t))dt.$$

Then f is C^∞ on $[0,1]$ and $f'(x) = f(x - x^2)$.

4-2-20 (a) $f_n'(x) = \dfrac{-2n}{(nx+1)^3}$ and $f_n^{''}(x) = \dfrac{6n^2}{(nx+1)^4}$.

$|f_n(x) \leq \dfrac{1}{(na+1)^2}$ and $|f_n'(x)| \leq \dfrac{2n}{(na+1)^3}$ and $|f_n^{''}(x)| \leq \dfrac{6n^2}{(na+1)^4}$, for all $x \in [a, +\infty[$. Then the series $\sum_{n\geq 0} f_n$, $\sum_{n\geq 0} f_n'$ and $\sum_{n\geq 0} f_n^{''}$ are uniformly convergent on $[a, +\infty[$.

(b) $F(\frac{1}{2}) = \sum_{n=1}^{+\infty} \dfrac{4}{(n+2)^2} = 4\sum_{n=3}^{+\infty} \dfrac{1}{n^2} = 4(\dfrac{\pi^2}{6} - 1 - \dfrac{1}{4}) = \dfrac{2\pi^2}{3} - 5.$

$F(1) = \sum_{n=1}^{+\infty} \dfrac{1}{(n+1)^2} = \dfrac{\pi^2}{6} - 1.$

$\dfrac{\pi^2}{6} = \sum_{n=1}^{+\infty} \dfrac{1}{n^2} = \dfrac{1}{4}\sum_{n=1}^{+\infty} \dfrac{1}{n^2} + F(2) - 1$. Then $F(2) = \dfrac{\pi^2}{12} + 1$.

(c) Since the series $\sum_{n\geq 0} f_n$, $\sum_{n\geq 0} f_n'$ and $\sum_{n\geq 0} f_n^{''}$ are uniformly convergent on any $[a, +\infty[$, for $a > 0$, F is $\mathcal{C}^2$ on $]0, +\infty[$. $F' \leq 0$ and $F^{''} \geq 0$.

(d) Since the series $\sum_{n\geq 0} f_n$ is uniformly convergent on $[1,+\infty[$, then $\lim_{x\to+\infty} F(x) = 0$.

$\lim_{x\to 0^+} F(x) \geq \lim_{x\to 0^+} \sum_{n=1}^{m} \frac{1}{(nx+1)^2} = m$, for all $m\in\mathbb{N}$, then $\lim_{x\to 0^+} F(x) = +\infty$.

4-2-21 (a) $\lim_{n\to+\infty} e^{-n^2x} = 0 \iff x > 0$ and for $x > 0$, $\lim_{n\to+\infty} n^2e^{-n^2x} = 0$.
Then $D_f =]0,+\infty[$.
The series $\sum_{n\geq 0} xe^{-n^2x}$ converges for $x = 0$, then $D_g = [0,+\infty[$.

(b) i. The functions $x \longmapsto e^{-n^2x}$ are decreasing on D_f, then f is decreasing on D_f.

ii. $\lim_{x\to 0^+} f(x) \geq \lim_{x\to 0^+} \sum_{n=0}^{m} e^{-n^2x} = m+1$, for all $m\in\mathbb{N}$, then $\lim_{x\to 0^+} f(x) = +\infty$.

(c) For all $a > 0$, the series $\sum_{n\geq 0} xe^{-n^2x}$ converges uniformly on $[a,+\infty[$ since e^{-n^2x} is decreasing and the series $\sum_{n\geq 0} xe^{-n^2a}$ is convergent.
Then f is continuous on D_f.

(d) i. If $f_n(x) = xe^{-n^2x}$, $f_n'(x) = e^{-n^2x}(1-n^2x)$, then $\sup_{x\geq 0} xe^{-n^2x} = \frac{1}{en^2}$.

ii. Since the series $\sum_{n\geq 0}^{+\infty} \frac{1}{en^2}$ is convergent, the series $\sum_{n\geq 0}^{+\infty} xe^{-n^2x}$ is uniformly convergent on D_g.

4-2-22 (a) $\lim_{n\to+\infty} (-1)^n \frac{e^{-nx}}{n+1} = 0$ if and ony if $x \in [0,+\infty[$. Moreover if $x \in [0,+\infty[$, the series $\sum_{n\geq 0} (-1)^n \frac{e^{-nx}}{n+1}$ converges since it is alternate and the sequence $(\frac{e^{-nx}}{n+1})_n$ is decreasing end tends uniformly to 0. The domain $D = [0,+\infty[$ and since the functions $f_n(x) = (-1)^n \frac{e^{-nx}}{n+1}$

are continuous on D, the series $\sum_{n\geq 0}(-1)^n \frac{e^{-nx}}{n+1}$ defines a continuous function on D.

(b) In the same way, the domain of convergence of the the series $\sum_{n\geq 0}(-1)^n \frac{e^{-nx}}{n^2+1}$ is $[0,+\infty[$ and the series converges normally on D. The functions g is continuous on D.

The functions $g_n(x) = (-1)^n \frac{e^{-nx}}{n^2+1}$ are C^∞ on D and $g_n^{'}(x) = (-1)^{n+1} \frac{ne^{-nx}}{n^2+1}$. This series converges also uniformly on D with the same arguments gives above. The g is $\mathcal{C}^1$ on D.

$g_n^{''}(x) = (-1)^n \frac{n^2 e^{-nx}}{n^2+1} = -g_n(x) + e^{-nx}$. The series $\sum_{n\geq 0} g_n^{''}(x)$ converges uniformly on any interval $[a,+\infty[$ for all $a>0$. Then g is $\mathcal{C}^2$ on $]0,+\infty[$ and fulfills

$$g^{''}(x) = \frac{1}{1+e^{-x}} - g(x).$$

Since the functions g and $\frac{1}{1+e^{-x}}$ are continuous on D, then is $\mathcal{C}^2$ on D and by induction g is C^∞ on D.

4-2-23 (a) For $x=0$ the series is convergent and for $x\neq 0$, $\lim_{n\to+\infty} n^2 f_n(x) = 0$, the series $\sum_{n\geq 0} f_n$ is pointwise convergent on $\mathbb{R}$.

(b) $f_n(\frac{1}{\sqrt{n}}) = \frac{\sqrt{n}}{e}$, then the series $\sum_{n\geq 0} f_n$ is not normally convergent on $\mathbb{R}$.

(c) On the interval $[a,+\infty[$ the function f_n is decreasing for n large enough, and since the series $\sum_{n\geq 0} f_n(a)$ is convergent, then the series $\sum_{n\geq 0} f_n$ is normally convergent on $[a,+\infty[$.

(d) Since the series $\sum_{n\geq 0} f_n$ is normally convergent on $[a,+\infty[$,

$$\int_a^x f(t)dt = \int_a^x \sum_{n=0}^{+\infty} f_n(t)dt = \sum_{n=0}^{+\infty} \int_a^x f_n(t)dt = -\frac{1}{2}\frac{e^{x^2}}{e^{x^2}-1} + \frac{1}{2}\frac{e^{a^2}}{e^{a^2}-1},$$

which yields that

$$f(x) = \frac{xe^{x^2}}{(e^{x^2}-1)^2}, \quad \text{for } x \neq 0.$$

4-2-24 Let $f_n(x) = \dfrac{x}{1+n^2x^2}$. f_n is odd, then we study the series $\sum_{n\geq 0} f_n$ on $[0,+\infty[$.

(a) $f_n(0) = 0$, $\sum_{n\geq 0} f_n(0)$ is convergent. For $x \neq 0$, $\lim_{n\to+\infty} n^2 f_n(x) = \dfrac{1}{x}$, then $\sum_{n\geq 0} f_n$ is pointwise convergent on $\mathbb{R}$.

(b) On the interval $[a,+\infty[$, the function f_n is decreasing for n large and $\sup_{x\in[a,+\infty[} f_n(x) = f_n(a)$. Since the series $\sum_{n\geq 0} f_n(a)$ is convergent, the series $\sum_{n\geq 0} f'_n$ converges normally on $[a,+\infty[$.

(c) $f'_n(x) = \dfrac{1-n^2x^2}{(1+n^2x^2)^2}$, but $|f'_n(\dfrac{2}{n})| = \dfrac{3}{5}$, then the series $\sum_{n\geq 0} f'_n$ is not uniformly convergent on $\mathbb{R}$.

(d) For $a > 0$, $\sup_{x\in[a,+\infty[} |f'_n(x)| \leq \frac{1}{1+n^2a^2}$. Then the series $\sum_{n\geq 0} f'_n$ converges uniformly on $[a,+\infty[$ and F is $\mathcal{C}^1$ on $\mathbb{R}^*$.
F is odd and if F is differentiable at 0, its derivative at 0 must be 0.
But $kF(\dfrac{1}{k}) \geq \sum_{n=0}^{k} \dfrac{k^2}{n^2+k^2} \geq \dfrac{(k+1)}{2}$. Then F is not differentiable at 0.

4-2-25 (a) $x^{\ln(y)} = e^{\ln(y)\ln(x)} = y^{\ln(x)}$.

(b) $f_n(x) = x^{\ln(n)} = n^{\ln(x)}$. Then the series $\sum_{n\geq 1} f_n(x)$ converges if and only if $\ln x < -1 \iff x < \dfrac{1}{e}$.

(c) i. For $x \in [a,b]$, $f_n(x) = x^{\ln(n)} \leq b^{\ln n}$, then the series $\sum_{n\geq 1} f_n$ is normally convergent on $[a,b]$.

ii. Since f_n are continuous and the series $\sum_{n\geq 1} f_n$ is normally convergent on $[a,b]$, for all $0 < a < b < \dfrac{1}{e}$, f is continuous on $]0, \dfrac{1}{e}[$.

(d) The sequence $(f_n(x))_n$ is decreasing for all $x < \frac{1}{e}$, then $x^{\ln(k+1)} \leq \int_k^{k+1} x^{\ln t} dt \leq x^{\ln k}$ and

$$f(x) - x^{\ln 2} = \sum_{n=1}^{+\infty} x^{\ln(n+1)} \leq \int_1^{+\infty} x^{\ln t} dt \leq \sum_{n=1}^{+\infty} x^{\ln(n)} = f(x).$$

$\displaystyle\int_1^{+\infty} x^{\ln t} dt = -\frac{1}{1+\ln x}$ and $x^{\ln 2} \leq 1$, then

$\forall x \in]0, \dfrac{1}{e}[, \qquad \dfrac{-1}{1+\ln(x)} \leq f(x) \leq \dfrac{\ln(x)}{1+\ln(x)}$

The function f is not bounded on $]0, \dfrac{1}{e}[$ since $\lim_{x\to\frac{1}{e}} f(x) = +\infty$.

4-2-26 (a) The domain of definition of f_n is $\mathbb{R} \setminus \{-n\}$.

(b) The series $\sum_{n\geq 0} f_n$ converges on $D = \mathbb{R} \setminus \{n \in \mathbb{Z};\ n \leq 0\}$.

(c) i. $f(1) = \displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+1)!} = 1 - \frac{1}{e}$.

ii.

$$\begin{aligned} xf(x) - f(x+1) &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(x+n)} - \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(x+n+1)} \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n(x+n-n)}{n!(x+n)} - \sum_{n=0}^{+\infty} \frac{(-1)^n x}{n!(x+n+1)} \\ &= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} - \sum_{n=1}^{+\infty} \frac{(-1)^n}{(n-1)!(x+n)} - \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(x+n+1)} \\ &= \frac{1}{e}. \end{aligned}$$

(d) $f_n'(x) = \dfrac{(-1)^{n+1}}{n!(x+n)^2}$ and $f_n''(x) = \dfrac{2(-1)^n}{n!(x+n)^3}$.

Let K be a compact of D, there exists $N \in \mathbb{N}$ such that $K \subset [-N, N]$. There exists $\delta > 0$ such that $d(-k, K) \geq \delta$, for all $0 \leq k \leq N$. Moreover for $N < p < q$

$$\lim_{p,q\to+\infty} \sum_{n=p}^{q} \sup_{x\in K} |f_n'(x)| \leq \lim_{p,q\to+\infty} \sum_{n=p}^{q} \frac{1}{n!(n-N)^2} = 0$$

and

$$\lim_{p,q\to+\infty} \sum_{n=p}^{q} \sup_{x\in K} |f_n^{''}(x)| \leq \lim_{p,q\to+\infty} \sum_{n=p}^{q} \frac{2}{n!(n-N)^3} = 0.$$

then $\sum_{n\geq 0} f_n^{''}$ converge uniformly on any compact of D, then f is $\mathcal{C}^2$ on D.

4-2-27 (a) i. $f_n'(x) = \dfrac{(-1)^{n+1}\ln^2 n}{n^x}$, and if $x \in [a, b]$, $|f_n'(x)| \leq \dfrac{\ln^2 n}{n^a}$. Since the series $\sum_{n\geq 1} \dfrac{\ln^2 n}{n^a}$ is convergent, then the series $\sum_{n\geq 1} f_n'(x)$ converges normally on the interval $[a, b] \subset]1, +\infty[$.

ii. The function f_n are C^∞ on $\mathbb{R}_+^*$ and $|f_n| \leq f_n'|$, then the series $\sum_{n\geq 1} f_n(x)$ converges and then f is $\mathcal{C}^1$ on $]1, +\infty[$.

(b) i. $f_n'(x) = \dfrac{(-1)^{n+1}\ln^2 n}{n^x}$. The series $\sum_{n\geq 1} f_n(x)$ is alternate, the sequence $(\dfrac{\ln^2 n}{n^x})_n$ is decreasing and converges uniformly on any interval $[\alpha, +\infty[$ to 0. Then the series $\sum_{n\geq 1} f_n'(x)$ converges uniformly on any interval $[\alpha, +\infty[$, with $\alpha > 0$.

ii. f is $\mathcal{C}^1$ on any interval $[\alpha, +\infty[$, then it is $\mathcal{C}^1$ on $]0, +\infty[$.

(c) Let $k \in \mathbb{N}$, $f_n^{(k)}(x) = \dfrac{(-1)^{n+k}\ln^{k+1} n}{n^x}$. The sequence $(\dfrac{\ln^{k+1} n}{n^x})_n$ is decreasing for n large and converges uniformly on any interval $[\alpha, +\infty[$ to 0. Then the series $\sum_{n\geq 1} f_n^{k+1}(x)$ converges uniformly on any interval $[\alpha, +\infty[$, with $\alpha > 0$, which proves that f is C^∞ on $]0, +\infty[$.

4-2-28 (a) i. For $x \geq 0$, $\dfrac{e^{-2nx}}{4n^2-1} \leq \dfrac{1}{4n^2-1}$, then the series $\displaystyle\sum_{n\geq 1} \frac{e^{-2nx}}{4n^2-1}$ converges uniformly on $[0,+\infty[$.

ii. $\displaystyle 3e^{2x}f(x) - 1 = \sum_{n=1}^{+\infty} \frac{3e^{-2(n-1)x}}{4n^2-1} - 1 = \sum_{n=2}^{+\infty} \frac{3e^{-2(n-1)x}}{4n^2-1} \leq 3e^{-2x} \sum_{n=2}^{+\infty} \frac{1}{4n^2-1}$.

iii. Since the series $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{4n^2-1}$ is convergent and $\displaystyle\lim_{x\to+\infty} 3e^{-2x} \sum_{n=2}^{+\infty} \frac{1}{4n^2-1} = 0$, then $f(x) \underset{+\infty}{\approx} \dfrac{1}{3}e^{-2x}$.

