Some New Classes and Techniques in the Theory of Bernstein Functions

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Abstract In this paper we provide some new properties that are complementary to the book of Schilling-Song-Vondraček (Bernstein functions, 2nd edn. De Gruyter, Berlin, 2012).

Keywords Bernstein functions · Complete Bernstein functions · Thorin Bernstein functions · Subordinators · Infinite divisible distributions · Generalized Gamma convolutions · Positive stable density

1 A Unified View on Subclasses of Bernstein Functions

In the sequel all measures will be understood on the space $(0, \infty)$ and their densities, if they have one, are with respect to Lebesgue measure on $(0, \infty)$ which will be denoted by dx. We recall that the *Mellin convolution* (or multiplicative convolution)

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of two measures ν and τ on $(0,\infty)$ is defined by:

$$\nu \circledast \tau(A) = \int_{(0,\infty)^2} 1_A(xy)\nu(dx)\tau(dy), \qquad \text{if } A \text{ is a Borel set of } (0,\infty)\,.$$

If ν is absolutely continuous with density function h, then $\nu \circledast \tau$ is the function given by

$$\nu \circledast \tau(x) = h \circledast \tau(x) = \int_{(0,\infty)} h\left(\frac{x}{y}\right)\frac{\tau(dy)}{y}, \quad x > 0.$$

Another nice property of the Mellin convolution is that if a is a real number, then

$$x^a\,(\nu \circledast \tau) = (x^a \nu) \circledast (x^a \tau) \tag{1.1}$$

Notice that all the integrals above may be infinite if ν and/or τ are not finite measures. A function f defined on $(0,\infty)$ is called *completely monotone*, and we denote $f \in \mathcal{CM}$, if it is infinitely differentiable there satisfies

$$(-1)^n f^{(n)}(x) \ge 0, \quad \text{for all } n = 0, 1, 2, \cdots,\ x > 0. \tag{1.2}$$

Bernstein's theorem says that $f \in \mathcal{CM}$ if, and only if, it is the Laplace transform of some measure τ on $[0,\infty)$:

$$f(\lambda) = \int_{[0,\infty)} e^{-\lambda x}\,\tau(dx), \quad \lambda > 0.$$

Denote $\check{\tau}$ the image of $\tau_{|(0,\infty)}$ by the function $x \mapsto 1/x$ and notice that f has the representation

$$f(\lambda) = \tau(\{0\}) + \int_{(0,\infty)} e^{-\frac{\lambda}{x}}\,\check{\tau}(dx) = e^{-x} \circledast (x\check{\tau})(\lambda). \tag{1.3}$$

A function ϕ is called a **Bernstein function**, and we denote $\phi \in \mathcal{BF}$, if it has the representation

$$\phi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\pi(dx), \quad \lambda \ge 0\,, \tag{1.4}$$

where $\mathrm{q},\ \mathrm{d} \ge 0$, the measure π, supported by $(0,\infty)$, satisfies

$$\int_{(0,\infty)} (x \wedge 1)\,\pi(dx) < \infty.$$

The usage is to call q the *killing* term and d the *drift* term. Any measure on $(0, \infty)$ that satisfies the preceding integrability condition is called a *Lévy measure*. As for completely monotone functions, notice that ϕ is represented by

$$\phi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + (1 - e^{-x}) \circledast (x\check{\pi})(\lambda).$$

Bernstein functions are more likely called by probabilists *Laplace exponents of infinite divisible sub-probability distributions* or *Laplace exponents of* (possibly killed) *subordinators*, and the previous representation is their Lévy-Khintchine representation. See [6] for more account on subordinators and Lévy processes.

Next theorem illustrates to what extent the Mellin convolution is involved into the most popular subclasses of infinitely divisible distributions. Roughly speaking, we will see that each of these subclass $\mathbf{C}$ is associated to a Lévy measures π of the form $\pi = \mathbf{c} \circledast \nu$, where $\mathbf{c}$ is a specified function and ν is some Lévy measure.

Theorem 1.1 *Let π be a Lévy measure.*

(1) *The measure π has a non increasing density if, and only if, π is of the form*

$$\pi = 1_{(0,1]}(x)dx \circledast \nu,$$

where ν is some a Lévy measure;

(2) *The measure $x\pi(dx)$ has a non increasing density if, and only if, π is of the form*

$$\pi = 1_{(0,1]}(x)\frac{dx}{x} \circledast \nu,$$

where ν is a measure which integrates the function $g_0(x) = x\, 1_{(0,1]}(x) + \log x\, 1_{[1,\infty)}(x)$ (in particular ν is a Lévy measure);

(3) *The measure π has a density of the form $x^{a-1}\, k(x)$ with $a \in (-1, \infty)$ and k a completely monotonic function such that $\lim_{x\to+\infty} k(x) = 0$ if, and only if, π has the expression*

$$\pi = x^{a-1}\, e^{-x} dx \circledast \nu,$$

where ν is a measure which integrates the function g_a given by

$$g_a(x) := \begin{cases} x\, 1_{(0,1]}(x) + x^{-a}\, 1_{[1,\infty)}(x) & \text{if } a \in (-1, 0), \\ x\, 1_{(0,1]}(x) + \log x\, 1_{[1,\infty)}(x) & \text{if } a = 0, \\ x\, 1_{(0,1]}(x) + \, 1_{[1,\infty)}(x) & \text{if } a \in (0, \infty)\,. \end{cases} \tag{1.5}$$

Consequently, ν is a Lévy measure in all cases. Moreover ν may be an arbitrary Lévy measure in case (1) and in case (3) with $a > 0$.

