Chapter 1: Introduction

<u>1.1:</u> **Suppose that 4 out of 12 buildings in a certain city violate the building code. A building engineer randomly inspects a sample of 3 new buildings in the city.**

given that $p = \frac{4}{12} = \frac{1}{3} \Rightarrow q = \frac{2}{3}$ and $n = 3$

(a) Find the probability distribution function of the random variable X representing the number of buildings that violate the building code in the sample.

$$\boldsymbol{P(X = x)} = \binom{n}{x} p^x q^{n-x}, \; x = 0,1,\dots,n$$

$$P(X = x) = \binom{3}{x}\left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{3-x}, \; x = 0,1,2,3$$

(b) Find the probability that

(i) **none of the buildings in the sample violating the building code**

$$\boldsymbol{P(X = 0)} = \binom{3}{0}\left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{3-0} = 0.2963$$

(ii) **one building in the sample violating the building code.**

$$\boldsymbol{P(X = 1)} = \binom{3}{1}\left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{3-1} = 0.4444$$

(iii) **at lease one building in the sample violating the building code.**

$$\boldsymbol{P(X \geq 1)} = 1 - P(X < 1) = 1 - P(X = 0) = 1 - 0.2963 = 0.7037$$

(c) Find the expected number of buildings in the sample that violate the building code.

$$\mu = np = 3\left(\frac{1}{3}\right) = 1$$

(d) Find Var(X).

$$\sigma^2 = npq = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$$

1.2: **On average, a certain intersection results in 3 traffic accidents per day. Assuming Poisson distribution**

λ: average number of traffic accident per day

λ: 3 per day

X: $number\ of\ traffic\ accident\ per\ day$

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-3}3^x}{x!}, \qquad x = 0,1,2,\ldots$$

(i) what is the probability that at this intersection

(a) no accidents will occur in a given day?

$$P(X = 0) = \frac{e^{-3}3^0}{0!} = 0.4979$$

(b) More than 3 accidents will occur in a given day?

$$P(X > 3) = 1 - P(X \le 3) = 1 - \left(\sum_{x=0}^{3} \frac{e^{-3}3^x}{x!}\right) = 0.3528$$

(c) Exactly 5 accidents will occur in a period of two days?

λ_1 = average number of traffic accidents per 2 days

$\lambda_1 = \lambda t = 3(2) = 6$ per two days

$$P(Y = y) = \frac{e^{-\lambda_1}\lambda_1{}^y}{y!} = \frac{e^{-6}6^y}{y!}, \quad y = 0,1,2,\ldots$$

$$P(Y = 5) = \frac{e^{-6}6^5}{5!} = 0.1606$$

(ii) what is the average number of traffic accidents in a period of 4 days?

λ_2 = average number of traffic accidents per 4 days

$\lambda_2 = \lambda t = 3(4) = 12$ per 4 days

1.3: **If the random variable X has a uniform distribution on the interval (0,10), then**

Given that $X \sim Uniform(0,10) \Rightarrow f_X(x) = \frac{1}{b-a} = \frac{1}{10-0} = \frac{1}{10}, \quad 0 \le x \le 10$

(a) $P(X < 6) = \int_0^6 \frac{1}{10}\,dx = \frac{3}{5}$

(b) **The mean of X is** $\mu = \frac{a+b}{2} = \frac{10}{2} = 5$

(c) **The variance X is** $\sigma^2 = \frac{(b-a)^2}{12} = 8.333$

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1.4: Suppose that Z is distributed according to the standard normal distribution.

(a) the area under the curve to the left of 1.43 is:

$P(Z < 1.43) = 0.9236$

(b) the area under the curve to the right of 0.89 is:

$P(Z > 0.89) = P(Z < -0.89) = 0.1867$

or $P(Z > 0.89) = 1 - P(Z < 0.89) = 1 - 0.8133 = 0.1867$

(c) the area under the curve between 2.16 and 0.65 is:

$P(0.65 < Z < 2.16) = P(Z < 2.16) - P(Z < 0.65) = 0.9846 - 0.7422 = 0.2424$

(d) the value of k such that $P(0.93 < Z < k) = 0.0427$ is:

$P(0.93 < Z < k) = 0.0427$

$\Leftrightarrow \; P(Z < k) - P(Z < 0.93) = 0.0427$

$\Leftrightarrow \; P(Z < k) - 0.8238 = 0.0427$

$\Leftrightarrow \; P(Z < k) = 0.8665$

$\Leftrightarrow \; k = 1.11$

1.5: The finished inside diameter of a piston ring is normally distributed with a mean of 12 centimeters and a standard deviation of 0.03 centimeter. $X \sim Normal(12, (0.03)^2)$

Find:

(a) the proportion of rings that will have inside diameter less than 12.05 centimeters.