(b) Let $\displaystyle g(x) = \sum_{n=1}^{+\infty} \frac{e^{-(2n+1)x}}{2n-1}$.

i. $\displaystyle\lim_{n\to+\infty} n^2 \frac{e^{-(2n+1)x}}{2n-1} = 0$ for all $x > 0$, then the series $\displaystyle\sum_{n\geq 2} \frac{e^{-(2n+1)x}}{2n-1}$ is pointwise convergent on $]0,+\infty[$. Moreover $\dfrac{e^{-(2n+1)x}}{2n-1} \leq \dfrac{e^{-(2n+1)a}}{2n-1}$ for all $x \in [a,+\infty[$. Since the series $\displaystyle\sum_{n\geq 2} \frac{e^{-(2n+1)a}}{2n-1}$ is convergent, then the series $\displaystyle\sum_{n\geq 2} \frac{e^{-(2n+1)x}}{2n-1}$ is uniformly convergent on $[a,+\infty[$ for any $a > 0$.

ii. Let $a > 0$, $\displaystyle e^{3x}g(x) - 1 = \sum_{n=2}^{+\infty} \frac{e^{-2(n-1)x}}{2n-1} \leq e^{-x} \sum_{n=2}^{+\infty} \frac{e^{-(2n-3)a}}{2n-1}$, $\forall x \in [a,+\infty[$.

iii. Since the series $\sum_{n=2}^{+\infty} \dfrac{e^{-(2n-3)a}}{2n-1}$ is convergent and $\displaystyle\lim_{x\to+\infty} e^{-x} \sum_{n=2}^{+\infty} \frac{e^{-(2n-3)a}}{2n-1} = 0$, then $g(x) \underset{+\infty}{\approx} e^{-3x}$.

(c) Let $\displaystyle u(x) = \sum_{n=1}^{+\infty} e^{-(2n-1)x}$.

i. Since $\lim_{n\to+\infty} n^2 e^{-(2n-1)x} = 0$ for all $x > 0$, then the series $\displaystyle\sum_{n\geq 1} e^{-(2n-1)x}$ is pointwise convergent on $]0,+\infty[$ and since the function $x \longmapsto e^{-(2n-1)x}$ is decreasing, this series converges uniformly on $[a,+\infty[$, $\forall a > 0$.

ii. The sequence $(e^{-(2n-1)x})_n$ is a geometric sequence, then $u(x) = \dfrac{e^{-1}}{1-e^{-2x}} = \dfrac{1}{2\sinh x}$.

(d) Let $F(x) = e^{-x}f(x)$ and $G(x) = e^{2x}g(x)$.

i. $F(x) = \displaystyle\sum_{n=1}^{+\infty} \frac{e^{-(2n+1)x}}{4n^2-1}$. Since $\lim_{n\to+\infty} n^2 \frac{e^{-(2n+1)x}}{2n-1} = 0$ for all $x > 0$, then the series $\displaystyle\sum_{n\geq 1} \frac{e^{-(2n+1)x}}{2n-1}$ is pointwise convergent on $]0,+\infty[$ and since the function $x \longmapsto \frac{e^{-(2n+1)x}}{2n-1}$ is decreasing, this series converges uniformly on $[a,+\infty[$, $\forall a > 0$. Then F is differentiable on $]0,+\infty[$ and $F'(x) = -\displaystyle\sum_{n=1}^{+\infty} \frac{e^{-(2n+1)x}}{2n-1} = -g(x)$.

$G(x) = e^{2x}g(x) = \displaystyle\sum_{n=1}^{+\infty} \frac{e^{-(2n-1)x}}{2n-1}$. With the same arguments G is differentiable on $]0,+\infty[$ and $G'(x) = -\displaystyle\sum_{n=1}^{+\infty} e^{-(2n-1)x} = -u(x) = -\frac{1}{2\sinh x}$.

ii. $\displaystyle\int_x^{+\infty} \frac{dt}{\sinh t} = \int_x^{+\infty} \frac{\sinh t dt}{\cosh^2 t - 1} = \frac{1}{2}\ln(\frac{\cosh x + 1}{\cosh x - 1})$.

$\displaystyle\int_x^{+\infty} \frac{1}{t^3} \ln\left(\frac{e^t-1}{e^t+1}\right) dt = \frac{e^{-2x}-1}{4} \ln\left(\frac{e^x-1}{e^x+1}\right) - \frac{e^{-x}}{2}$.

(e) $g(x) = \dfrac{e^{-2x}}{4} \ln(\dfrac{\cosh x + 1}{\cosh x - 1})$

$$\begin{aligned} F(x) &= \frac{1}{4}\int_x^{+\infty} e^{-2t} \ln(\frac{\cosh t - 1}{\cosh t + 1}) dt \\ &\overset{s=e^t}{=} \frac{1}{2}\int_{e^x}^{+\infty} \ln(\frac{s-1}{s+1}) \frac{ds}{s^3} \\ &= \frac{e^{-2x}-1}{4} \ln(\frac{e^x-1}{e^x+1}) - \frac{e^{-x}}{2}. \end{aligned}$$

and $f(x) = \dfrac{\sinh(x)}{2} \ln(\dfrac{e^x-1}{e^x+1}) - \dfrac{1}{2}$.

4-2-29 (a) $\varphi_n(x,t) = \displaystyle\sum_{k=0}^{n} C_n^k e^{\frac{kt}{n}} x^k (1-x)^{n-k} = \sum_{k=0}^{n} C_n^k \left(e^{\frac{t}{n}} x\right)^k (1-x)^{n-k} = (e^{\frac{t}{n}} x + 1 - x)^n$.

i. $\dfrac{\partial \varphi_n}{\partial t}(x,t) = \sum_{k=1}^{n} C_{n-1}^{k-1} e^{\frac{kt}{n}} x^k (1-x)^{n-k}$

and $\dfrac{\partial^2 \varphi_n}{\partial t^2}(x,t)$.

ii. $\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = (x+1-x)^n = 1$,

$\sum_{k=0}^{n} k C_n^k x^k (1-x)^{n-k} = nx \sum_{k=1}^{n} C_{n-1}^{k-1} x^{k-1} (1-x)^{n-k} = nx(x+1-x)^{n-1} = nx$.

$\sum_{k=0}^{n} k^2 C_n^k x^k (1-x)^{n-k} = nx \sum_{k=1}^{n} k C_{n-1}^{k-1} x^{k-1} (1-x)^{n-k} = nx(1-x) \sum_{k=0}^{n-1} (k+1) C_{n-1}^{k} x^k (1-x)^{n-1-k} = nx + n(n-1)x^2$.

$$B_n(x) = \sum_{k=0}^{n} C_n^k f(\frac{k}{n}) x^k (1-x)^{n-k}.$$

$$\begin{aligned}
|f(x) - B_n(x) &= \Big|\sum_{k=0}^{n} C_n^k (f(x) - f(\frac{k}{n})) x^k (1-x)^{n-k}\Big| \\
&\leq \sum_{k=0}^{n} C_n^k \Big|f(x) - f(\frac{k}{n})\Big| x^k (1-x)^{n-k} \\
&= \sum_{|x-\frac{k}{n}|<\alpha} C_n^k \Big|f(x) - f(\frac{k}{n})\Big| x^k (1-x)^{n-k} + \sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k |f(x) - f(\frac{k}{n})| x^k (\\
&\leq \varepsilon + 2||f||_\infty \sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k (1-x)^{n-k}
\end{aligned}$$

$$\sum_{|x-\frac{k}{n}|\geq\alpha} C_n^k x^k (1-x)^{n-k} \leq \frac{1}{\alpha^2} \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2.$$

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x-\frac{k}{n})^2 = x^2 - \frac{2x}{n} \sum_{k=0}^{n} C_n^k k x^k (1-x)^{n-k} + \frac{1}{n^2} \sum_{k=0}^{n} C_n^k k^2 x^k (1-x)^{n-k}.$$

Since $\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} = 1$, then we differentiate with respect to x and we set $h(x) = \sum_{k=0}^{n} C_n^k k x^k (1-x)^{n-k}$, we get: $h(x) = nx$. We reiterate this precess, we get:

$$\sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 = \frac{x(1-x)}{n}.$$

Then

$$\frac{1}{\alpha^2} \sum_{k=0}^{n} C_n^k x^k (1-x)^{n-k} (x - \frac{k}{n})^2 \leq \frac{1}{4n\alpha^2}$$

Then the sequence $(B_n)_n$ converges uniformly to f on $[0, 1]$.

4.8 Solutions of Exercises on Chapter 5

5-1-1 1) Recall that if $h(x) = \sum_{n=1}^{+\infty} \frac{x^{2n-1}}{2n-1}$, with $|x| < 1$, then

$h'(x) = \sum_{n=1}^{+\infty} x^{2n-2} = \frac{1}{1-x^2}$ and $h(x) = \frac{1}{2} \ln \frac{1+x}{1-x}$.

Let $f(x) = \sum_{n=0}^{+\infty} \frac{x^n}{2n-1}$, $R = \lim_{n \to +\infty} \frac{2n+1}{2n-1} = 1$ and for $x \geq 0$ we set

$x = t^2$ $f(x) = -1 + t \sum_{n=1}^{+\infty} \frac{t^{2n-1}}{2n-1} = -1 + \frac{\sqrt{x}}{2} \ln \frac{1+\sqrt{x}}{1-\sqrt{x}}$.

For $x \leq 0$, we set $x = -t^2$.

---

2) Recall that for $|x| < 1$ $\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n$. If $f(x) = \sum_{n=1}^{+\infty} x^n = \frac{1}{1-x}$,

$f'(x) = \sum_{n=1}^{+\infty} n x^{n-1} = \sum_{n=1}^{+\infty} n x^n + \sum_{n=0}^{+\infty} x^n$. Then $\sum_{n=1}^{+\infty} n x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x}$.

$f^{''}(x) = \sum_{n=1}^{+\infty} n^2 x^{n-1} + \sum_{n=1}^{+\infty} n x^{n-1} = \sum_{n=0}^{+\infty} (n^2+2n+1) x^n + \sum_{n=0}^{+\infty} (n+1) x^n$.

Then

$$\sum_{n=1}^{+\infty} n^2 x^n = \frac{2}{(1-x)^3} - 3 \sum_{n=0}^{+\infty} n x^n - 2 \sum_{n=0}^{+\infty} x^n = \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x}.$$

---

3) $R = \lim_{n \to +\infty} \frac{(n^2+1)(n+1)}{(n+1)^2+1} = +\infty$.

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{n^2+1}{n!} x^n &= \sum_{n=1}^{+\infty} \frac{n}{(n-1)!} x^n + e^x = e^x + x \sum_{n=0}^{+\infty} \frac{n+1}{n!} x^n \\
&= e^x + x e^x + x \sum_{n=1}^{+\infty} \frac{x^n}{(n-1)!} \\
&= e^x + x e^x + x^2 \sum_{n=0}^{+\infty} \frac{1}{n!} x^n = e^x (1 + x + x^2).
\end{aligned}$$

---

4) $R = \lim_{n\to+\infty} \frac{(n+2)(n+4)}{(n+1)(n+3)} = 1.$
For $x \neq 0$

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{x^n}{(n+1)(n+3)} &= \frac{1}{2}\sum_{n=0}^{+\infty} \frac{x^n}{n+1} + \frac{1}{2}\sum_{n=0}^{+\infty} \frac{x^n}{n+3} \\
&= \frac{1}{2x}\sum_{n=0}^{+\infty} \frac{x^{n+1}}{n+1} + \frac{1}{2x^3}\sum_{n=0}^{+\infty} \frac{x^{n+3}}{n+3} \\
&= -\frac{1}{2x}\ln(1-x) + \frac{1}{2x^3}(x - \frac{x^2}{2} - \ln(1-x)).
\end{aligned}$$

---

5) If $x^2 = x$, the radius of convergence of the power series $\sum_{n\geq 0} \frac{(-1)^n x^n}{4n^2-1}$ is $R = \lim_{n\to+\infty} \frac{4(n+1)^2-1}{4n^2-1} = 1$. Then the radius of convergence of the power series $\sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{4n^2-1}$ is 1. For $x \neq 0$

$$\frac{d}{dx}\sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n+1}}{4n^2-1} = \sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n}}{2n-1} = x\sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n-1}}{2n-1} = x\tan^{-1} x.$$

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{4n^2-1} &= -1 + \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int_0^x \frac{t^2}{1+t^2}dt \\
&= -1 + \frac{x^2}{2}\tan^{-1} x - \frac{x}{2} + \frac{1}{2}\tan^{-1} x.
\end{aligned}$$

---

6) $R = \lim_{n\to+\infty} \frac{(n+1)\cosh(na)}{n\cosh((n+1)a)} = e^a.$

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{x^n}{n}\cosh(na) &= \frac{1}{2}\sum_{n=1}^{+\infty} \frac{x^n e^{na}}{n} + \frac{1}{2}\sum_{n=1}^{+\infty} \frac{x^n e^{-na}}{n} \\
&= -\frac{1}{2}\ln(e^a - x) - \frac{1}{2}\ln|e^{-a} - x| = -\frac{1}{2}\ln(e^a - x)|e^{-a} - x|.
\end{aligned}$$

---

7) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n \sin n\theta}{2^n} = \operatorname{Im} \sum_{n=1}^{+\infty} \frac{x^n e^{\mathrm{i}n\theta}}{2^n},\ R = 2.$

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{x^n \sin n\theta}{2^n} &= \operatorname{Im} \sum_{n=1}^{+\infty} \frac{x^n e^{\mathrm{i}n\theta}}{2^n} \\
&= \frac{2x\cos\theta}{x^2+4-4x\cos\theta}
\end{aligned}$$

---

8) $\displaystyle\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n2^n} = \operatorname{Re} \sum_{n=1}^{+\infty} \frac{x^n e^{\mathrm{i}n\theta}}{n2^n},\ R = 2.$

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{x^n \cos n\theta}{n2^n} &= \operatorname{Re} \sum_{n=1}^{+\infty} \frac{x^n e^{\mathrm{i}n\theta}}{n2^n} \\
&= \ln 2 - \ln(x^2+4-4x\cos\theta)
\end{aligned}$$

---

9) $R = 2.$

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{nx^n \sin^2(n\theta)}{2^n} &= \sum_{n=1}^{+\infty} \frac{nx^n(1+\cos(2n\theta))}{2^{n+1}} \\
&= \frac{x}{4(1-x)^2} + \operatorname{Re} \frac{xe^{2\mathrm{i}\theta}}{2(2-xe^{2\mathrm{i}\theta})} \\
&= \frac{x}{4(1-x)^2} + \frac{2x\cos(2\theta)-x^2}{2(x^2+4-4x\cos(2\theta))}.
\end{aligned}$$

---

10) $R = 1.$

$$\begin{aligned}
\sum_{n=0}^{+\infty} \frac{n^2+1}{n+1} x^n &= \sum_{n=0}^{+\infty} (n+1)^2 x^n - 2\sum_{n=0}^{+\infty} x^n + 2\sum_{n=0}^{+\infty} \frac{x^n}{n+1} \\
&= -\frac{2}{1-x} - 2\frac{\ln(1-x)}{x}.
\end{aligned}$$

---

11) $R = +\infty$.

$$\sum_{n=0}^{+\infty} \frac{x^n}{(2n)!} = \begin{cases} \cosh(\sqrt{x}) & \text{if } x \geq 0 \\ \cos(\sqrt{-x}) & \text{if } x \leq 0 \end{cases}$$

---

12) $R = +\infty$.

$$\begin{aligned} \sum_{n=0}^{+\infty} \frac{\sin^2(n\theta)}{n!} x^{2n} &= \frac{1}{2} \sum_{n=0}^{+\infty} \frac{1-\cos(2n\theta)}{n!} x^{2n} = \frac{1}{2} e^{x^2} - \frac{1}{2} \operatorname{Re} e^{x^2 e^{2i\theta}} \\ &= \frac{1}{2} e^{x^2} - \frac{1}{2} e^{x^2 \cos(2\theta)} \cos(x^2 \sin(2\theta)). \end{aligned}$$

---

13) $R = 1$

$$\sum_{n=0}^{+\infty} (2n+1) x^n = \frac{2x}{(1-x)^2} + \frac{1}{1-x}.$$

---

14) $R = +\infty$.