Proof Notice that if μ has a density h, then $\mu \circledast \nu$ has a density, denoted by $h \circledast \nu$, and taking values in $[0, \infty]$:

$$h \circledast \nu(x) = \int_{(0,\infty)} \frac{1}{y} h\left(\frac{x}{y}\right) \nu(dy), \quad x > 0.$$

(1) Using the last expression for $h(x) = u_0(x) = 1_{(0,1]}(x)$, we have

$$u_0 \circledast \nu(x) = \int_x^\infty \frac{\nu(dy)}{y}.$$

Notice that any non-increasing function (taking values in $[0, \infty]$) is of the form $u_0 \circledast \nu$ and conversely. Since

$$\int_0^\infty (x \wedge 1)\,(u_0 \circledast \nu)(x)\,dx = \int_0^1 \frac{x^2}{2}\,\frac{\nu(dx)}{x} + \frac{1}{2}\int_1^\infty \frac{\nu(dy)}{y} + \int_1^\infty \frac{z-1}{z}\nu(dz)\,,$$

we deduce that the measure with density $\int_x^\infty \nu(dy)/y$ is a Lévy measure if, and only if, ν integrates the function $x \wedge 1$ or, in other words, ν is a Lévy measure.

(2) Using the expression of $h \circledast \nu$ with $h(x) = u_1(x) = 1_{(0,1]}(x)/x$, we have:

$$u_1 \circledast \nu(x) = \frac{\nu(x, \infty)}{x}, \quad x > 0.$$

Notice that any function π, valued in $[0, \infty]$, such that $x\pi(x)$ is non increasing is of the form $u_1 \circledast \nu$ and conversely. After that, note that

$$\int_0^\infty (x \wedge 1) u_1 \circledast \nu(x) dx = \int_0^1 x\,\nu(dx) + \nu(1, \infty) \int_1^\infty \log x\ \nu(dx)\,.$$

Thus, the measure with density $u_1 \circledast \nu(x)$ is a Lévy measure if, and only if, $\nu(dx)$ integrates $g_0(x) = x\,1_{(0,1]}(x) + \log x\, 1_{(1,\infty)}(x)$.

(3) Without surprise, one is tempted to use the fact (1.1) together with representation (1.3) and write that for some measure τ

$$x^{a-1}k(x) = x^{a-1}(e^{-x} \circledast (x\check{\tau})) = (x^{a-1}e^{-x}) \circledast (x^a\check{\tau}),$$

where the transform $\check{\tau}$ of τ is given right before (1.3). We will do this in detail and provide the integrability conditions for the involved measures: let $a \in (-1, \infty)$, k be a completely monotone function such that $k(\infty) = 0$ and

$\pi(dx) = x^{a-1}\,k(x)\,1_{(0,\infty)}dx$. By Bernstein theorem, k is the Laplace transform of a measure on $(0,\infty)$, and may be written in the form

$$k(x) := \int_{(0,\infty)} e^{-xu}\ u^a \sigma(du), \quad x > 0\,. \tag{1.6}$$

Defining

$$h_a(u) := \int_0^\infty (x \wedge u)\, x^{a-1}\, e^{-x} dx, \quad u > 0,$$

and using Fubini's theorem, write

$$\int_{(0,\infty)} (x \wedge 1)\pi(dx) = \int_{(0,\infty)} (x \wedge 1) x^{a-1}\, k(x) dx = \int_{(0,\infty)} h_a(u) \frac{\sigma(du)}{u}\,.$$

We will now find the necessary and sufficient conditions on σ insuring that the last integral is finite. First, notice that $h_a(u) \nearrow \Gamma(a+1)$ when $u \to \infty$ and then h_a is bounded for any $a > -1$. Then, elementary computations give the following behavior of h_a in a neighborhood of 0,

$$\begin{aligned}
\lim_{0+} \frac{h_a(u)}{u} &= \Gamma(a), \ \text{ if } a > 0\,;\\
0 < \liminf_{0+} \frac{h_a(u) - u}{u|\log u|} \le \limsup_{0+} \frac{h_a(u) - u}{u|\log u|} &< \infty, \ \text{ if } a = 0\,;\\
\lim_{0+} \frac{h_a(u)}{u^{1-|a|}} = \frac{1}{|a|} + \frac{1}{1-|a|}&, \ \text{ if } -1 < a < 0.
\end{aligned}$$

and then π is a Lévy measure iff $\int_1^\infty \frac{\sigma(du)}{u} du < \infty$ and

$$\begin{aligned}
\sigma([0,1]) &< \infty, \text{ if } a > 0;\\
\int_{(0,1]} |\log u|\, \sigma(du) &< \infty, \text{ if } a = 0;\\
\int_{(0,1]} \frac{\sigma(du)}{u^{|a|}} &< \infty, \text{ if } -1 < a < 0.
\end{aligned}$$

Notice that in each case $\sigma([0,1]) < \infty$ and then $\sigma(du)/u$ is a Lévy measure. Also notice that the measure ν, defined as the image of $\sigma(du)$ induced by the function $u \mapsto 1/u$, is also a Lévy measure, so that the integrability properties of the measure σ are equivalent to ν integrates the function g_a in (1.5). In order to conclude, write

$$x^{a-1}\, k(x) = x^{a-1} \int_{(0,\infty)} e^{-xu}\, u^a\, \sigma(du) = \int_{(0,\infty)} \left(\frac{x}{y}\right)^{a-1} e^{-\frac{x}{y}}\ \frac{\nu(dy)}{y} = (y^{a-1} e^{-y} \circledast \nu\,)\,(x)\,.$$

(4) After the above developments, the last assertion becomes obvious.

$\square$

Remark 1.2 Below are some classes of Bernstein functions which can be defined via the correspondence between π and μ obtained in Theorem 1.1:

(i) The class $\mathcal{JB}$ of Bernstein function whose Lévy measure is of type (1) is often called the class *the Jurek class of Bernstein functions*. It is also characterized by those function

$$\phi \geq 0 \quad s.t. \quad \lambda \mapsto (x \mapsto x\phi(x))'(\lambda) \in \mathcal{BF}.$$

(ii) Bernstein functions whose Lévy measure is of type (2) is called *self-decomposable Bernstein functions*, and we denote $\mathcal{SDBF}$ their set. It is easy to check (see [9, Theorem 2.6 ch. VI], for instance), that

$$\phi \in \mathcal{SDBF} \quad \Longleftrightarrow \phi(0) \geq 0 \quad \text{and} \quad \lambda \mapsto \lambda\phi'(\lambda) \in \mathcal{BF}.$$

The class $\mathcal{SDBF}$ functions corresponds to self-decomposable distributions: namely, a r.v. X has a self-decomposable distribution if there exists a family of positive r.v. $(Y_c)_{0<c<1}$, each Y_c is independent from X such that the identity in distribution holds: $X \stackrel{d}{=} cX + Y_c$.

(iii) In [7, pp. 49], the class $\mathcal{CBF}$ of *complete Bernstein functions* corresponds to the Bernstein functions appearing in point (3) of Theorem 1.1 when the parameter a equals 1. In matrix analysis and operator theory, the name "operator monotone function" is more common for $\mathcal{CBF}$-functions. Another feature is that $\mathcal{CBF}$ is included into the class of $\mathcal{SBF}$ of special Bernstein functions, i.e. the class of Bernstein functions ϕ such that $\lambda \mapsto \lambda/\phi(\lambda) \in \mathcal{BF}$. The class $\mathcal{CBF}$ will be deeply investigated in next section.

(iv) The class $\mathcal{TBF}$ [7, pp. 73] of *Thorin Bernstein functions* corresponds to $a = 0$. The class $\mathcal{TBF}$ corresponds to the Laplace exponents of the generalized Gamma distributions, shortly GGC, introduced by Bondesson [1, 2] and the GGC subordinators studied by James, Roynette and Yor [5]. For more developments on $\mathcal{TBF}$, see [7].

2 Investigating the Class $\mathcal{CBF}_a$

We have seen that the well known Thorin class $\mathcal{TBF}$ corresponds to $\mathcal{CBF}_0$, and we will not go into further investigations in it. The simplest $\mathcal{CBF}$-function is given by $\lambda \mapsto \lambda/(\lambda+1)$.

Point (3) of Theorem 1.1 suggests a generalization of the notion of $\mathcal{CBF}$ and $\mathcal{TBF}$ for any parameter $a > -1$ by introducing the set class $\mathcal{CBF}_a$ of Bernstein functions such that the corresponding Lévy measure π has a density of the form

$x^{a-1}k(x)$ such that k is a completely monotonic function and $k(\infty) = 0$. It is clear that $\mathcal{CBF}_a \subset \mathcal{CBF}_b$ for every $a \leq b$, and that $\mathcal{TBF} \subset \mathcal{CBF}_a \cap \mathcal{SDBF} \subset \mathcal{CBF}$ for every $0 \leq a \leq 1$. The simplest functions in $\mathcal{CBF}_a$ are given when taking the complete monotonic functions k of the form $k(x) = e^{-cx}$, which is the Laplace transform of the Dirac measure at point $c > 0$. Then the associated Bernstein function is

$$\varphi_{a,b}(\lambda) = \int_0^\infty (1 - e^{-\lambda x})\, x^{a-1}\, e^{-bx}\, dx = \begin{cases} \Gamma(a) \left(\dfrac{1}{b^a} - \dfrac{1}{(b+\lambda)^a} \right) & \textit{if } a \neq 0 \\ \log(1 + \frac{\lambda}{b}) & \textit{if } a = 0\,. \end{cases} \tag{2.1}$$

Notice that for $a \in (-1, 0)$, these Bernstein functions are those associated to the so-called *tempered stable processes* of index $\alpha = -a$ and, for $a = 0$, it is associated to the normalized *Gamma process*. As stated in the next theorem, any $\mathcal{CBF}_a$ function is a conic combination of these simple ones. Next theorem is a straightforward consequence of Theorem 1.1:

Theorem 2.1 (Representation of $\mathcal{CBF}_a$-functions) *Let $a > -1$, $\phi : [0,\infty) \to [0,\infty)$, $\mathrm{q} = \phi(0)$ and $\mathrm{d} = \lim_{+\infty} \phi(x)/x < \infty$. Then ϕ belongs to $\mathcal{CBF}_a$ if, and only if, it $\lambda \mapsto \phi(\lambda) - \mathrm{q} - \mathrm{d}\lambda$ is the Mellin convolution of $\varphi_{a,1}$ defined in (2.1) with some measure. Namely,*

$$\phi(\lambda) = \begin{cases} \mathrm{q} + \mathrm{d}\lambda + \Gamma(a) \displaystyle\int_{(0,\infty)} \left(1 - \frac{u^a}{(u+\lambda)^a}\right) \sigma(du), & \textit{if } a \neq 0 \\ \mathrm{q} + \mathrm{d}\lambda + \displaystyle\int_{(0,\infty)} \log\left(1 + \frac{\lambda}{u}\right) \sigma(du) & \textit{if } a = 0\,, \end{cases} \tag{2.2}$$

where σ is a measure that integrates the function $g_a(1/t)$ given by (1.5). In this case, the Lévy measure associated to ϕ has the density function

$$x^{a-1} \int_{(0,\infty)} e^{-xt}\, t^a\, \sigma(dt), \qquad x > 0.$$

Example 2.2 The stable Bernstein function given by the power function $\lambda \mapsto \lambda^\alpha$, $\alpha \in (0, 1)$, is a trivial example of a function in $\mathcal{CBF}_\alpha$, because

$$\lambda^\alpha = \int_0^\infty (1 - e^{-\lambda x})\, \frac{c_\alpha}{x^{\alpha+1}}\, dx, \quad \text{where } c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)} \quad \text{and} \quad x \mapsto k(x) = x^{-2\alpha} \in \mathcal{CM},$$