If $j = e^{\frac{2i\pi}{3}}$,

$$\sum_{n=0}^{+\infty} \frac{(jx)^n}{n!} = e^{jx} = \sum_{n=0}^{+\infty} \frac{x^{3n}}{(3n)!} + j \sum_{n=0}^{+\infty} \frac{x^{3n+1}}{(3n+1)!} + j^2 \sum_{n=0}^{+\infty} \frac{x^{3n+2}}{(3n+2)!}.$$

---

15) $R = +\infty$

$$\sum_{n=0}^{+\infty} (n^2+1) \frac{x^n}{n!} = e^x (1 + x + x^2).$$

---

16) $R = +\infty$

$$\sum_{n=0}^{+\infty} \frac{x^n \cos n\theta}{n!} = \operatorname{Re} e^{x e^{i\theta}} = e^{x \cos\theta} \cos(x \sin\theta).$$

---

17) $R = +\infty$

$$\sum_{n=0}^{+\infty} \frac{x^n \sin n\theta}{n!} = \operatorname{Im} e^{x e^{i\theta}} = e^{x \cos\theta} \sin(x \sin\theta).$$

---

18) $R=3$. For $x\neq 0$

$$\sum_{n=0}^{+\infty}\frac{nx^n}{3^n(n+1)}=\frac{3}{3-x}+\frac{3\ln(3-x)-3\ln 3}{x}.$$

---

19) $R=1$, and set $x=t^3$. For $x\neq 0$

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{(-1)^n x^n}{3n+1} &= \frac{1}{t}\sum_{n=0}^{+\infty}\frac{(-1)^n t^{3n+1}}{3n+1}\\ &= \frac{1}{t}\int_0^t\frac{ds}{1+s^3}=\frac{1}{3t}\ln(1+t)+\frac{1}{3t}\int_0^t\frac{-bs+2}{1-s+s^2}\\ &= \frac{1}{3t}\ln(1+t)-\frac{1}{6t}\ln(1-t+t^2)+\frac{1}{t\sqrt{3}}\tan^{-1}(\frac{2t-1}{\sqrt{3}})+\frac{\pi}{6t\sqrt{3}}.\end{aligned}$$

5-1-2 (a) Remarque that $u_{n+1}-\sqrt{2}v_{n+1}=(1-\sqrt{2})(u_n-\sqrt{2}v_n)$ and $u_{n+1}+\sqrt{2}v_{n+1}=(1+\sqrt{2})(u_n+\sqrt{2}v_n)$. Then

$$u_n=\sqrt{2}v_n+(1-\sqrt{2})^n,\quad u_n=-\sqrt{2}v_n+(1+\sqrt{2})^n.$$

We deduce that $u_n=\frac{1}{2}(1-\sqrt{2})^n+\frac{1}{2}(1+\sqrt{2})^n$. The radius of convergence of the series $\sum_{n\geq 0}u_nx^n$ is $R=1+\sqrt{2}$ and

$$\sum_{n=0}^{+\infty}u_nx^n=\frac{1}{2}\frac{1}{1-(1-\sqrt{2})x}+\frac{1}{2}\frac{1}{1-(1+\sqrt{2})x}.$$

(b) 1 is the radius of convergence of the power series $\sum_{n\geq 0}a_nx^n$.

$f'(x)=-1-x\tan^{-1}x$, then $f(x)=-\frac{x}{2}-\frac{(1+x^2)}{2}\tan^{-1}(x)$. We can also compute $f(x)$ as follows:

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{(2n-1)(2n+1)} &= \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{2n-1}-\frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\\ &= -\frac{x}{2}-\frac{(1+x^2)}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\\ &= -\frac{x}{2}-\frac{(1+x^2)}{2}\tan^{-1}(x).\end{aligned}$$

$\boxed{\text{5-1-3}}$ (a) If R is the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$, then the power series $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} (-1)^n a_n x^n$ are absolutely convergent if $|x| < R$ and for $|x| > R$ the series are divergent because $|a_n x^n| = |(-1)^n a_n x^n|$.

(b) Yes, because $|a_n x^n| = ||a_n| x^n|$.

(c) No, the domain of convergence of the series $\sum_{n\geq 0} \frac{1}{n} x^n$ is $[-1, 1[$ but the domain of convergence of the series $\sum_{n\geq 0} \frac{(-1)^n}{n} x^n$ is $]-1, 1]$.

(d) No, the radius of convergence of the power series $\sum_{n\geq 0} \frac{x^n}{n!}$ is infinite, but the series is not uniformly convergent on $\mathbb{R}$ since $\sup_{x\in\mathbb{R}} \frac{x^n}{n!} = +\infty$.

(e) Yes, for all $p \in \mathbb{N}$, $f(x) \geq a_{p+1} x^{p+1}$, then $\lim_{x\to+\infty} \frac{f(x)}{x^p} = +\infty$.

$\boxed{\text{5-1-4}}$ $f(x) = \dfrac{x}{1-x-x^2} = \dfrac{-x}{(x-\alpha)(x-\beta)} = \dfrac{a}{x-\alpha} + \dfrac{b}{x-\beta}$,

where $a = -\dfrac{1+\sqrt{5}}{2\sqrt{5}}$, $b = -\dfrac{1-\sqrt{5}}{2\sqrt{5}}$, $\alpha = -\dfrac{1+\sqrt{5}}{2}$, and $\beta = -\dfrac{1-\sqrt{5}}{2}$.

$$
\begin{aligned}
f(x) = \frac{x}{1-x-x^2} &= -\frac{a}{\alpha} \sum_{n=0}^{+\infty} \frac{x^n}{\alpha^n} - \frac{a}{\beta} \sum_{n=0}^{+\infty} \frac{x^n}{\beta^n} \\
&= -\frac{1}{2\sqrt{5}} \sum_{n=0}^{+\infty} \frac{(-2)^n x^n}{(1+\sqrt{5})^n} - \frac{1}{2\sqrt{5}} \sum_{n=0}^{+\infty} \frac{(-2)^n x^n}{(1-\sqrt{5})^n}.
\end{aligned}
$$

$\boxed{\text{5-1-5}}$ 1) $\dfrac{1}{1+x} = \sum_{n=0}^{+\infty} (-1)^n x^n$, $\ln(1+x) = \sum_{n=0}^{+\infty} (-1)^n \dfrac{x^{n+1}}{n+1}$, then the product of the series yields

$$\frac{\ln(1+x)}{1+x} = x \sum_{n=0}^{+\infty} c_n x^n, \text{ with } c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{k+1}.$$

2) The function f fulfills the following differential equation

$$(1-x^2)y'' - xy' = 2.$$

If $f(x) = \sum_{n=0}^{+\infty} a_n x^n$, then $a_0 = a_1 = 0$ and $a_2 = 2$. Moreover $(n+1)(n+2)a_{n+2} = n^2 a_n$ for $n \geq 1$. then Thus $a_{2n} = 0$, $a_{2n+1} = \frac{(2n-1)^2 a_{2n-1}}{2n(2n+1)}$, and $a_{2n+1} = 2\frac{(2^n n!)^2}{(2n+1)!}$. Then

$$(\sin^{-1} x)^2 = \sum_{n=0}^{+\infty} 2\frac{(2^n n!)^2}{(2n+1)!} x^{2n+2}.$$

3) $\frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}} = -\frac{d}{dx}(\sin^{-1}\sqrt{x})^2 = 2\sum_{n=0}^{+\infty} \frac{(2^n(n+1)!)^2}{(2n+1)!} x^{n+1}$

4) If $g(x) = \ln(1 - 2x\cos\alpha + x^2)$, then $g'(x) = \frac{1}{x - e^{\mathrm{i}\alpha}} + \frac{1}{x - e^{-\mathrm{i}\alpha}}$. As

$$\frac{1}{x - e^{\mathrm{i}\alpha}} = -e^{-\mathrm{i}\alpha} \sum_{n=0}^{+\infty} x^n e^{-\mathrm{i}n\alpha}.$$

Then

$$g(x) = \ln(1 - 2x\cos\alpha + x^2) = \sum_{n=0}^{+\infty} -2\frac{x^{n+1}}{n+1}\cos(n+1)\alpha.$$

5) $e^{2x}\cos x = \operatorname{Re} e^{x(2+\mathrm{i})} = \operatorname{Re}\sum_{n=0}^{+\infty} \frac{(2+\mathrm{i})^n}{n!} x^n$. Set $(2+\mathrm{i}) = \sqrt{5}e^{\mathrm{i}\theta}$, thus

$$e^{2x}\cos x = \sum_{n=0}^{+\infty} \frac{(\sqrt{5})^n}{n!} x^n \cos n\theta.$$

5-1-6

1) $\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=0}^{+\infty}(n+1)x^n$, $R = 1$.

2) $\frac{1}{(x-2)(x-3)} = \frac{1}{x-3} - \frac{1}{x-2} = \sum_{n=0}^{+\infty}\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)x^n$, $R = 2$.

3) For $|x| < 1$, $\ln(1 + x + x^2) + \ln(1 - x) = \ln(1 - x^3) = -\sum_{n=1}^{+\infty}\frac{x^{3n}}{n}$.

Then

$$\ln(1 + x + x^2) = -\sum_{n=1}^{+\infty}\frac{x^{3n}}{n} + \sum_{n=1}^{+\infty}\frac{x^n}{n} = \sum_{n=1}^{+\infty} a_n\frac{x^n}{n},$$

where $a_{3n} = 0$, $a_{3n+1} = a_{3n+2} = -1$.

4) $R=+\infty$, we linearize $\sin^3 x$,

$$\begin{aligned}\sin^3 x &= \sin x\Big(\frac{1-\cos(2x)}{2} = \frac{1}{2}\sin x - \frac{1}{2}\sin x\cos(2x)\\ &= \frac{1}{2}\sin x - \frac{1}{4}\sin(3x) + \frac{1}{4}\sin(x) = \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)\end{aligned}$$

Or

$$\begin{aligned}\sin^3 x &= \frac{-1}{8\mathrm{i}}(e^{\mathrm{i}x}-e^{-\mathrm{i}x})^3 = \frac{-1}{8\mathrm{i}}(e^{3\mathrm{i}x}-3e^{\mathrm{i}x}+3e^{-\mathrm{i}x}-e^{-3\mathrm{i}x})\\ &= \frac{1}{4}\left(3\sum_{n=0}^{+\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} - \sum_{n=0}^{+\infty}\frac{(-1)^n 3^{2n+1}x^{2n+1}}{(2n+1)!}\right).\end{aligned}$$

5) $R=+\infty$,

$$\begin{aligned}\sinh^3 x &= \frac{1}{8}(e^{x}-e^{-x})^3 = \frac{1}{8}(e^{3x}-3e^{x}+3e^{-x}-e^{-3x})\\ &= \frac{1}{4}\left(\sum_{n=0}^{+\infty}\frac{3^{2n+1}x^{2n+1}}{(2n+1)!} - 3\sum_{n=0}^{+\infty}\frac{x^{2n+1}}{(2n+1)!}\right).\end{aligned}$$

6) $R=2$,

$$\begin{aligned}(x-1)\ln(x^2-5x+6) &= (x-1)\ln(2-x)(3-x)\\ &= (x-1)(\ln 2 - \sum_{n=1}^{+\infty}\frac{x^n}{n2^n}) + (x-1)(\ln 3 - \sum_{n=1}^{+\infty}\frac{x^n}{n3^n})\\ &= (x-1)\left(\ln 6 - \sum_{n=1}^{+\infty}\left(\frac{1}{2^n}+\frac{1}{3^n}\right)\frac{x^n}{n}\right).\end{aligned}$$

7)

$$x\ln(x+\sqrt{x^2+1}) = \sum_{n=0}^{+\infty}\frac{(-1)^n C_{2n}^n}{4^n}\frac{x^{2n+2}}{2n+1}.$$

8)

$$\begin{aligned}\frac{x-2}{x^3-x^2-x+1} &= \frac{x-2}{(1-x)^2(1+x)} = -\frac{3}{4(1-x)} - \frac{1}{2(1-x)^2} - \frac{3}{4(1+x)}\\ &= -\frac{3}{4}\sum_{n=0}^{+\infty}x^n - \frac{1}{2}\sum_{n=0}^{+\infty}(n+1)x^n - \frac{3}{4}\sum_{n=0}^{+\infty}(-1)^n x^n.\end{aligned}$$

9)

$$\begin{aligned}
\frac{1}{1+x-2x^3} &= \frac{1}{(1-x)(1+2x+2x^2)} = \frac{1}{5(1-x)} + \frac{x+4}{5(1+2x+2x^2)} \\
&= \frac{1}{5}\sum_{n=0}^{+\infty} x^n + \frac{1}{10}\sum_{n=0}^{+\infty}(-1)^n 2^{\frac{n+1}{2}}(\cos(n+1)\frac{\pi}{4} + 7\sin(n+1)\frac{\pi}{4})x^n
\end{aligned}$$

10)

$$\begin{aligned}
\frac{1-x}{(1+2x-x^2)^2} &= -\frac{1}{2}\frac{d}{dx}\frac{1}{1+2x-x^2} = -\frac{d}{dx}\sum_{n=0}^{\infty}(\frac{(2-\sqrt{2})^n}{2^{n+2}} + \frac{(2+\sqrt{2})^n}{2^{n+2}})x^n \\
&= -\sum_{n=0}^{\infty} n(\frac{(2-\sqrt{2})^n}{2^{n+2}} + \frac{(2+\sqrt{2})^n}{2^{n+2}})x^{n-1}.
\end{aligned}$$

11)

$$\begin{aligned}
\tan^{-1}(x+1) &= \frac{\pi}{4} + \int_0^x \frac{dt}{2+2t+t^2} \\
&= \frac{\pi}{4} + \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{1+\mathrm{i}+t} - \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{1-\mathrm{i}+t} \\
&= \frac{\pi}{4} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(n+1)(1+\mathrm{i})^{n+1}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(n+1)(1-\mathrm{i})^{n+1}} \\
&= \frac{\pi}{4} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n e^{-\mathrm{i}\frac{(n+1)\pi}{4}} x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n e^{\mathrm{i}\frac{(n+1)\pi}{4}} x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} \\
&= \frac{\pi}{4} + \sum_{n=0}^{+\infty}\frac{(-1)^n \sin(\frac{(n+1)\pi}{4})x^{n+1}}{(n+1)2^{\frac{n+1}{2}}}
\end{aligned}$$

12)

$$\begin{aligned}
\tan^{-1}(x+\sqrt{3}) &= \frac{\pi}{3} + \int_0^x \frac{dt}{4+2\sqrt{3}t+t^2} \\
&= \frac{\pi}{3} + \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{\sqrt{3}+\mathrm{i}+t} - \frac{\mathrm{i}}{2}\int_0^x \frac{dt}{\sqrt{3}-\mathrm{i}+t} \\
&= \frac{\pi}{3} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(n+1)(\sqrt{3}+\mathrm{i})^{n+1}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n x^{n+1}}{(n+1)(\sqrt{3}-\mathrm{i})^{n+1}} \\
&= \frac{\pi}{3} + \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n e^{-\mathrm{i}\frac{(n+1)\pi}{6}} x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} - \frac{\mathrm{i}}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n e^{\mathrm{i}\frac{(n+1)\pi}{6}} x^{n+1}}{(n+1)2^{\frac{n+1}{2}}} \\
&= \frac{\pi}{3} + \sum_{n=0}^{+\infty}\frac{(-1)^n \sin(\frac{(n+1)\pi}{6})x^{n+1}}{(n+1)2^{\frac{n+1}{2}}}
\end{aligned}$$

13) $R = \frac{1}{2}$.

$$\begin{aligned}
\int_0^x \frac{\ln(t^2 - \frac{5}{2}t + 1)}{t}\, dt &= \int_0^x \frac{\ln(2-t)(\frac{1}{2}-t)}{t}\, dt \\
&= -\sum_{n=1}^{+\infty} \int_0^x \left(\frac{t^{n-1}}{n2^n} + \frac{2^n t^{n-1}}{n}\right) dt \\
&= -\sum_{n=1}^{+\infty} \left(\frac{x^n}{n^2 2^n} + \frac{2^n x^n}{n^2}\right) dt.
\end{aligned}$$

14) $\left(\frac{(1+x)\sin x}{x}\right)^2 = (1+x)^2 \frac{1-\cos(2x)}{2x^2} = \frac{1}{2}(1+x)^2 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1} x^{2(n-1)}}{2n!}$.

$R = +\infty$.

15) $R = +\infty$.

$\int_x^{2x} e^{-t^2}\, dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(2n+1)} (2^{2n+1} - 1) x^{2n+1}$

16) $R = +\infty$.

$e^{-2x^2} \int_0^x e^{2t^2}\, dt = \sum_{n=0}^{+\infty} c_n x^{2n+1}$, where

$c_n = \frac{2^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(2k+1)(n-k)} = \frac{2^n}{n!(n+1)}$.

17) $R = 1$.

$\frac{e^x}{1-x} = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} c_n x^n$, where $c_n = \sum_{k=0}^{n} \frac{1}{k!}$.

18) $R = 1$.

$\frac{e^{x^2}}{1-x} = \sum_{n=0}^{+\infty} \frac{x^{2n}}{n!} \sum_{n=0}^{+\infty} x^n = \sum_{n=0}^{+\infty} c_n x^n$, where $c_{2n} = \sum_{k=0}^{n} \frac{1}{k!}$, and $c_{2n+1} = \sum_{k=0}^{n} \frac{1}{k!}$.

19) $R = +\infty$.

$\cos t = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{2n!}$ and

$\int_0^x \frac{\cos t - 1}{t^2} dt = \sum_{n=1}^{+\infty} \frac{(-1)^n x^{2n-1}}{2n!(2n-1)}$.

20) $R = 1$.
$$\ln\left(\frac{1+x}{2-x}\right) = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}x^n}{n} - \ln 2 + \sum_{n=1}^{+\infty}\frac{x^n}{n2^n}.$$
21) If $a > 0$, $R = e^{-a}$.

$$\begin{aligned}
\ln\sqrt{1-2x\cosh a + x^2} &= \frac{1}{2}\ln(e^{-a}-x)(e^{a}-x) \\
&= -\frac{1}{2}\sum_{n=1}^{+\infty}\frac{e^{na}x^n}{n} - \frac{1}{2}\sum_{n=1}^{+\infty}\frac{e^{-na}x^n}{n} \\
&= -\sum_{n=1}^{+\infty}\frac{\cosh(na)x^n}{n}.
\end{aligned}$$

5-1-7 (a) We know that $\sin^{-1}x = \sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}\frac{x^{2n+1}}{2n+1}$ for $|x| < 1$ and $\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{+\infty}\frac{C_{2n}^n}{4^n}x^{2n}$ for $|x| < 1$, then the expansion in power series in a neighborhood of 0 of f is the power series product of these series for $|x| < 1$.

(b) $(1-x^2)f'(x) - xf(x) = 1$.

Since f is odd, $f(x) = \sum_{n=0}^{+\infty}a_n x^{2n+1}$, then for all $n \geq 1$, $a_n = \frac{2n}{2n+1}a_{n-1} = \frac{(2^n n!)^2}{(2n+1)!}$.
$$f(x) = \sum_{n=0}^{+\infty}\frac{(2^n n!)^2}{(2n+1)!}x^{2n+1}.$$
(c) If $g(x) = (\sin^{-1})^2(x)$, $g'(x) = 2f(x)$ and
$$(\sin^{-1})^2(x) = 2\sum_{n=0}^{+\infty}\frac{(2^n n!)^2}{(2n+2)!}x^{2n+2}.$$

5-1-8 (a) $f(x) = \cos x$, $(x_0 = \frac{\pi}{4})$,

$$\begin{aligned}
\cos x &= \cos(x - \frac{\pi}{4} + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\cos(x-\frac{\pi}{4}) - \sin(x-\frac{\pi}{4})) \\
&= \frac{\sqrt{2}}{2}(\sum_{n=0}^{+\infty}\frac{(-1)^n(x-\frac{\pi}{4})^{2n}}{(2n)!} - \sum_{n=0}^{+\infty}\frac{(-1)^n(x-\frac{\pi}{4})^{2n+1}}{(2n+1)!}).
\end{aligned}$$

(b) $f(x) = (1-x^3)^{-\frac{1}{2}}$, $(x_0 = 0)$,

$$(1-x^3)^{-\frac{1}{2}} = \sum_{n=0}^{+\infty} \frac{C_{2n}^n}{4^n} x^{3n}.$$

5-1-9 The power series $\sum_{n\geq 0} a_n x^n$ is convergent if and only if the power series $\sum_{n\geq 0} a_{2n} x^{2n}$ and the series $\sum_{n\geq 0} a_{2n+1} x^{2n+1}$ is convergent. Then the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is $\inf(\sqrt{R}, \sqrt{R'})$.

5-1-10 (a) By the Abel Lemma (1.2) the power series $\sum_{n\geq 0} a_n x^n$ is convergent for $|x| < 1$. If the radius of convergence of the power series $\sum_{n\geq 0} a_n x^n$ is $R > 1$, then the series $\sum_{n\geq 0} a_n$ is absolutely convergent, which is absurd, then $R = 1$.

(b) For $x = 1$, the power series $\sum_{n\geq 0} a_n x^n$ is divergent, but for $x = -1$, the power series $\sum_{n\geq 0} a_n x^n$ is convergent, by the Abel Lemma (1.2).

5-1-11 (a) i.

$$\begin{aligned}
S_p(x) - S_n(x) &= \sum_{k=n+1}^{p} a_k x^k = \sum_{k=n+1}^{p} (R_{k-1} - R_k) x^k \\
&= \sum_{k=n}^{p} R_k x^{k+1} - \sum_{k=n+1}^{p} R_k x^k \\
&= R_n x^{n+1} - R_p x^p + \sum_{k=n+1}^{p-1} R_k (x^{k+1} - x^k).
\end{aligned}$$

ii. $\sup_{x\in[0,1]} |S_p(x) - S_n(x)| \leq |R_n| + |R_p| + \sup_{k\geq n+1} |R_k| \sup_{x\in[0,1]} \sum_{k=n+1}^{p-1} (x^k - x^{k+1}) \leq |R_n| + |R_p| + \sup_{k\geq n+1} |R_k|$. Then $\lim_{n,p\to+\infty} \sup_{x\in[0,1]} |S_p(x) - S_n(x)| = 0$.

(b) i. Define $a_n = b_n x_0^n$. The series $\sum\limits_{n\geq 0} a_n$ is convergent. $f(x) = \sum\limits_{n=0}^{+\infty} b_n x^n = \sum\limits_{n=0}^{+\infty} a_n (\frac{x}{x_0})^n$. Then by the previous question f is continuous on the interval $[0, x_0]$ and then $\lim\limits_{\substack{x \longmapsto x_0 \\ x\in[0,x_0]}} f(x) = \sum\limits_{n=0}^{+\infty} b_n x_0^n$.

ii. On the interval $[0,1[$, $\ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}x^n}{n}$ and the series $\sum\limits_{n\geq 1} \frac{(-1)^n}{n}$ is convergent, then $\sum\limits_{n=1}^{+\infty} \frac{(-1)^n}{n} = -\ln 2$.

5-1-12 (a) The power series $\sum\limits_{n\geq 0} \frac{x^{4n}}{(4n)!}$ is convergent on $\mathbb{R}$.

$f(x) = \sum\limits_{n=0}^{+\infty} \frac{x^{4n}}{(4n)!}$, $f'(x) = \sum\limits_{n=1}^{+\infty} \frac{x^{4n-1}}{(4n-1)!}$, $f''(x) = \sum\limits_{n=1}^{+\infty} \frac{x^{4n-2}}{(4n-2)!}$, $f'''(x) = \sum\limits_{n=1}^{+\infty} \frac{x^{4n-3}}{(4n-3)!}$, $f^{(4)}(x) = \sum\limits_{n=1}^{+\infty} \frac{x^{4n-4}}{(4n-4)!} = \sum\limits_{n=0}^{+\infty} \frac{x^{4n}}{(4n)!} = f(x)$.

(b) The power series $\sum\limits_{n\geq 0} \frac{x^n}{(n!)^2}$ is convergent on $\mathbb{R}$.

$f(x) = \sum\limits_{n=0}^{+\infty} \frac{x^n}{(n!)^2}$, $f'(x) = \sum\limits_{n=1}^{+\infty} \frac{x^{n-1}}{n!(n-1)!}$, $f''(x) = \sum\limits_{n=2}^{+\infty} \frac{x^{n-2}}{n!(n-2)!}$.

$$\begin{aligned} xf''(x) + f'(x) - f(x) &= \sum_{n=1}^{+\infty} \frac{(n-1)x^{n-1}}{n!(n-1)!} + \sum_{n=1}^{+\infty} \frac{x^{n-1}}{n!(n-1)!} - \sum_{n=0}^{+\infty} \frac{x^n}{(n!)^2} \\ &= \sum_{n=1}^{+\infty} \frac{x^{n-1}}{((n-1)!)^2} - \sum_{n=0}^{+\infty} \frac{x^n}{(n!)^2} = 0. \end{aligned}$$

(c) The power series $\sum\limits_{n\geq 0} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$ is convergent on $\mathbb{R}$.

$f(x) = \sum\limits_{n=0}^{+\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$, $f'(x) = \sum\limits_{n=1}^{+\infty} \frac{(-1)^n 2^{2n} x^{2n-1}}{(2n-1)!}$, $f''(x) = \sum\limits_{n=1}^{+\infty} \frac{(-1)^n 2^{2n} x^{2n-2}}{(2n-2)!} = -\sum\limits_{n=0}^{+\infty} \frac{(-1)^n 2^{2n+2} x^{2n}}{(2n)!} = -4f(x)$.

$f(x) = \cos(2x)$.

5-1-13 (a) Let $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$, $y''(x) = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2}$, $x^2 y''(x) = \sum_{n=0}^{+\infty} n(n-1)a_n x^n$, $xy''(x) = \sum_{n=1}^{+\infty} n(n+1)a_{n+1}x^n$ and $xy'(x) = \sum_{n=0}^{+\infty} n a_n x^n$.

$$x(x-1)y'' + 3xy' + y = a_0 + \sum_{n=1}^{+\infty}(n+1)((n+1)a_n - n a_{n+1})x^n.$$

Then $a_0 = 0$ and $(n+1)a_n - n a_{n+1} = 0$ for all $n \geq 1$, which yields that

$$y(x) = a_1 \sum_{n=1}^{+\infty} n x^n = \frac{a_1 x}{(1-x)^2}.$$

(b) The radius of convergence of the obtained series is 1.

5-1-14 (a) Let $y(x) = \sum_{n=0}^{+\infty} a_n x^{2n}$, $y'(x) = \sum_{n=1}^{+\infty} 2n a_n x^{2n-1}$, $y''(x) = \sum_{n=1}^{+\infty} 2n(2n-1)a_n x^{2n-2} = \sum_{n=0}^{+\infty} 2(n+1)(2n+1)a_{n+1}x^{2n}$ and $xy'(x) = \sum_{n=0}^{+\infty} 2n a_n x^{2n}$.

$$y''(x) - 2xy'(x) + 2\lambda y(x) = \sum_{n=0}^{+\infty} 2((n+1)(2n+1)a_{n+1} - (2n-\lambda)a_n)x^{2n}.$$

y is a solution of (6.6) with $a_0 = 1$ if and only if

$$(n+1)(2n+1)a_{n+1} = (2n-\lambda)a_n, \tag{7.12}$$

for all $n \in \mathbb{N}$. This condition gives a unique even solution.

(b) Let $y(x) = \sum_{n=0}^{+\infty} a_n x^{2n+1}$, $y'(x) = \sum_{n=0}^{+\infty}(2n+1)a_n x^{2n}$, $y''(x) = \sum_{n=1}^{+\infty} 2n(2n+1)a_n x^{2n-1} = \sum_{n=0}^{+\infty} 2(n+1)(2n+3)a_{n+1}x^{2n+1}$, and $xy'(x) = \sum_{n=0}^{+\infty}(2n+1)a_n x^{2n+1}$.

$$y''(x) - 2xy'(x) + 2\lambda y(x) = \sum_{n=0}^{+\infty} 2((n+1)(2n+3)a_{n+1} - ((2n+1)-\lambda)a_n)x^{2n+1}.$$

y is a solution of (6.6) with $a_0 = 1$ if and only if

$$(n+1)(2n+3)a_{n+1} = ((2n+1)-\lambda)a_n, \tag{7.13}$$

for all $n \in \mathbb{N}$. This condition gives a unique odd solution.

(c) The conditions (7.12) and (7.13) yield that if $\lambda \in \mathbb{N}$ is an integer, then the equation (6.6) has a non vanishing polynomial solution.