In order to prove Proposition 2.4 below, we need some formalism and a Lemma. Let S_α, $\alpha \in (0,1)$, denotes a normalized positive stable random variable with density function f_α, i.e.

$$\mathbb{E}[e^{-\lambda S_\alpha}] = \int_0^\infty e^{-\lambda x} f_\alpha(x)dx = e^{-\lambda^\alpha}, \quad \lambda \geq 0, \tag{2.3}$$

and observe that for any $t > 0$,

$$e^{-t\lambda^\alpha} = t^{-1/\alpha} \int_0^\infty e^{-\lambda x} f_\alpha(xt^{-1/\alpha})dx\,. \tag{2.4}$$

Also, let γ_t denotes a normalized gamma distributed random variable with parameter $t > 0$, i.e.

$$\mathbb{E}[e^{-\lambda\gamma_t}] = \frac{1}{(1+\lambda)^t}, \quad \lambda \geq 0.$$

For any positive r.v. S satisfying $\mathbb{E}[S^s] < \infty$, $s \in \mathbb{R}$, we adopt the notation $S^{(s)}$ for a version of the size biased distribution of order s:

$$\mathbf{P}(S^{[s]} \in dx) = \frac{x^s}{\mathbb{E}[S^s]}\mathbf{P}(S \in dx). \tag{2.5}$$

Shanbhag and Sreehari [8] showed the remarkable identity in law

$$\gamma_t^{1/\alpha} \stackrel{d}{=} \frac{\gamma_{\alpha t}}{S_\alpha^{[-\alpha t]}}.$$

from which we can extract from, when taking two independent and identically distributed random variables S_α and S'_α, that

$$\gamma_1^{1/\alpha} S_\alpha \stackrel{d}{=} \gamma_1 X_\alpha, \quad \text{where} \quad X_\alpha = \frac{S_\alpha}{S'_\alpha} \stackrel{d}{=} \frac{1}{X_\alpha} \tag{2.6}$$

$$\stackrel{d}{=} \gamma_\alpha Y_\alpha, \quad \text{where} \quad Y_\alpha = \frac{S_\alpha}{(S'_\alpha)^{[-\alpha]}} \stackrel{d}{=} \frac{1}{Y_\alpha^{[-\alpha]}} \tag{2.7}$$

$$\gamma_{1/\alpha}^{1/\alpha} S_\alpha \stackrel{d}{=} \gamma_1 Z_\alpha, \quad \text{where} \quad Z_\alpha = \frac{S_\alpha}{(S'_\alpha)^{[-1]}} \stackrel{d}{=} \frac{1}{Z_\alpha^{[-1]}}, \tag{2.8}$$

where, in each product, we have used the notation of (2.5) and the r.v.'s involved in the identities in law are assumed to be independent. Last identities are used in the following lemma:

Lemma 2.3 *Let $\alpha \in (0, 1)$. With the above notations, we have*

(1) *The function $\phi_\alpha(\lambda) := \dfrac{\lambda^\alpha}{\lambda^\alpha + 1}$ belongs to $\mathcal{CBF}_\alpha$ and is represented by*

$$\phi_\alpha(\lambda) = \mathbb{E}\left[\frac{\lambda X_\alpha}{\lambda + X_\alpha}\right] = \mathbb{E}\left[\left(\frac{\lambda}{1 + \lambda\, Y_\alpha}\right)^\alpha\right] = 1 - \mathbb{E}\left[\frac{1}{(1 + \lambda Y_\alpha)^\alpha}\right], \quad \lambda \geq 0; \tag{2.9}$$

(2) *The function $\varphi_\alpha(\lambda) := 1 - \dfrac{1}{(\lambda^\alpha + 1)^{1/\alpha}}$ belongs to $\mathcal{CBF}$ and is represented by*

$$\varphi_\alpha(\lambda) = \mathbb{E}\big[\frac{\lambda Z_\alpha}{1 + \lambda Z_\alpha}\big], \quad \lambda \geq 0. \tag{2.10}$$

Proof

(1) Since

$$\frac{1}{1 + \lambda^\alpha} = \mathbb{E}\left[e^{-\lambda^\alpha \gamma_1}\right] = \mathbb{E}\left[e^{-\lambda \gamma_1^{1/\alpha} S_\alpha}\right] = \mathbb{E}\left[e^{-\lambda \gamma_1 X_\alpha}\right] = \mathbb{E}\left[\frac{1}{1 + \lambda X_\alpha}\right]. \tag{2.11}$$

The first equality in (2.9) comes from

$$\phi_\alpha(\lambda) = 1 - \frac{1}{1 + \lambda^\alpha} = 1 - \mathbb{E}\left[\frac{1}{1 + \lambda X_\alpha}\right] = \mathbb{E}\left[\frac{\lambda X_\alpha}{1 + \lambda X_\alpha}\right].$$