5-1-15 (a) i. Let $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1)a_{n+1}x^n$

$xy(x) = \sum_{n=0}^{+\infty} a_n x^{n+1} = \sum_{n=1}^{+\infty} a_{n-1}x^n$, and $a_0 = 1$.

$$y' - 2xy = a_1 + \sum_{n=1}^{+\infty} ((n+1)a_{n+1} - 2a_{n-1})x^n.$$

y is a solution of the differential equation $y' - 2xy = 0$, $y(0) = 1$ if and only if $a_1 = 0$, $a_0 = 1$ and $(n+1)a_{n+1} = 2a_{n-1}$ for all $n \geq 1$. Then $a_{2n+1} = 0$, $a_{2n} = \frac{1}{n!}$ and $y = e^{x^2}$.

ii. Let $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$, $xy'(x) = \sum_{n=0}^{+\infty} n a_n x^n$

$y''(x) = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{+\infty} (n+1)(n+2)a_{n+2}x^n$.

$$y'' + xy' + y = \sum_{n=0}^{+\infty} (n+1)((n+2)a_{n+2} + a_n)x^n.$$

y is a solution of the differential equation $y'' + xy' + y = 0$ if and only if $(n+2)a_{n+2} + a_n = 0$ for all $n \in \mathbb{N}$. Then $a_{2n} = \dfrac{(-1)^n}{2^n n!}a_0$, $a_{2n+1} = \dfrac{(-1)^n 2^n n!}{(2n+1)!}a_1$ and

$$y(x) = a_0 \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!}x^{2n} + a_1 \sum_{n=0}^{+\infty} \frac{(-1)^n 2^n n!}{(2n+1)!}x^{2n+1}.$$

iii. Let $y(x) = \sum_{n=0}^{+\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} = \sum_{n=0}^{+\infty} (n+1) a_{n+1} x^n$

$y''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2}$, and $xy''(x) = \sum_{n=0}^{+\infty} n(n+1) a_{n+1} x^n$.

$$4xy'' + 2y' - y = \sum_{n=0}^{+\infty} (2(n+1)(2n+1)a_{n+1} - a_n) x^n.$$

y is a solution of the differential equation $4xy'' + 2y' - y = 0$ if and only if $2(n+1)(2n+1)a_{n+1} - a_n = 0$ for all $n \in \mathbb{N}$. Then $a_n = \dfrac{a_0}{2n!}$ and

$$y(x) = a_0 \sum_{n=0}^{+\infty} \frac{1}{2n!} x^n = a_0 \cosh(\sqrt{x}).$$

(b) $\displaystyle\int_0^x e^{\frac{t^2}{2}} dt = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}$ and $e^{\frac{-x^2}{2}} = \displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{2^n n!}$. Then

$f(x) = x \displaystyle\sum_{n=0}^{+\infty} c_n x^{2n}$, where $c_n = \dfrac{(-1)^n}{2^n n!} \displaystyle\sum_{k=0}^{n} \frac{C_n^k}{2k+1}$.

We remark that $(1+t^2)^n = \displaystyle\sum_{k=0}^{n} C_n^k t^{2k}$ and $\displaystyle\int_0^x (1+t^2)^n dt = \sum_{k=0}^{n} \frac{C_n^k x^{2k+1}}{2k+1}$.

Then

$$c_n = \frac{(-1)^n}{2^n n!} \int_0^1 (1+t^2)^n dt.$$

If $I_n = \displaystyle\int_0^1 (1+t^2)^n dt$, by an integration by parts, I_n fulfills the following induction relation $(2n+1)I_n = 2^n + 2nI_{n-1}$.

5-1-16 Define $u_n(x) = (-1)^n \dfrac{x^n}{n(n-1)}$, for $n \geq 2$.

(a) The radius of convergence of the power series $\displaystyle\sum_{n\geq 2} u_n(x)$ is 1 and the series converges on the interval $[-1, 1]$.

(b) $u_n'(x) = (-1)^n \dfrac{x^{n-1}}{(n-1)}$ and $u_n''(x) = (-1)^n x^{n-2}$.

(c) $\displaystyle\sum_{n=2}^{+\infty} u_n''(x) = \frac{1}{1+x}$, $\displaystyle\sum_{n=2}^{+\infty} u_n'(x) = \ln(1+x)$ and $\displaystyle\sum_{n=2}^{+\infty} u_n(x) = \ln(1+x) - x$.

5-1-17 (a) i.

$$\begin{aligned}
F(x) &= \sum_{n=0}^{+\infty} a_n x^n \\
&= 1 + 2x + \sum_{n=0}^{+\infty} a_{n+2} x^{n+2} \\
&= 1 + 2x + 7x \sum_{n=0}^{+\infty} a_{n+1} x^{n+1} - 12x^2 \sum_{n=0}^{+\infty} a_n x^n \\
&= 1 + 2x - 12x^2 F(x) + 7x \sum_{n=1}^{+\infty} a_n x^n \\
&= 1 + 2x - 12x^2 F(x) + 7x(F(x) - 1).
\end{aligned}$$

Then $F(x) = \dfrac{1-5x}{1-7x+12x^2} = \dfrac{2}{1-3x} - \dfrac{1}{1-4x}$.

ii. $F(x) = \dfrac{2}{1-3x} - \dfrac{1}{1-4x} = 2\sum_{n=0}^{+\infty} 3^n x^n - \sum_{n=0}^{+\infty} 4^n x^n = \sum_{n=0}^{+\infty} (2.3^n - 4^n) x^n$. Then $a_n = (2.3^n - 4^n)$ for all $n \in \mathbb{N}$.

(b) Consider the function $G(x) = \sum_{n=0}^{+\infty} a_n x^n$.

$$\begin{aligned}
G(x) &= \sum_{n=0}^{+\infty} a_n x^n \\
&= 1 + 2x + \sum_{n=0}^{+\infty} a_{n+2} x^{n+2} \\
&= 1 + 2x + 7x \sum_{n=0}^{+\infty} a_{n+1} x^{n+1} - 12x^2 \sum_{n=0}^{+\infty} a_n x^n + \sum_{n=0}^{+\infty} n x^n \\
&= 1 + 2x - 12x^2 G(x) + 7x \sum_{n=1}^{+\infty} a_n x^n + \frac{x}{(1-x)^2} \\
&= 1 + 2x - 12x^2 G(x) + 7x(G(x) - 1) + \frac{x}{(1-x)^2}.
\end{aligned}$$

Then $G(x) = -\dfrac{1}{4(1-3x)} + \dfrac{7}{9(1-4x)} + \dfrac{11}{36(1-x)} + \dfrac{1}{6(1-x)^2}$.

$$\begin{aligned}G(x) &= -\frac{1}{4}\sum_{n=0}^{+\infty} 3^n x^n + \frac{7}{9}\sum_{n=0}^{+\infty} 4^n x^n + \frac{11}{36}\sum_{n=0}^{+\infty} x^n + \frac{1}{6}\sum_{n=0}^{+\infty}(n+1)x^n \\ &= \sum_{n=0}^{+\infty}\left(\frac{n+1}{6} + \frac{11}{36} + \frac{7.4^n}{9} - \frac{3^n}{4}\right)x^n.\end{aligned}$$

Then $a_n = (\frac{n+1}{6} + \frac{11}{36} + \frac{7.4^n}{9} - \frac{3^n}{4})$ for all $n \in \mathbb{N}$.

(c) Consider the function $H(x) = \sum_{n=0}^{+\infty} a_n x^n$.

$$\begin{aligned}H(x) &= \sum_{n=0}^{+\infty} a_n x^n \\ &= 1 + 2x + \sum_{n=0}^{+\infty} a_{n+2}x^{n+2} \\ &= 1 + 2x + 8x\sum_{n=0}^{+\infty} a_{n+1}x^{n+1} - 16x^2\sum_{n=0}^{+\infty} a_n x^n \\ &= 1 + 2x - 12x^2 H(x) + 8x\sum_{n=1}^{+\infty} a_n x^n \\ &= 1 + 2x - 12x^2 H(x) + 8x(H(x) - 1).\end{aligned}$$

Then $H(x) = \dfrac{1-6x}{12x^2 - 8x + 1} = \dfrac{1}{(1-2x)} = \sum_{n=0}^{+\infty} 2^n x^n$ and $a_n = 2^n$ for all $n \in \mathbb{N}$.

5-1-18 (a) If $a > 0$, $R = \dfrac{1}{\overline{\lim}_{n\to+\infty} \frac{|a_n|^{\frac{1}{n}}}{(n!)^{\frac{1}{n}}}} = +\infty$.

If $a = 0$, $R \geq \dfrac{1}{\overline{\lim}_{n\to+\infty} \frac{1}{(n!)^{\frac{1}{n}}}} = +\infty$.

Then the radius of convergence of the power series $\sum_{n\geq 0} \frac{a_n x^n}{n!}$ is $+\infty$.

(b) For $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, $a-\varepsilon \leq a_n \leq a+\varepsilon$.

$$(a-\varepsilon)e^{-t}(e^t - \sum_{n=0}^{N-1}\frac{a_n t^n}{n!}) \leq e^{-t}\sum_{N}^{+\infty}\frac{a_n t^n}{n!} \leq (a+\varepsilon)e^{-t}(e^t - \sum_{n=0}^{N-1}\frac{a_n t^n}{n!})$$

Then $\lim_{t \longmapsto +\infty} e^{-t} f(t) = a$.

$\boxed{\text{5-1-19}}$ $y = \sum_{n=0}^{+\infty} a_n x^n$, $y' = \sum_{n=1}^{+\infty} n a_n x^{n-1}$

$$3xy' + (2-5x)y - x = 2a_0 + \sum_{n=1}^{+\infty} ((3n+2)a_n - 5a_{n-1})x^n - x.$$

y is a solution of the differential equation if and only if $a_0 = 0$, $a_1 = \frac{1}{5}$ and
$(3n+2)a_n - 5a_{n-1} = 0$ for all $n \geq 2$. Then

$$a_n = \frac{5^{n-1}}{\prod_{k=2}^{n}(3k+2)} a_1 = \frac{5^{n-2}}{\prod_{k=2}^{n}(3k+2)}, \quad n \geq 2.$$

The radius of convergence is $+\infty$ since $\frac{a_{n-1}}{a_n} = \frac{3n+2}{5}$.

$\boxed{\text{5-1-20}}$ (a) $y(x) = \sum_{n=1}^{+\infty} a_n x^n$, $y'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2}$.

$$\begin{aligned} x^2 y'' + xy' - (x^2+x+1)y &= \sum_{n=2}^{+\infty} (n^2-1)a_n x^n - \sum_{n=2}^{+\infty} a_{n-2} x^n - \sum_{n=2}^{+\infty} a_{n-1} x^n \\ &= \sum_{n=2}^{+\infty} ((n^2-1)a_n - a_{n-2} - a_{n-1})x^n. \end{aligned}$$

If $\varphi(x) = \sum_{n=1}^{+\infty} a_n x^n$ is a solution of the equation (6.1) with $\varphi'(0) = 1$, then $(n^2-1)a_n - a_{n-2} - a_{n-1} = 0$ for all $n \geq 2$, $a_1 = 1$ and $a_0 = 0$, then $a_n = \frac{2n(n+1)}{((n+1)!)^2}$.

(b) $a_n = \frac{2n(n+1)}{((n+1)!)^2} = \frac{2}{(n-1)!(n+1)!} \leq \frac{1}{(n-1)!}$. Then the radius of convergence of φ is $+\infty$.

(c) $y = \frac{e^{-x}}{x} z$, $y' = \frac{e^{-x}}{x} z' - \frac{e^{-x}(1+x)}{x^2} z$ and $y'' = \frac{e^{-x}}{x} z'' - 2\frac{e^{-x}(1+x)}{x^2} z' + \frac{e^{-x}(2+2x+x^2)}{x^3} z$.

$x^2y'' + xy' - (x^2+x+1)y = e^{-x}\left(xz'' - (1+2x)z'\right)$. Then $z = \lambda e^{2x}(2x-1) + C$ and $y = (e^x - \dfrac{\sinh x}{x})$.

$\boxed{\text{5-1-21}}$ (a) $a_0 = y(0) = 0$, $a_1 = y'(0) = 0$ and $a_2 = \lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$.

(b) $y(x) = \sum_{n=2}^{+\infty} a_n x^n$,

$$x^2y'(x) - y(x) + x^2 = \sum_{n=3}^{+\infty}((n-1)a_{n-1} - a_n)x^n.$$

Then $a_{n+1} = na_n$ for all $n \geq 2$.

(c) The previous question yields that $a_n = (n-1)!$. The series $\sum_{n\geq 4} n!x^n$ diverges for $x \neq 0$.

4.9 Solutions of Exercises on Chapter 6

6-1-1 Let $t \in \mathbb{R} \setminus \mathbb{Z}$ and $f(x) = \cos tx$, for $-\pi \le x \le \pi$ and $2\pi-$periodic.

(a) $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \cos tx dx = \dfrac{\sin \pi t}{\pi t}$ and for $n \ge 1$, $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \cos tx \cos nx dx = \dfrac{(-1)^n 2t \sin \pi t}{\pi(t^2 - n^2)}$.

(b) Since the function f is continuous and piecewise differentiable, then for any $x \in [-\pi, \pi]$,

$$\cos tx = \frac{\sin \pi t}{\pi t}\Big(1 + \sum_{n=1}^{+\infty} \frac{(-1)^n 2t^2}{t^2 - n^2} \cos(nx)\Big).$$

(c) i. We take this relation at $x = 0$, we find that for $t \notin \mathbb{Z}$:

$$\frac{\pi}{\sin \pi t} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{(-1)^n 2t}{t^2 - n^2}.$$

ii. We take the same relation at $x = \pi$, we find that for $t \notin \mathbb{Z}$:

$$\pi \text{cotan} \pi t = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}.$$

iii. Since $\pi \text{cotan} \pi t = \dfrac{1}{t} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2t}{t^2 - n^2} = \dfrac{1}{t} + \displaystyle\sum_{n=1}^{\infty} (\dfrac{1}{t-n} + \dfrac{1}{t+n})$ and the series $\displaystyle\sum_{n \ge 1} (\dfrac{1}{t-n} + \dfrac{1}{t+n})$ is pointwise convergent on $\mathbb{R} \setminus \mathbb{Z}$ and the series of the derivative converges uniformly on any compact subset of $\mathbb{R} \setminus \mathbb{Z}$, thus

$$\frac{\pi^2}{\sin^2 \pi t} = \frac{1}{t^2} + \sum_{n=1}^{\infty} (\frac{1}{(t-n)^2} + \frac{1}{(t+n)^2}) = \sum_{-\infty}^{+\infty} \frac{1}{(t+n)^2}.$$

(The series $\displaystyle\sum_{n \in \mathbb{Z}} \dfrac{1}{(t+n)^2}$ converges uniformly on any compact subset $K \subset \mathbb{R} \setminus \mathbb{Z}$, because $\lim_{N \to +\infty} \sup_{z \in K} \sum_{|n| \ge N} \frac{1}{|z+n|^2} \le \lim_{N \to +\infty} \sum_{|n| \ge N} \frac{1}{(n-R)^2} = 0$, with $K \subset D(0, R)$.)

6-1-2 (a) $a_0 = \frac{2}{\pi}\int_0^{2\delta} \frac{2\pi}{\delta}(1-\frac{x}{2\delta})dx = 8$. $a_n = \frac{4\sin^2(n\delta)}{n^2\delta^2}$.

The series $4+\sum_{n\geq 1}\frac{4\sin^2(n\delta)}{n^2\delta^2}\cos(nx)$ converges uniformly on $\mathbb{R}$. Since f is continuous and piecewise differentiable continuous, the Fourier series of f converges to f.

$$\frac{2\pi}{\delta}(1-\frac{x}{2\delta}) = 4+\sum_{n=1}^{+\infty}\frac{4\sin^2(n\delta)}{n^2\delta^2}\cos(nx), \quad \forall 0\leq x\leq 2\delta. \quad (7.14)$$

(b) For $x=0$, the relation (7.14) yields $\sum_{n=1}^{+\infty}\frac{\sin^2 n\delta}{n^2} = \frac{\delta(\pi-2\delta)}{2}$. For the second sum we use the Parseval identity.