Going back to (2.11) and using again (2.6), we obtain the second and third representations in (2.9) by writing

$$\phi_\alpha(\lambda) = \lambda^\alpha\, \mathbb{E}[e^{-\lambda\, \gamma_1^{1/\alpha}\, S_\alpha}] = \lambda^\alpha\, \mathbb{E}[e^{-\lambda\, \gamma_\alpha Y_\alpha}] = \mathbb{E}\left[\left(\frac{\lambda}{1 + \lambda\, Y_\alpha}\right)^\alpha\right],$$

and also

$$\phi_\alpha(\lambda) = 1 - \frac{\phi_\alpha(\lambda)}{\lambda^\alpha} = 1 - \mathbb{E}\left[\frac{1}{(1 + \lambda\, Y_\alpha)^\alpha}\right].$$

Since the third representation of ϕ_α meets the one of Theorem 2.1, we deduce that $\phi_\alpha \in \mathcal{CBF}_\alpha$.

(2) Similarly, write

$$\varphi_\alpha(\lambda) = 1 - \mathbb{E}\left[e^{-\lambda\, \gamma_{1/\alpha}^{1/\alpha}\, S_\alpha}\right] = 1 - \mathbb{E}\left[e^{-\lambda\, \gamma_1\, Z_\alpha}\right] = \mathbb{E}\left[\frac{\lambda\, Z_\alpha}{1 + \lambda\, Z_\alpha}\right],$$

and deduce that $\varphi_\alpha \in \mathcal{CBF}$.

□

We are now able to exhibit additional links between $\mathcal{CBF}_a,\ a > 0$, and $\mathcal{CBF}$:

Proposition 2.4 *The following implications are true:*

(1) *If* $0 < a \le 1$ *and* $\phi \in \mathcal{CBF}$, *then* $\lambda \mapsto \phi(\lambda^a) \in \mathcal{CBF}_a$ *and* $\phi(\lambda^a)^{1/a} \in \mathcal{CBF}$.
(2) *If* $a \ge 1$ *and* $\varphi \in \mathcal{CBF}_a$, *then* $\lambda \mapsto \varphi(\lambda^{1/a}) \in \mathcal{CBF}$.

Remark 2.5 The first assertion of Proposition 2.4 is a refinement of [7, Corollary 7.15]:

$$a \ge 1 \quad \text{and} \quad \phi(\lambda^a)^{1/a} \in \mathcal{CBF} \Longrightarrow \phi \in \mathcal{CBF}.$$

The latter could be also obtained by a Pick-Nevanlinna argument as in Remark 3.6 below.

Proof of Proposition 2.4 The second assertion in (1) can be found in [7, Corollary 7.15]. In Example 2.2, we have seen that $\lambda \mapsto \lambda^\alpha \in \mathcal{CBF}_\alpha$ for every $0 < \alpha \le 1$, so, we may suppose that ϕ has no killing nor drift term. The assertions are a conic combination argument together with the result of Lemma 2.3. For the first assertion of (1), use the function $\phi_a \in \mathcal{CBF}_a$ given by (2.9), for the assertion (2), use the function $\varphi_{1/a} \in \mathcal{CBF}$ given by (2.10), and get the representations

$$\phi(\lambda^a) = \int_0^\infty \phi_a\left(\frac{\lambda}{u}\right) \nu(du) \quad \text{and} \quad \varphi(\lambda^a) = \int_0^\infty \varphi_{1/a}\left(\frac{\lambda}{u}\right) \mu(du), \qquad \lambda \ge 0,$$

where ν and μ are some measure. □

3 A New Injective Mapping from $\mathcal{BF}$ onto $\mathcal{CBF}$

We recall that a $\mathcal{CBF}$ function is a Bernstein function whose Lévy measure has a density which is a completely monotonic function. We recall the connection between $\mathcal{CBF}$-functions; ϕ is a $\mathcal{CBF}$-function if, and only if, it admits the representation:

$$\phi(\lambda) = \mathrm{q} + \mathrm{d} + \int_{(0,\infty)} \frac{\lambda}{\lambda + x}\, \nu(dx), \qquad \lambda \ge 0, \tag{3.1}$$

where $\mathrm{q},\ \mathrm{d} \ge 0$ and ν is a measure which integrates $1 \wedge 1/x$.

Another characterization of $\mathcal{CBF}$ functions is given by the *Pick-Nevanlinna characterization of* $\mathcal{CBF}$*-functions*:

Theorem 3.1 (Theorem 6.2 [7]) *Let ϕ a non-negative continuous function on $[0,\infty)$ is a $\mathcal{CBF}$ function if, and only if, it has an analytic continuation on $\mathbb{C}(-\infty,0]$ such that*

$$\Im\big(\phi(z)\big) \geq 0, \quad \textit{for all } \ z \ \textit{ s.t. } \ \Im(z) > 0.$$

Notice that any $\phi \in \mathcal{BF}$ has an analytic continuation on the half plane $\{z,\ \Re(z) > 0\}$ which can be extended by continuity to the closed half plane $\{z,\ \Re(z) \geq 0\}$ and we still denote by ϕ this continuous extension. In next theorem we state a representation similar to (3.1) and valid for any Bernstein function ϕ. Part (1) of this theorem is also quoted as [7, Proposition 3.6].[1]

Theorem 3.2

(1) *Let $\phi \in \mathcal{BF}$ represented by (1.4), then, for all $\lambda \geq 0$,*

$$\phi(\lambda) = \mathrm{d}\lambda + \int_0^\infty \frac{\lambda}{\lambda^2 + u^2}\, v(u)\, du\,, \tag{3.2}$$

where v, given by $v(u) := 2\Re\big(\phi(iu)\big)/\pi$, is a negative definite function (in the sense of [7, Definition 4.3]) satisfying the integrability condition

$$\int_1^\infty \frac{v(u)}{u^2} < \infty. \tag{3.3}$$

(2) *Conversely, let $\mathrm{d} \geq 0$ and $v : \mathbb{R} \to \mathbb{R}_+$ be a negative definite function satisfying (3.3), then*

$$\lambda \mapsto \mathrm{d}\lambda + \int_0^\infty \frac{\lambda}{\lambda^2 + u^2}\, v(u)\, du \in \mathcal{BF}\,.$$

Proof

(1) We suppose without loss of generality that $\mathrm{q} = \mathrm{d} = 0$. In this proof, we denote by $(C_t)_{t\geq 0}$ a standard Cauchy process, i.e. a Lévy process such that $\mathbb{E}[e^{iuC_t}] = e^{-t|u|}$, $u \in \mathbb{R}$. Since

$$\phi(ix) = \int_{(0,\infty)} (1 - e^{-ixs})\pi(ds), \quad x \in \mathbb{R},$$

[1] The results in Theorem 3.2 (i), Corollaries 3.4, 3.5 and Proposition 3.8 below can be found, with different proofs, in [7, Proposition 3.6, Proposition 7.22]. As is stated in [7, pp. 34 & 108 and reference entries 119, 120], *the statements of these results are due to S. Fourati and W. Jedidi and were, with a different proof, communicated by S. Fourati and W. Jedidi* in 2010.

then, for all $\lambda > 0$, we can write

$$\begin{aligned}\phi(\lambda) &= \int_{(0,\infty)} (1 - e^{-\lambda s})\,\pi(ds) = \int_{(0,\infty)} \mathbb{E}[1 - e^{-isC_\lambda}]\,\pi(ds)\\ &= \int_{(0,\infty)} \left(\int_{\mathbb{R}} (1 - e^{-ius}) \frac{\lambda}{\pi(u^2+\lambda^2)} du\right) \pi(ds) = \int_{\mathbb{R}} \frac{\lambda}{\pi(u^2+\lambda^2)}\,\phi(iu)\,du\\ &= \int_0^\infty \frac{\lambda}{\pi(u^2+\lambda^2)} \big(\phi(iu) + \phi(-iu)\big) du = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{(u^2+\lambda^2)}\,\Re\big(\phi(iu)\big)\,du.\end{aligned}$$

Notice that $v(u) := 2\Re\big(\phi(iu)\big)/\pi$ is an even function on $\mathbb{R}$, is a $[0, \infty)$-valued negative definite function in the sense of [7, Definition 4.3], and representation (3.2) proves that it necessarily satisfies (3.3).

(2) By [4, Corollary 1.1.6], every $[0, \infty)$-valued, negative definite function v, has necessarily the form

$$v(u) = \mathrm{q} + \mathrm{c}u^2 + \int_{\mathbb{R}\{0\}} (1 - \cos ux)\,\mu(dx),$$

where $\mathrm{q}, \mathrm{c} \geq 0$ and the Lévy measure μ is symmetric and integrates $x^2 \wedge 1$. We deduce that v is an even function and necessarily $\mathrm{c} = 0$ because of the integrability condition (3.3). So, v is actually represented by

$$v(u) = \mathrm{q} + 2\int_{(0,\infty)} (1 - \cos ux)\,\mu(dx).$$

Then, observe that

$$\int_1^\infty \frac{v(u)}{u^2}\,du = \mathrm{q} + 2\int_{(0,\infty)} \theta(x)\,\mu(dx) < \infty$$

where

$$\theta(x) = \int_1^\infty \frac{1-\cos(xt)}{t^2}\,dt = x\int_x^\infty \frac{1-\cos t}{t^2}\,dt \leq 2.$$

Since $\lim_{x\to 0}\theta(x)/x = \pi/2$, deduce that μ necessarily integrates $x \wedge 1$. Finally, v is the real part of some Bernstein function ϕ and conclude with part (1) of this theorem.

□

Remark 3.3

(i) Note that condition (3.3) on the negative definite function v was obtained as an immediate consequence of representation (3.2) and is equivalent, in our context, to the usual integrability condition (on the Lévy measure at 0) for a Lévy process to have finite variation paths, see the book of Breiman [3, Exercise 13 p. 316]. In Vigon's thesis, [10, Proposition 1.5.3] one can also find a nice proof of condition (3.3) based on a Fourier single-integral formula.

(ii) In (3.2), it is not clear that the constant functions belong to $\mathcal{CBF}^-$. They actually do, since for all $\mathrm{q} \geq 0$ and $\lambda > 0$,

$$\mathrm{q} = \frac{2}{\pi} \int_0^\infty \frac{\lambda\, \mathrm{q}}{\lambda^2 + u^2}\, du,$$

then $\lambda \mapsto \phi(\lambda) = \mathrm{q} \in \mathcal{CBF}^-$.

Now, it appears natural to introduce the class of functions $\mathcal{CBF}^-$ associated to negative definite functions :

$$\mathcal{CBF}^- := \left\{ \lambda \mapsto \varphi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_0^\infty \frac{\lambda}{\lambda + u^2}\, v(u)\, du \right\},$$

where $\mathrm{q},\ \mathrm{d} \geq 0$ and $v : [0,\infty) \to [0,\infty)$ is a negative definite function, necessarily satisfying the integrability condition (3.3). It is obvious that $\mathcal{CBF}^-$ is a (strict) subclass of $\mathcal{CBF}$.