$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{8\pi}{3\delta} = \frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{+\infty}a_n^2$. Then

$$\sum_{n=1}^{+\infty}\frac{\sin^4 n\delta}{n^4} = \frac{(\pi-6\delta)\delta^3}{3}.$$

6-1-3 We recall some results on the Fejer Theorem.

Let $S_N(x) = \frac{a_0}{2}+\sum_{n=1}^{N}(a_n\cos(nx)+b_n\sin(nx))$, be the partial sum of the Fourier series of a piecewise continuous $2\pi-$periodic function f (we can take a Riemann integrable function on $[0,2\pi]$) and $\Lambda_N(f,x) = \frac{S_0(x)+\ldots S_N(x)}{N+1}$. Then

$$\lim_{N\to+\infty}\Lambda_N(f,x) = \frac{f(x+)+f(x-)}{2},$$

for all x where $f(x+)$ and $f(x-)$ exist. In particular if f is continuous function and $2\pi-$periodic and the sequence $(S_n(x))_n$ converges, then $\Lambda_N(f,x)$ converges to the same limit. Then f is the sum of its Fourier series.

6-1-4 (a) i. Let $f(x)=x-\pi$, for $x\in]0,2\pi]$, odd and 2π-periodic, $b_n = \frac{-2}{n}$.

Thus $x = \pi - 2\sum_{n=1}^{+\infty}\frac{\sin(nx)}{n}$ for $0<x<2\pi$.

ii. Let $f(x)=x$, for $x\in]-\pi,\pi[$, 2π-periodic, $b_n=\frac{-2(-1)^n}{n}$.

Thus $x=-2\sum_{n=1}^{+\infty}\frac{(-1)^n\sin(nx)}{n}$ for $-\pi<x<\pi$.

iii. Let $f(x)=x-\frac{\pi}{2}$, for $x\in]0,\pi]$, even and 2π-periodic, $a_0=0$, $a_{2n=0}$ and $a_{2n+1}=\frac{-4}{\pi(2n+1)^2}$.

Thus $x=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$ for $0\le x\le\pi$.

iv. Let $f(x)=x$, for $x\in[0,\frac{\pi}{2}]$, $f(x)=\pi-x$, for $x\in[\frac{\pi}{2},\pi]$ odd and 2π-periodic, $b_{2n=0}$ and $b_{2n+1}=\frac{4(-1)^n}{\pi(2n+1)^2}$.

Thus $x=\frac{4}{\pi}\sum_{n=0}^{+\infty}\frac{(-1)^n\sin(2n+1)x}{(2n+1)^2}$ for $\frac{-\pi}{2}\le x\le\frac{\pi}{2}$.

v. Let $f(x)=x$, for $x\in[0,\pi[$ and zero on $]-\pi,0]$ and 2π-periodic, $a_0=\frac{\pi}{2}$ $a_{2n=0}$, $a_{2n+1}=\frac{-2}{\pi(2n+1)^2}$ and $b_n=\frac{(-1)^{n+1}}{n}$. Thus $x=\frac{\pi}{4}-\frac{2}{\pi}\sum_{n=0}^{+\infty}\frac{\cos(2n+1)x}{(2n+1)^2}+\sum_{n\ge1}\frac{(-1)^{n+1}\sin(nx)}{n}$ for $0\le x<\pi$.

(b) Using b) for $x=\frac{\pi}{2}$, we shall have: $\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$.

using c) for $x=0$, we shall have: $\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$.

$\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2}+\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}$, thus $\sum_{n=1}^{+\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

By the Parseval identity in the relation b), we shall have: $\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^4}=\frac{\pi^4}{96}$.

(c) i. In the relation c) the series $\frac{\pi}{2}-\frac{4}{\pi}\sum_{n\ge0}\frac{\cos(2n+1)x}{(2n+1)^2}$ converges normally on $[0,\pi]$, thus by integration on the interval $[0,x]$, with $x\in[0,\pi]$, we shall have: $\sum_{n=0}^{+\infty}\frac{\sin(2n+1)\pi x}{(2n+1)^3}=\frac{\pi^2x}{8}-\frac{x^2}{2}$.

ii. Let $h(x)=\frac{\pi^2x}{8}-\frac{x^2}{2}$, for $x\in[0,\pi]$, h is odd, then $b_{2n+1}=\frac{1}{(2n+1)^3}$.

6-1-5 (a) $a_0=0$, $a_{2n=0}$ and $a_{2n+1}=\frac{4(-1)^n}{\pi(2n+1)}$.

(b) As f is continuous on $]-\frac{\pi}{2},\frac{\pi}{2}[$ and piecewise continuously differentiable, then $f(0)=1=\sum_{n=0}^{+\infty}\frac{4(-1)^n}{\pi(2n1)}$. Thus $\sum_{n\geq 0}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$.

6-1-6 (a) Since the series $\sum_{n\geq 1}\frac{1}{n}$ is divergent, then by the Bessel inequality, there exists no locally Riemann integrable function f such that its Fourier series is $\sum_{n\geq 1}\frac{\sin(nx)}{\sqrt{n}}$.

(b) Let $g(x)=\sum_{n=1}^{+\infty}\frac{\sin(nx)}{n^3}$. As the series which defines g converges normally on $\mathbb{R}$, thus g is continuous and the Fourier coefficients of g, $a_n=0$ for any $n\geq 0$ and $b_n=\frac{1}{n^3}$ for any $n\geq 1$.

6-1-7 (a) If $z=e^{\mathrm{i}x}$, then

$$f(x)=\frac{1}{\cosh(a)-\cos(x)}=\frac{-2z}{z^2-2z\cosh a+1}=\frac{1}{\sinh a}\Big(\frac{-e^a}{z-e^a}+\frac{e^{-a}}{z-e^{-a}}\Big).$$

In making the expansion in power series of the two functions $\frac{-e^a}{z-e^a}$ and $\frac{e^{-a}}{z-e^{-a}}$ and remark that $|z|=1$, one shall have:

$$f(x)=\frac{1}{\sinh(a)}\sum_{n=-\infty}^{+\infty}e^{inx}e^{-|n|a}=\frac{1}{\sinh(a)}+\frac{2}{\sinh(a)}\sum_{n=1}^{+\infty}e^{-na}\cos(nx).$$

(b)

$$\int_0^{2\pi}\frac{dx}{\cosh(a)-\cos(x)}=2\pi C_0=\frac{2\pi}{\sinh(a)}.$$

6-1-8 (a)

i. f is odd and $b_n=\frac{2}{n}$ for $n\geq 1$. The Fourier series of f is $\sum_{n\geq 1}\frac{2}{n}\sin(nx)$.

ii. $a_0=\pi$, for $n\geq 1$, $a_{2n}=0$ and for $n\geq 0$, $a_{2n+1}=\frac{4}{(2n+1)^2\pi}$. The Fourier series of g is $\frac{\pi}{2}+\frac{4}{\pi}\sum_{n\geq 0}\frac{\cos((2n+1)x)}{(2n+1)^2}$. The Fourier series converges uniformly on $\mathbb{R}$ to f.

(b) h is piecewise continuously differentiable. $h' = g$ on $[0, \pi]$. Then the Fourier series of h is

$$\frac{\pi x}{2} + \frac{4}{\pi} \sum_{n \geq 0} \frac{\sin((2n+1)x)}{(2n+1)^3}.$$

$\boxed{\text{6-1-9}}$ (a) $a_n = \dfrac{2(-1)^n \sinh \pi}{\pi(1+n^2)}$ and $b_n = -n^2 a_n$.

(b) The Fourier series of f is

$$\frac{\sinh \pi}{\pi} + \sum_{n=1}^{+\infty} \frac{2(-1)^n \sinh \pi \cos(nx)}{\pi(1+n^2)} + \sum_{n=1}^{+\infty} \frac{2(-1)^{n+1} n^2 \sinh \pi \sin(nx)}{\pi(1+n^2)}.$$

For $x = \pi$, $\dfrac{\pi \cosh \pi}{\sinh \pi} - 1 = 2 \displaystyle\sum_{n=1}^{+\infty} \frac{1}{1+n^2}$. Then

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2} = \frac{\pi(\cosh \pi + \sinh \pi)}{2 \sinh \pi}.$$

$\boxed{\text{6-1-10}}$ (a) $a = \frac{\pi^2}{3}$, $a_n = \dfrac{2(-1)^n}{n^2}$ and $b_n = -\dfrac{\pi(-1)^n}{n} - \dfrac{2(1-(-1)^n)}{n^3 \pi}$. The Fourier series of f is

$$\frac{\pi^2}{6} + \sum_{n=1}^{+\infty} \frac{2(-1)^n}{n^2} \cos(nx) - \sum_{n=1}^{+\infty} \left(\frac{\pi(-1)^n}{n} - \frac{2(1-(-1)^n)}{n^3 \pi}\right) \sin(nx).$$

(b) For $x = \pi$, $\frac{\pi^2}{2} = \frac{\pi^2}{6} + \displaystyle\sum_{n=1}^{+\infty} \frac{2}{n^2}$, then $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

For $x = 0$, $0 = \frac{\pi^2}{6} + \displaystyle\sum_{n=1}^{+\infty} \frac{2(-1)^n}{n^2}$, then $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$.

$\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n^2} + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} = 2 \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}$, then

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$\boxed{\text{6-1-11}}$ (a) $b_{2n} = 0$ and $b_{2n+1} = \dfrac{4}{\pi(2n+1)^3}$.

(b) $\dfrac{1}{2\pi}\displaystyle\int_0^{\pi} x^2(\pi-x)^2dx = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{16}{\pi^2(2n+1)^6}.$

$$\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^6}=\frac{\pi^6}{240}.$$

$\boxed{\text{6-1-12}}$ (a) $\displaystyle\sum_{n=1}^{+\infty} r^n\cos n\theta = \mathrm{Re}\sum_{n=1}^{+\infty} r^n e^{\mathrm{i}n\theta} = \frac{re^{\mathrm{i}\theta}}{1-re^{\mathrm{i}\theta}} = \frac{r\cos\theta - r^2}{1+r^2-2r\cos\theta}.$

(b)

$$\begin{aligned}
\sum_{-\infty}^{+\infty} r^{|n|}e^{in\theta} &= 1+2\sum_{n=1}^{+\infty} r^n\cos n\theta \\
&= 1+\frac{2r\cos\theta-2r^2}{1+r^2-2r\cos\theta} = \frac{1-r^2}{1+r^2-2r\cos\theta} \\
&= Q_r(\theta).
\end{aligned}$$

(c) The function Q_r is C^∞ and $2\pi-$periodic, then $I_n(r)=\displaystyle\int_0^{2\pi}\frac{\cos n\theta}{1-2r\cos\theta+r^2}d\theta = \frac{\pi a_n}{1-r^2}=\frac{2\pi r^n}{1-r^2}.$

$\boxed{\text{6-1-13}}$ (a) i. Let $\alpha=\sqrt{3}+2$ and $\beta=-\sqrt{3}+2$ the roots of the polynomial x^2-4x+1.

$$\begin{aligned}
h(x) &= \frac{x^2-1}{x^2-4x+1} = 1+\frac{4x-2}{x^2-4x+1} \\
&= \frac{a}{x-\sqrt{3}-2}+\frac{b}{x+\sqrt{3}-2} \\
&= 1+\frac{\alpha}{x-\alpha}+\frac{\beta}{x-\beta} \\
&= 1-\sum_{n=0}^{+\infty}\frac{x^n}{\alpha^n}-\sum_{n=0}^{+\infty}\frac{x^n}{\beta^n}.
\end{aligned}$$

ii. $R=\beta=2-\sqrt{3}$.

(b) Let a and z be two complex numbers, such that $|a|\neq|z|$ and $az\neq 0$.

Recall that $\dfrac{1}{1-z}=\displaystyle\sum_{n=0}^{+\infty} z^n$, for $|z|<1$.

If $|a|<|z|$, $\dfrac{1}{z-a}=\dfrac{1}{z(1-\frac{a}{z})}=\dfrac{1}{z}\displaystyle\sum_{n=0}^{+\infty}\frac{a^n}{z^n}.$

If $|z| < |a|$, $\dfrac{1}{z-a} = \dfrac{-1}{a(1-\frac{z}{a})} = \dfrac{-1}{a}\displaystyle\sum_{n=0}^{+\infty}\frac{z^n}{a^n}$.

(c) If $|z| \in]\beta, \alpha[$,

$$\begin{aligned} h(z) &= 1 + \frac{\alpha}{z-\alpha} + \frac{\beta}{z-\beta} \\ &= 1 - \sum_{n=0}^{+\infty}\frac{z^n}{\alpha^n} + \sum_{n=1}^{+\infty}\frac{\beta^n}{z^n} \\ &= 1 - \sum_{n=0}^{+\infty}\beta^n z^n + \sum_{n=1}^{+\infty}\frac{\beta^n}{z^n} \end{aligned}$$

(d) $h(e^{\mathrm{i}t}) = \dfrac{e^{2\mathrm{i}t}-1}{e^{2\mathrm{i}t}-4e^{\mathrm{i}t}+1} = \dfrac{e^{\mathrm{i}t}-e^{-\mathrm{i}t}}{e^{\mathrm{i}t}-4+e^{-\mathrm{i}t}} = -\mathrm{i}f(t)$.

(e) i. $f(t) = \dfrac{-1}{\mathrm{i}}(1 - \displaystyle\sum_{n=0}^{+\infty}\beta^n e^{int} + \sum_{n=1}^{+\infty}\beta^n e^{-int}) = 2\sum_{n=1}^{+\infty}\beta^n \sin(nt)$.

ii. Since the Fourier series of f converges normally to f, $\displaystyle\int_0^{2\pi}\frac{\sin^2 x}{2-\cos x}dx = \pi b_1 = 2\pi\beta$.

iii. F is C^∞ and $2\pi-$ periodic, then the Fourier series of F converges uniformly to F.

iv. $F'(t) = f(t) = 2\displaystyle\sum_{n=1}^{+\infty}\beta^n\sin(nt)$ and

$$F(t) = C - 2\sum_{n=1}^{+\infty}\frac{\beta^n}{n}\cos(nt).$$

$F(0) = \ln 2$, $-2\displaystyle\sum_{n=1}^{+\infty}\frac{\beta^n}{n} = \frac{-2\beta}{(1-\beta)^2} = -1$, then $C = 1 + \ln 2$

v. $\displaystyle\int_0^{\pi}\ln(2-\cos x)dx = \pi C = \pi\ln(2e)$.

6-1-14 (a) For $A > 0$ and $n \geq 1$

$$\sup_{|t|\leq A} f_n(t) = \begin{cases} \frac{1}{a^2} & \text{if } 2\pi n \leq A \\ \frac{1}{a^2+(A-2n\pi)^2} & \text{if } 2\pi n \geq A \end{cases},$$

then the series $\displaystyle\sum_{n\geq 1} f_n(t)$ converges normally on any interval $[-A, A] \subset \mathbb{R}$.

(b) i. $|f_n'(t)| = \frac{1}{a}|\frac{-2a(t+2n\pi)}{(a^2+(t+2n\pi)^2)^2}| \overset{\text{Cauchy-Schwarz}}{\leq} \frac{1}{a}\frac{a^2+(t+2n\pi)^2}{(a^2+(t+2n\pi)^2)^2} = \frac{f_n(t)}{a}$.

ii. Since the series $\sum_{n\geq 1} f_n(t)$ converges normally on any interval $[-A, A] \subset \mathbb{R}$, the series $\sum_{n=1}^{+\infty} f_n'(t)$ converges normally on any interval $[-A, A] \subset \mathbb{R}$.