A reformulation of last theorem gives the following two corollaries quoted as [7, Proposition 7.22 and Propositon 3.6] respectively. The reader is also addressed to the footnote before Theorem 3.2.

Corollary 3.4 (Classes $\mathcal{BF}$ and $\mathcal{CBF}^-$ are one-to-one)

(1) *If ϕ is in $\mathcal{BF}$, then $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$ is in $\mathcal{CBF}^-$;*
(2) *Conversely, any function in $\mathcal{CBF}^-$ is of the form $\lambda \mapsto \sqrt{\lambda}\phi(\sqrt{\lambda})$, where ϕ is in $\mathcal{BF}$.*

Corollary 3.5 *Any Bernstein function leaves globally invariant the cônes*

$$\left\{\rho e^{i\pi\theta};\quad \rho \geq 0,\ \alpha \in [-\sigma,\sigma]\right\}, \qquad \textit{for any} \quad \sigma \in [0, \frac{1}{2}].$$

Proof The function $\psi_u(\lambda) = \lambda/(\lambda^2 + u^2),\ u > 0$, maps the half-line $\left\{\rho e^{i\pi\sigma};\quad \rho \geq 0\right\}$ onto in the cône $\left\{\rho e^{i\pi\theta};\quad \rho \geq 0,\ \theta \in [-\sigma,\sigma]\right\}$. Since this cone is convex and closed by any conic combination of ψ_u, deduce that the integral

$$\frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + u^2}\, \Re\big(\phi(iu)\big)\, du$$

is in the same cône. □

Remark 3.6 The property in the last corollary has to be compared with the much stronger property fulfilled by a $\mathcal{CBF}$-function: any $\mathcal{CBF}$-function has an analytic continuation on $\mathbf{C} \setminus (-\infty, 0]$ and this continuation leaves globally invariant the cônes

$$\left\{\rho e^{i\pi\theta};\quad \rho \geq 0,\ \theta \in [0, \sigma]\right\}, \tag{3.4}$$

for any $\sigma \in [0, 1)$. Moreover this property fully characterizes the class of $\mathcal{CBF}$-functions: the Pick-Nevanlinna characterization given in Theorem 3.2 is equivalent to invariance of the cone (3.4) for $\sigma = 1$. This property is not satisfied by all Bernstein functions. For instance, $\phi(\lambda) = 1 - e^{-\lambda} \in \mathcal{BF} \setminus \mathcal{CBF}$, because $\Im\left(5e^{i\pi/4}\right) > 0$ but $\Im\left(\phi(5e^{i\pi/4})\right) < 0$.

In the following results, we give some extension of Theorem 3.4 by replacing the function $\lambda \mapsto \sqrt{\lambda}$ by other functions:

Corollary 3.7 *Let $\phi \in \mathcal{BF}$. Then,*

(1) *For any function ψ such that $\lambda \mapsto \psi(\lambda^2)$ is in $\mathcal{BF}$, we have $\psi^2 \in \mathcal{BF}$ and $\phi(\psi) \in \mathcal{CBF}^-$.*
(2) *For any function ψ such that ψ^2 is in $\mathcal{CBF}$, the following functions are in $\mathcal{CBF}$:*

$$\phi(\psi) \cdot \psi, \qquad \frac{\psi}{\phi(\psi)}, \qquad \phi(1/\psi) \cdot \psi, \qquad \frac{\psi}{\phi(1/\psi)}$$

and also

$$\lambda \cdot \frac{\phi(\psi)}{\psi}, \qquad \frac{\lambda}{\phi(\psi) \cdot \psi}, \qquad \lambda \cdot \frac{\phi(1/\psi)}{\psi}, \qquad \frac{\lambda}{\phi(1/\psi) \cdot \psi}.$$

Proof

(1) Since $\psi_1(\lambda) := \psi(\lambda^2) \in \mathcal{BF}$, then $\psi_2(\lambda) := \psi^2(\lambda) = \psi_1(\sqrt{\lambda})^2 \in \mathcal{BF}$. To get the last claim, just check the complete monotonicity of the derivative of ψ_2. The second assertion is seen by stability by composition of the class $\mathcal{BF}$: since $\lambda \mapsto \phi(\psi_1(\lambda)) = \phi(\psi(\lambda^2)) \in \mathcal{BF}$, then Corollary 3.4 applies on the last function.
(2) Recall $\mathcal{S}$ is the class of Stieltjes functions, i.e. the class of functions obtained by a double Laplace transform (see [7]) and observe that

$$\varphi \in \mathcal{CBF} \Longleftrightarrow \lambda \mapsto \frac{\varphi(\lambda)}{\lambda} \in \mathcal{S}.$$

As consequence of [7, (7.1), (7.2), (7.3) pp. 96], obtain that

$$\sqrt{\lambda}\,\phi(\sqrt{\lambda}) \in \mathcal{CBF} \iff \frac{\sqrt{\lambda}}{\phi(\sqrt{\lambda})} \in \mathcal{CBF} \iff \sqrt{\lambda}\,\phi(1/\sqrt{\lambda}) \in \mathcal{CBF} \iff \frac{\sqrt{\lambda}}{\phi(1/\sqrt{\lambda})} \in \mathcal{CBF} \tag{3.5}$$

$$\Updownarrow$$

$$\frac{\phi(\sqrt{\lambda})}{\sqrt{\lambda}} \in \mathcal{S} \iff \frac{1}{\sqrt{\lambda}\phi(\sqrt{\lambda})} \in \mathcal{S} \iff \frac{\phi(1/\sqrt{\lambda})}{\sqrt{\lambda}} \in \mathcal{S} \iff \frac{1}{\sqrt{\lambda}\phi(1/\sqrt{\lambda})} \in \mathcal{S}. \tag{3.6}$$

To get the first claim, compose the four $\mathcal{CBF}$-functions in (3.5) with $\psi^2 \in \mathcal{CBF}$, and use the stability by composition of the class $\mathcal{CBF}$ [7, Corollary 7.9.]. To obtain the last claim, also compose the four $\mathcal{S}$-functions in (3.6) with $\psi^2 \in \mathcal{CBF}$, use [7, Corollary 7.9], to get that the compositions stays in $\mathcal{S}$, and finally multiply by λ to get the announced $\mathcal{CBF}$-function in the Corollary.