(c) We deduce that the function $g(x) = \sum_{n=0}^{+\infty} \frac{1}{a^2+(t+2n\pi)^2}$ is $\mathcal{C}^1$ on $\mathbb{R}$. Moreover the function $h(x) = \sum_{n=-\infty}^{0} \frac{1}{a^2+(t+2n\pi)^2} = g(-t)$ is also $\mathcal{C}^1$ on $\mathbb{R}$. Then the function

$$f(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{a^2+(t+2n\pi)^2}$$

is even, 2π-periodic and by Dirichlet Theorem f is equal to its Fourier series.

(d) i. By Residue Theorem

$$I_k(a) = \int_{-\infty}^{+\infty} \frac{\cos kx}{a^2+x^2}dx = \pi\frac{e^{-|k|a}}{a}.$$

ii. Since the series $\sum_{n\geq 1} f_n(t)$ converges normally on any interval $[-A, A] \subset \mathbb{R}$;

$$\int_0^{2\pi} f(x)\cos kx dx = \sum_{-\infty}^{+\infty}\int_0^{2\pi} \frac{\cos kx}{a^2+(x+2n\pi)^2}dx = \sum_{-\infty}^{+\infty}\int_{2n\pi}^{2(n+1)\pi} \frac{\cos kx}{a^2+x^2}dt = I$$

iii.

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi} I_0(a) + \frac{1}{\pi}\sum_{k=1}^{+\infty} I_k(a)\cos(kx) \\
&= \frac{1}{2a} + \frac{1}{a}\sum_{k=1}^{+\infty} e^{-ka}\cos(kx) \\
&= \frac{1}{2a} + \frac{1}{a}\operatorname{Re}\sum_{k=1}^{+\infty} e^{-k(a+\mathrm{i}x)} \\
&= \frac{\sinh a}{2a(\cosh a - \cos x)}.
\end{aligned}
$$

4.10 Solutions of Exercises on Chapter 7

7-1-1 If $\mathscr{A}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$, it contains three elements, there is a set $A \in \mathscr{A}$ such that $A \neq \emptyset$ and $A \neq \mathbb{R}$. But $A^c \in \mathscr{A}$. $A^c \neq \emptyset$ and $A^c \neq \mathbb{R}$, this is impossible. So there is no $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$ which contains three elements.

If $\mathscr{A}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$ which contains four elements. Let $A \in \mathscr{A}$ such that $A \neq \emptyset$ and $A \neq \mathbb{R}$. The set $\mathscr{A} = \{\emptyset, \mathbb{R}, A, A^c\}$ is a $\sigma-$algebra in $\mathscr{P}(\mathbb{R})$. It contains four elements. This is the general form $\sigma-$algebra that contains four elements.

7-1-2 The set $\mathbb{R} \in \mathscr{A}$ and so $\emptyset \in \mathscr{A}$.
If $A \in \mathscr{A}$, then $A^c \subset f^{-1}(f(A^c))$. and if there exists $x \in A \cap f^{-1}(f(A^c))$, then there is $y \in A^c$ such that $f(x) = f(y)$. Hence $y \in f^{-1}(f(A)) = A$. This is a contradiction. Then $A^c \in \mathscr{A}$.
Let $(A_n)_n$ be a sequence in $\mathscr{A}$. As $f^{-1}(f(\cup_{n=1}^{\infty} A_n)) = \cup_{n=1}^{\infty} f^{-1}(f(A_n)) = \cup_{n=1}^{\infty} A_n$. Then $\cup_{n=1}^{\infty} A_n \in \mathscr{A}$, therefore $\mathscr{A}$ is a $\sigma-$algebra.

7-1-3 • $\mathbb{R} \in \mathscr{A}$.

• Let $A \in \mathscr{A}$ such that $A \neq \mathbb{R}$. If there exists $x \notin A$ and $f(x) \in A$, then $x \in f^{-1}(A) \subset A$, and this is impossible. Then $f(x) \notin A$ and therefore $f(A^c) \subset A^c$. On the other hand if $x \in f^{-1}(A^c)$ and $x \in A$, then $x \in A$ and $f(x) \in A^c$ and this is impossible. Then $f^{-1}(A^c) \subset A^c$ and $A^c \in \mathscr{A}$.

• Let $(A_n)_n$ be a sequence in $\mathscr{A}$. $f\left(\cup_{n=1}^{\infty} A_n\right) = \cup_{n=1}^{\infty} f(A_n) \subset \cup_{n=1}^{\infty} A_n$ and $f^{-1}\left(\cup_{n=1}^{\infty} A_n\right) = \cup_{n=1}^{\infty} f^{-1}(A_n) \subset \cup_{n=1}^{\infty} A_n$. Then $\mathscr{A}$ is a $\sigma-$algebra.

7-1-4 Any $\sigma-$algebra containing $\mathscr{C}$ must contains $\mathscr{A} = \mathscr{C} \cup \mathscr{C}^c$, where $\mathscr{C}^c = \{F : \ F^c \in \mathscr{C}\}$. We will prove that the set $\mathscr{A}$ is a $\sigma-$algebra.
It is obvious that $\mathbb{R} \in \mathscr{A}$ and if $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$.
Let $(A_n)_n$ be a sequence in $\mathscr{A}$.

If there exists A_j such that $E \subset A_j$, then $\displaystyle\bigcup_{k=1}^{\infty} A_k \in \mathscr{C} \subset \mathscr{A}$. But if $A_j \in \mathscr{C}^c$, for every $j \in \mathbb{N}$, then $A_j \subset E^c$, for every $j \in \mathbb{N}$. Therefore $\displaystyle\bigcup_{j=1}^{\infty} A_j \subset E^c$. If $\bigcup_{j=1}^{\infty} A_j \in \mathscr{A}$.

$\mathscr{A}$ is the $\sigma-$algebra generated by $\mathscr{C}$.

7-1-5 Let $\sigma(S)$ be the $\sigma-$algebra generated by S.
• If the set E finite or countable, as $\sigma(S)$ is closed under countable union, then $\sigma-$algebra $\sigma(S)$ contains the finite or countable sets. Since the subsets of E are finite or countable, then $\sigma(S) = \mathscr{P}(E)$.
• If the set E is not countable.

Let $\mathscr{A}$ be the family of finite or countable sets in E and $\mathscr{B} = \{A^c, A \in \mathscr{A}\}$. As $\sigma(S)$ is closed under countable union, then $\mathscr{A} \subset \sigma(S)$. Also since $\sigma(S)$ is closed under the complement, then $\mathscr{B} \subset \sigma(S)$. Now, we prove that $\mathscr{A} \cup \mathscr{B}$ is a $\sigma-$algebra and therefore $\sigma(S) = \mathscr{A} \cup \mathscr{B}$.
$\emptyset \in \mathscr{A} \subset \mathscr{A} \cup \mathscr{B}$ and $\mathscr{A} \cup \mathscr{B}$. Let $(A_n)_n$ be a sequence in $\mathscr{A} \cup \mathscr{B}$.
If $A_n \in \mathscr{A}$, for every $n \in \mathbb{N}$, then $\displaystyle\bigcup_{n=1}^{+\infty} A_n \in \mathscr{A} \subset \mathscr{A} \cup \mathscr{B}$.
If there is $n \in \mathbb{N}$ such that $A_n \in \mathscr{B}$, then $A_n^c \in \mathscr{A}$ and then A_n^c is finite or countable and $\displaystyle\left(\bigcup_{k=1}^{+\infty} A_k\right)^c = \bigcap_{k=1}^{+\infty} A_k^c \subset A_n^c$ is finite or countable and this proves that $\displaystyle\left(\bigcup_{n=1}^{+\infty} A_n\right)^c \in \mathscr{A}$ and $\displaystyle\bigcup_{n=1}^{+\infty} A_n \in \mathscr{B} \subset \mathscr{A} \cup \mathscr{B}$. Hence $\mathscr{A} \cup \mathscr{B}$ is closed under countable union and then $\mathscr{A} \cup \mathscr{B}$ is a $\sigma-$algebra.

7-1-6 (a) Let $\mathscr{A} = \mathscr{P}(A^c) \cup \mathscr{C}$.
$\mathbb{R} \in \mathscr{C}$ and if $B \in \mathscr{C}$, then $B^c \in \mathscr{P}(A^c)$ and if $B \in \mathscr{P}(A^c)$, then $B^c \in \mathscr{C}$. On the other hand, if $(A_n)_n$ is a sequence in $\mathscr{A}$ and if there exists k such that $A \subset A_k$, then $A \subset \cup_{n=1}^{\infty} A_n$. Moreover, if $A_k \in \mathscr{P}(A^c)$ for every $k \in \mathbb{N}$, then $\cup_{n=1}^{\infty} A_n \in \mathscr{P}(A^c)$. Then $\mathscr{A}$ is a $\sigma-$algebra and it is the $\sigma-$algebra generated by $\mathscr{C}$.

(b) $\mathscr{A} = \mathscr{P}(\mathbb{R})$ unless if $A = \emptyset$ or $A = \mathbb{R}$, because $\mathscr{A} = \mathscr{P}(A) \cup \mathscr{P}(A^c)$.

7-2-1 $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$. The subsets $A \setminus B$, $B \setminus A$ and $A \cap B$ are disjoint. Then $\mu(A \cup B) = \mu(A \setminus B) + \mu(B \setminus A) + \mu(A \cap B)$.
$A = (A \setminus B) \cup (A \cap B)$, therefore $\mu(A) = \mu(A \setminus B) + \mu(A \cap B)$. The same for $B = (B \setminus A) \cup (A \cap B)$, and therefore $\mu(B) = \mu(B \setminus A) + \mu(A \cap B)$.
Then $\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B)$.

7-2-2 Let $(A_n = [n, +\infty[)_n$ for $n \in \mathbb{N}$. $\lambda(A_n) = +\infty$ and $\bigcap_{n \geq 0} A_n = \emptyset$.

7-2-3 As $\mathbb{Q}$ is countable, we can consider $\mathbb{Q} = \{x_n : \ n \in \mathbb{N}\}$. Define the set

$$\mathcal{O} = \bigcup_{n=1}^{\infty}]x_n - \frac{\varepsilon}{2n(n+1)}, x_n + \frac{\varepsilon}{2n(n+1)}[.$$

$\mathcal{O}$ is an open set and $\lambda(\mathcal{O}) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{n(n+1)} = \varepsilon$.

7-2-4 For $x \leq y$, $f(y) - f(x) = \lambda(A \cap]x, y]) \leq y - x$. Then f is continuous.

7-2-5 The two limits $A = \lim_{x \to -\infty} f(x)$ and $B = \lim_{x \to +\infty} f(x)$ exist.
As the function f is increasing, the set $f^{-1}(] - \infty, a])$ is an interval for all $a \in \mathbb{R}$ (If $x < y$ and $f(x) \leq a$ and $f(y) \leq a$, the for all $x < t < y$, $f(t) \leq f(y) \leq a$). So the function f is measurable.

7-2-6 $\{x \in \mathbb{R} : \ f(x) \neq 0\} = f^{-1}([-\infty, 0[) \cup f^{-1}(]0, + infty])$ is measurable.

7-2-7 $\mu(\emptyset) = \displaystyle\int_{\emptyset} \frac{1}{1+x^2} d\lambda(x) \leq \lambda(\emptyset) = 0$.

As λ is a measure, then if A and B are measurable and $A \subset B$, then $\displaystyle\int_A \frac{1}{1+x^2} d\lambda(x) \leq \int_B \frac{1}{1+x^2} d\lambda(x)$. Then $\mu(A) \leq \mu(B)$.

Let $(A_n)_n$ be an increasing sequence in $\mathscr{B}$ and let $A = \bigcup_{n=1}^{+\infty} A_n$. The sequence $(f_n)_n$ defined by: $f_n(x) = \dfrac{\chi_{A_n}(x)}{1+x^2}$ is increasing and using the monotone convergence theorem, $\lim_{n \to +\infty} \mu(A_n) = \displaystyle\int_A \frac{1}{1+x^2} d\lambda(x)$.

7-2-8 For $n \in \mathbb{N}$, $\mu\Big(\{x \in \mathbb{R} : \ |f(x)| \geq n\}\Big) \leq \dfrac{1}{n} \displaystyle\int_{\mathbb{R}} |f(x)| d\mu(x)$. Then

$$\mu\Big(\{x \in \mathbb{R} : \ f(x) = \pm\infty\}\Big) = 0.$$

7-2-9 For all $n \in \mathbb{N}$, the set $E_n = \{x \in \mathbb{R} : \ f(x) \geq \frac{1}{n}\}$ is measurable. Since $\displaystyle\int_{E_n} f(x)d\mu(x) = 0$, then E_n is a null set. So $E = \{x \in \mathbb{R} : \ f(x) > 0\}$ is a null set. Also the set $\{x \in \mathbb{R} : \ f(x) < 0\}$ is a null set, then $f = 0$ a.e.

7-2-10 As the function $\sin(x^2)$ is continuous, $\displaystyle\int_{\mathbb{R}} |\sin(x^2)|d\lambda(x) = \int_{-\infty}^{+\infty} |\sin(x^2)|dx$.

$$\begin{aligned}\int_1^{+\infty} |\sin(x^2)|dx &= \int_1^{+\infty} \frac{|\sin(x)|}{\sqrt{x}}dx \geq \int_1^{+\infty} \frac{\sin^2(x)}{\sqrt{x}}dx \\ &= \int_1^{+\infty} \frac{1-\cos(2x)}{2\sqrt{x}}dx.\end{aligned}$$

and since the integral $\displaystyle\int_1^{+\infty} \frac{\cos(2x)}{\sqrt{x}}dx$ is convergent, (We can use Abel's theorem) and the integral $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{x}}dx$ is divergent, then the function $\sin(x^2)$ is not integrable.

With the same arguments the function $\cos(x^2)$ is not integrable.

7-3-1 (a)

$$\begin{aligned}\int_0^{+\infty}\frac{|\sin(e^x)|}{1+nx^2}dx &\overset{t=x\sqrt{n}}{=} \frac{1}{\sqrt{n}}\int_0^{+\infty}\frac{\left|\sin(e^{\frac{t}{n}})\right|}{1+t^2}dt\\ &\leq \frac{1}{\sqrt{n}}\int_0^{+\infty}\frac{1}{1+t^2}dt.\end{aligned}$$

Then

$$\lim_{n\to+\infty}\int_0^{+\infty}\frac{\sin(e^x)}{1+nx^2}dx=0.$$

(b) $\lim_{n\to+\infty}\chi_{[0,n]}(x)(1+\frac{x}{n})^{-n}\cos x=\chi_{[0,+\infty[}(x)e^{-x}\cos x$ and $\chi_{[0,n]}(x)(1+\frac{x}{n})^{-n}|\cos x|\leq e^{-\frac{x}{2}}$, and this function is integrable. So using the dominate convergence Theorem

$$\lim_{n\to+\infty}\int_0^n(1+\frac{x}{n})^{-n}\cos x dx=\int_0^{+\infty}e^{-x}\cos x dx=\frac{1}{2}.$$

(c) $\lim_{n\to+\infty}\chi_{[0,n]}(x)(1+\frac{x}{n})^n e^{-2x}=\chi_{[0,+\infty[}(x)e^{-x}$. On the other hande, as $\frac{x}{2}\leq\ln(1+x)\leq x$, for every $x\in[0,1]$, then $\chi_{[0,n]}(x)(1+\frac{x}{n})^n e^{-2x}\leq e^{-x}$, and they are integrative. So using Capped Convergence Theorem

$$\lim_{n\to+\infty}\int_0^n(1+\frac{x}{n})^n e^{-2x}dx=\int_0^{+\infty}e^{-x}dx=1.$$

(d) $(1-\frac{x}{n})^{-n}e^{\frac{x}{2}}\geq e^{\frac{x}{2}}$ for all $x\in[0,n]$, and therefore:

$$\lim_{n\to+\infty}\int_0^n(1-\frac{x}{n})^{-n}e^{\frac{x}{2}}dx=+\infty.$$

(e) $\lim_{n\to+\infty}\chi_{[0,n]}(x)(1-\frac{x}{n})^n\frac{1+nx}{n+x}\cos x = xe^{-x}\cos x$ on the interval $[0,+\infty[$. On the other hand since $\ln(1-x)\leq -x$, for every $x\in[0,1[$, then
$\chi_{[0,n]}(x)(1-\frac{x}{n})^n\frac{1+nx}{n+x}|\cos x|\leq e^{-x}(1+x)$, and they are integrable. So using the dominate convergence theorem

$$\begin{aligned}\lim_{n\to+\infty}\int_0^n (1-\frac{x}{n})^n\frac{1+nx}{n+x}\cos x dx &= \int_0^{+\ infty} xe^{-x}\cos x dx\\ &= \frac{1}{2}e^{-x}(-x\cos x+(1+x)\sin x\Big]_0^\infty = !0.\end{aligned}$$

(f) $\lim_{n\to+\infty}(1+\frac{x}{n})^{n^2}e^{-nx} = \chi_{[0,+\infty[}(x)e^{-\frac{x^2}{2}}$. On the other hand, $\ln(1+x)\leq x-\frac{x^2}{4}$ for every $x\in[0,1]$, then $\chi_{[0,n]}(x)(1+\frac{x}{n})^{n^2}e^{-nx} = \chi_{[0,n]}(x)e^{n^2\ln(1+\frac{x}{n})-nx}\leq e^{-\frac{x^2}{4}}$. Using the dominate convergence theorem

$$\lim_{n\to+\infty}\int_0^n (1+\frac{x}{n})^{n^2}e^{-nx}dx = \int_0^{+\infty} e^{-\frac{x^2}{2}}dx = \sqrt{\frac{\pi}{2}}.$$

As $\ln(1+x)\leq\frac{1}{2}x$ for every $x\geq 1$, then $\chi_{[n,+\infty[}(x)(1+\frac{x}{n})^{n^2}e^{-nx}\leq\chi_{[n,+\infty[})x)e^{-\frac{n}{2}x}$ and

$$\lim_{n\to+\infty}\int_n^{+\infty}(1+\frac{x}{n})^{n^2}e^{-nx}dx\leq\int_n^{+\infty}e^{-\frac{n}{2}x}dx = 0.$$

Then

$$\lim_{n\to+\infty}\int_0^{+\infty}(1+\frac{x}{n})^{n^2}e^{-nx}dx = \sqrt{\frac{\pi}{2}}.$$

7-3-2 (a) Let $g(x) = \sum_{n=1}^{+\infty}\frac{f(nx)}{n^\alpha}$. Using the monotone convergence theorem,

$$\int_{\mathbb{R}}\sum_{n=1}^{+\infty}\frac{|f(nx)|}{n^\alpha}d\lambda(x) = \sum_{n=1}^{+\infty}\int_{\mathbb{R}}\frac{|f(nx)|}{n^\alpha}d\lambda(x) = \sum_{n=1}^{+\infty}||f||_1\frac{1}{n^{\alpha+1}} < +\infty.$$

So the function g is integrable and finite a.e. Then $\lim_{n\to+\infty}\frac{f(nx)}{n^\alpha} = 0$ a.e.

(b) i. Let $h(x) = \sum_{n=1}^{+\infty}\frac{f(nx)}{n^2}$. Using the monotone convergence theorem,

$$
\begin{aligned}
\int_0^T \sum_{n=1}^{+\infty} \frac{|f(nx)|}{n^2} d\lambda(x) &= \sum_{n=1}^{+\infty} \int_0^T \frac{|f(nx)|}{n^2} d\lambda(x) \\
&= \sum_{n=1}^{+\infty} \int_0^{nT} \frac{|f(x)|}{n^3} d\lambda(x) \\
&= \sum_{n=1}^{+\infty} \int_0^{T} \frac{|f(x)|}{n^2} d\lambda(x) < +\infty.
\end{aligned}
$$

So the function h is integrable on $[0, T]$, $T-$ periodic and bounded a.e. Then $\lim\limits_{n\to+\infty} \dfrac{f(nx)}{n^2} = 0$ a.e. $x \in \mathbb{R}$.

ii. $\displaystyle\int_0^\pi (\ln|\cos x|)^2 dx = \int_{-1}^1 \frac{(\ln|x|)^2}{\sqrt{1-x^2}} dx < +\infty$ and $(\ln|\cos x|)^2$ is $\pi-$periodic. Then $\lim\limits_{n\to+\infty} \dfrac{(\ln|\cos nx|)^2}{n^2} = 0$ a.e. and $\lim\limits_{n\to+\infty} (|\cos nx|)^{\frac{1}{n}} =$ 1 a.e.

7-3-3 $\dfrac{e^{-2x}}{1+e^x} = \dfrac{e^{-3x}}{1+e^{-x}} = \displaystyle\sum_{n=0}^{+\infty} (-1)^n e^{-(3+n)x}$. The sequence $\left(\displaystyle\sum_{k=0}^{n} (-1)^k e^{-(3+k)x}\right)_n$ is a sequence of integrable functions and bounded above by the function e^{-3x}. Then it is integrable and

$$
\int_0^{+\infty} \frac{e^{-2x} dx}{1+e^x} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{3+n} = \ln 2 - \frac{1}{2}.
$$

7-3-4 If $x \geq e$, $\ln t \geq 1$ for every $t \geq x$ and $\dfrac{1}{t^2 \ln^n(t)} \leq \dfrac{1}{t^2}$ and $\lim\limits_{n\to+\infty} \dfrac{1}{t^2 \ln^n(t)} =$ 0. So using the dominate convergence theorem $\lim_{n\to+\infty} I_n(x) = 0$.

If $x \leq e$, $\displaystyle\int_x^{+\infty} \frac{dt}{t^2 \ln^n(t)} = \int_e^{+\infty} \frac{dt}{t^2 \ln^n(t)} + \int_x^{e} \frac{dt}{t^2 \ln^n(t)}$.

The first integral $\displaystyle\int_e^{+\infty} \frac{dt}{t^2 \ln^n(t)}$ tends to 0.

For every, $x \leq t \leq e$, $\ln t \leq 1$ and the sequence $(\dfrac{1}{t^2 \ln^n(t)})_n$ is increasing.

Using the dominate convergence theorem $\lim\limits_{n\to+\infty} \displaystyle\int_x^{e} \frac{dt}{t^2 \ln^n(t)} = +\infty$.

7-3-5 Let $f(x) = \dfrac{xe^{-ax}}{1-e^{-bx}} = \displaystyle\sum_{n=0}^{+\infty} xe^{-(a+nb)x}$.

The function f is continuous on the interval $[0, +\infty[$ and non negative.

$(f(0) = \lim_{x\to 0} f(x) = \dfrac{1}{b})$. On the other hand, $\lim_{x\to+\infty} e^{-bx} = 0$, then there exists $A > 0$ such that $1 - e^{-bx} \geq \dfrac{1}{2}$ for all $x \geq A$ and then $f(x) \leq 2xe^{-ax}$ which is integrable.

Using the dominate convergence theorem, $\displaystyle\int_0^{+\infty} f(x)dx = \sum_{n=0}^{+\infty} \int_0^{+\infty} xe^{-(a+nb)x}dx = \sum_{n=0}^{+\infty} \frac{1}{(a+nb)^2}$.

7-3-6 The sequence $(\tan^n x)_n$ is decreasing for each $x \in [0, \frac{\pi}{4}[$ and $0 \leq \tan x < 1$ a.e. Therefore $\lim_{n\to+\infty} \tan^n(x) = 0$.

We note that $I_0 = \frac{\pi}{4}$, $I_1 = \ln\sqrt{2}$, and for each $n \geq 1$, $I_n + I_{n+2} = \dfrac{1}{n+1}$. define the sequences $(u_n = I_{2n})_n$ and $(v_n = I_{2n+1})_n$.
We have $u_n + u_{n+1} = \frac{1}{2n+1}$ and $v_n + v_{n+1} = \frac{1}{2n+2}$. Then $u_0 = \dfrac{\pi}{4} = \displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1}$ and then $2v_0 = \ln 2 = \displaystyle\sum_{k=1}^{+\infty} \frac{(-1)^k}{k}$.

7-3-7 (a) As $\ln(1-t) \leq -t$ for every $t \in [0,1[$, then $(1 - \dfrac{x}{n})^n = e^{n\ln(1-\frac{x}{n})} \leq e^{-x}$, for every $t \in [0, n]$. Then $\lim_{n\longrightarrow+\infty} (1 - \dfrac{x}{n})^n \leq e^{-x}$. Using the monotone convergence theorem

$$\lim_{n\longrightarrow+\infty} \int_0^n \sqrt{x}(1 - \frac{x}{n})^n \ln x dx = \int_0^\infty \sqrt{x}e^{-x} \ln x dx.$$

(b) $\dfrac{|\sin x|^{\frac{2}{n}}}{1+x^2} \leq \dfrac{1}{1+x^2}$ and $\lim_{n\longrightarrow+\infty} \dfrac{|\sin x|^{\frac{2}{n}}}{1+x^2} = \dfrac{1}{1+x^2}$. Using the dominate convergence theorem

$$\lim_{n\longrightarrow+\infty} \int_{\mathbb{R}} \frac{|\sin x|^{\frac{2}{n}}}{1+x^2} dx = \int_{\mathbb{R}} \frac{1}{1+x^2} dx = \pi.$$

(c) $\lim_{n\longrightarrow+\infty} \dfrac{1}{(1+x^p)^n} = 0$ a.e. and the $\dfrac{1}{(1+x^p)^n} \leq \dfrac{1}{(1+x^p)^{\frac{2}{p}}}$ and they are integrable. Then

$$\lim_{n\longrightarrow+\infty} \int_0^{+\infty} \frac{dx}{(1+x^p)^n} = 0.$$

(d) $\lim\limits_{n \longrightarrow +\infty} e^{-n\sin^2 x} f(x) = f(x)$ and $e^{-n\sin^2 x}|f|(x) \leq |f|(x)$, therefore

$$\lim_{n \longrightarrow +\infty} \int_0^{+\infty} e^{-n\sin^2 x} f(x)\ dx = \int_0^{+\infty} f(x)dx.$$

(e)

$$\int_\alpha^{+\infty} \frac{hf(x)}{h^2+x^2}dx = \int_{\frac{\alpha}{h}}^{+\infty} \frac{f(ht)}{1+t^2}dt = \int_{[0,+\infty[} \chi_{[\frac{\alpha}{h},+\infty[}(t)\frac{f(ht))}{1+t^2}dt.$$

$\lim\limits_{h\to 0^+} \chi_{[\frac{\alpha}{h},+\infty[}(t)\dfrac{f(ht)}{1+t^2} = 0$ a.e and bounded by the function $|f|$ which is integrable. Then $\lim\limits_{h\longrightarrow 0^+} \displaystyle\int_\alpha^{+\infty} \frac{hf(x)}{h^2+x^2}dx = 0.$

$$\int_0^{+\infty} \frac{hf(x)}{h^2+x^2}dx = \int_0^\alpha \frac{hf(x)}{h^2+x^2}dx + \int_\alpha^{+\infty} \frac{hf(x)}{h^2+x^2}dx.$$

$$\int_0^\alpha \frac{hf(x)}{h^2+x^2}dx = \int_0^{\frac{\alpha}{h}} \frac{f(ht)}{1+t^2}dt = \int_{[0,+\infty[} \chi_{[0,\frac{\alpha}{h}[)}(t)\frac{f(ht)}{1+t^2}dt$$

$\lim\limits_{h\to 0^+} \chi_{[0,\frac{\alpha}{h}[}(t)\dfrac{f(ht)}{1+t^2} = \dfrac{f(0)\chi_{[0,+\infty[}(t)}{1+t^2}$ a.e. and bounded by the function $|f|$ which is integrable. Then

$\lim\limits_{h\longrightarrow 0^+} \displaystyle\int_0^{+\infty} \frac{hf(x)}{h^2+x^2}dx = \frac{\pi f(0)}{2}$.

7-4-1 (a) $f(x) = \frac{1}{\sqrt{x}}$ a.e. and this function is integrable and $\int_{[0,1]} \frac{1}{\sqrt{x}} d\lambda(x) = 2$.

$g(x) = \cos x$ a.e. and this function is integrable $\int_{[0,1]} \cos x d\lambda(x) = \sin 1$

(b) For every $x \geq 1$, $\frac{|\sin x|}{x} \geq \frac{\sin^2 x}{x} = \frac{1-\cos(2x)}{2x} = \frac{1}{2x} - \frac{\cos(2x)}{2x}$.

The integral of the function $\frac{\cos(2x)}{2x}$ on $[1, +\infty[$ is convergent, but the integral of the function $\frac{1}{2x}$ on $[1, +\infty[$ is divergent. Then the integral of the function $\frac{|\sin x|}{x}$ is not convergent on $[1, +\infty[$.

$\frac{1}{\sqrt{|\sin x|}}$ is integrable on $[0, \frac{\pi}{2}]$ because $\sin x \leq \frac{2x}{\pi}$ on the interval $[0, \frac{\pi}{2}]$.

Let $I = \int_{[0,\frac{\pi}{2}]} \frac{dx}{\sqrt{|\sin x|}}$.

$\int_{n\pi}^{(n+1)\pi} g(x)dx \leq 2I \frac{1}{(1+(n\pi)^2)}$, therefore

$\int_0^{+\infty} g(x)dx \leq 2I \sum_{n=0}^{+\infty} \frac{1}{(1+(n\pi)^2)} < +\infty$. So the function g is integrable. We note that $\int_0^1 h(x)dx = \int_1^{+\infty} h(x)dx$. (We can make the substitution $t = \frac{1}{x}$). and for $x \geq 2$, $h(x) \leq \frac{1}{x \ln^2 x}$ which is integrable on $[2, +\infty[$. ($\int_a^{+\infty} \frac{1}{x \ln^2 x} dx = \frac{1}{\ln a}$, for $a > 1$). Then h is integrable.

7-4-2 (a) $e^{-[x]} = \sum_{n=0}^{+\infty} e^{-n} \chi_{[n,n+1[}(x)$. Then using the monotone convergence theorem $\int_{[0,+\infty[} e^{-[x]} d\lambda(x) = \sum_{n=0}^{+\infty} e^{-n} = \frac{e}{e-1}$.

(b) $f(x) = \cos x$ a.e., then $\int_{[0,\pi]} f(x) d\lambda(x) = \int_{[0,\pi]} \cos x d\lambda(x) = 0$.

(c) $\chi_{\mathbb{Q}} = 0$ a.e., then $\int_{[0,1]} \chi_{\mathbb{Q}}(x) d\lambda(x) = 0$.

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