□

Notice that if ψ belongs to $\mathcal{BF}$, then $\psi(\sqrt{\lambda})$ satisfies property (1). If further ψ belongs to $\mathcal{CBF}$ then $\psi(\sqrt{\lambda})$ and $\sqrt{\psi(\lambda)}$ both satisfy property (2).

Now, we summarize the properties that can be stated when composing a Bernstein function ϕ with the stable Bernstein function of Example 2.2.

Proposition 3.8 *Let $\alpha \in (0, 1]$, $\phi \in \mathcal{BF}$, π be the Lévy measure of ϕ and $\overline{\pi}$ the right tail of π: $\overline{\pi}(x) := \pi(x, \infty)$, $x > 0$. Then,*

(1) *$\lambda \mapsto \phi_\alpha(\lambda) := \lambda^{1-\alpha}\phi(\lambda^\alpha) \in \mathcal{BF}$. Further, $\phi_\alpha \in \mathcal{CBF}$ whenever*

$$x \mapsto \overline{\pi}_\alpha(x) := \alpha\, x^{\alpha-1}\,\overline{\pi}(x^\alpha) \in \mathcal{CM} \quad (\textit{which is true if } \phi \in \mathcal{CBF});$$

(2) *$\lambda \mapsto \lambda^\gamma\phi(\lambda^\alpha) \in \mathcal{CBF}$ (resp. $\mathcal{CBF}^-$) if $\alpha \le \frac{1}{2}$ and $\gamma \in (\alpha, 1-\alpha]$ (resp. $\gamma = \frac{1}{2}$).*

Proof Recall that f_α, the density function of normalized positive stable r.v., is given by (2.3).

(1) Since $\lambda \mapsto \lambda^\alpha$, $\lambda^{1-\alpha}$ are both in $\mathcal{CBF}$, there is no loss of generality to take $\mathrm{q} = \mathrm{d} = 0$ in the Lévy-Khintchine representation (1.4) of ϕ, and then to write

$$\lambda^{1-\alpha}\phi(\lambda^\alpha) = \lambda \int_0^\infty e^{-\lambda^\alpha t}\overline{\pi}(t)dt.$$

It is sufficient to prove that $\lambda \mapsto \int_0^\infty e^{-\lambda^\alpha t}\overline{\pi}(t)dt$ is the Laplace transform of a non increasing function. For that, use (2.4) and Fubini's theorem and get

$$\int_0^\infty e^{-\lambda^\alpha t}\overline{\pi}(t)dt = \int_0^\infty e^{-\lambda x}\overline{\Pi}_\alpha(x)dx,$$

where

$$\overline{\Pi}_\alpha(x) := \int_0^\infty f_\alpha\left(\frac{x}{t^{1/\alpha}}\right)\overline{\pi}(t)\,\frac{dt}{t^{1/\alpha}} = \int_0^\infty \overline{\pi}_\alpha\left(\frac{x}{z}\right) f_\alpha(z)\,\frac{dz}{z},$$

and the second representation by the change of variables $z = xt^{-\frac{1}{\alpha}}$. Since for each $z > 0$, the functions $z \mapsto \overline{\pi}_\alpha(x/z)$ are non-increasing (respectively completely monotone), deduce the same for $\overline{\Pi}_\alpha$.

(2) Observe that for $\alpha \le 1/2$, the function $\lambda \mapsto \lambda^\alpha$ satisfies the properties of Corollary 3.7: point (2) yields that the function $\lambda \mapsto \lambda^{1-\alpha}\phi(\lambda^\alpha)$ is in $\mathcal{CBF}$, and point (2) yields that $\sqrt{\lambda}\phi(\lambda^\alpha)$ is in $\mathcal{CBF}^-$. Taking representation of ϕ in Theorem 3.2, we obtain

$$\lambda^\gamma\phi(\lambda^\alpha) = \int_0^\infty \frac{\lambda^{\gamma+\alpha}}{\lambda^{2\alpha}+u^2}v(u)du, \quad \text{where } v(u) = \frac{2}{\pi}\Re\big(\phi(iu)\big).$$

Since $0 < 2\alpha \le \gamma + \alpha \le 1$, the function $\lambda \mapsto \lambda^{\gamma+\alpha}/(\lambda^{2\alpha}+u^2)$ leaves the half plane $\{\Im(\lambda) > 0\}$ globally invariant, and then, is a $\mathcal{CBF}$-function for every $u > 0$. By the argument of conic combination, this property remains true for the function $\lambda \mapsto \lambda^\gamma\phi(\lambda^\alpha)$ is $\mathcal{CBF}$. Now, the function $\psi(\lambda) = \lambda^\alpha$ satisfies property (2) of Corollary 3.7, and then, $\sqrt{\lambda}\phi(\lambda^\alpha) \in \mathcal{CBF}^-$.

□

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