$\boldsymbol{P(X < 12.05)} = P\left(Z < \frac{12.05-12}{0.03}\right) = P(z < 1.67) = 0.9525$

(b) the proportion of rings that will have inside diameter exceeding 11.97 centimeters.

$\boldsymbol{P(X > 11.97)} = P\left(Z > \frac{11.97-12}{0.03}\right) = P(Z > -1) = P(Z < 1) = 0.8413$

(c) the probability that a piston ring will have an inside diameter between 11.95 and 12.05 centimeters.

$P(11.95 < X < 12.05) = P\left(\frac{11.95-12}{0.03} < Z < \frac{12.05-12}{0.03}\right)$

$= P(z < -1.67) - P(z < 1.67)$

$= 0.9525 - 0.0475 = 0.905$

1.6: **Let X be $N(\mu,\sigma^2)$ so that $P(X<89)=0.90$ and $P(X<94)=0.95$. find μ and $\sigma 2$.**

$X \sim Normal(\mu, \sigma^2)$

$P(X < 89) = 0.9$

$P\left(Z < \frac{89-\mu}{\sigma}\right)=0.9$

$\frac{89-\mu}{\sigma} = 1.28$

$89 - \mu = 1.28\,\sigma$

$\mu = 89 - 1.28\,\sigma$ ...(1)

$P(X < 94) = 0.95$

$P\left(Z < \frac{94-\mu}{\sigma}\right)=0.95$

$\frac{94-\mu}{\sigma} = 1.645$

$94 - \mu = 1.645\,\sigma$

$\mu = 94 - 1.645\,\sigma$ ...(2)

$P(Z < 1.64) = 0.9495$

$P(Z < 1.65) = 0.9505$

$\frac{1.65 + 1.64}{2} = 1.645$

Then, $89 - 1.28\,\sigma = 94 - 1.645\,\sigma$

$(1.645-1.28)\,\sigma=94-89$

$\Rightarrow \sigma = 13.6986,$

$\Rightarrow \sigma^2=187.65$

We substitute in (1) or (2) by $\sigma = 13.6986$ we get

$\Rightarrow \mu = 71.46575$

1.7: **Assume the length (in minutes) of a particular type of a telephone conversation is a random variable with a probability density function of the form:**

$\boldsymbol{f(x) = 0.2e^{-0.2x}, x > 0} \Rightarrow X \sim \exp\left(\frac{1}{\theta} = \frac{1}{0.2}\right)$

Calculate:

(a) $\boldsymbol{P(3 < x < 10)} = \int_3^{10} 0.2e^{-0.2x}dx = 0.2476$

(b)The cdf of X. $\boldsymbol{F(x)} = 1 - e^{-\theta x} = 1 - e^{-0.2x}$

(c) The mean and the variance of X.

$\mu = \frac{1}{\theta} = \frac{1}{0.2} = 5 \qquad \sigma^2 = \frac{1}{\theta^2} = \frac{1}{0.2^2} = 25$

1.8: Find the moment-generating function of X, if you know that

$f(x) = 2e^{-2x},\ x > 0 \Rightarrow X \sim \exp\left(\frac{1}{\theta} = \frac{1}{2}\right)$

$M_X(t) = \frac{\theta}{\theta - t} = \frac{2}{2-t}$ or $M_X(t) = \left(1 - \frac{t}{\theta}\right)^{-1} = \left(1 - \frac{t}{2}\right)^{-1}$ where, $t < \theta \Rightarrow t < 2$

1.9: For a chi-squared distribution, find

$\chi^2_{0.025,15} = 27.49$

$\chi^2_{0.01,7} = 18.48$

$\chi^2_{0.99,22} = 9.54$

1.10: If $(1 - 2t)^{-6}$,t<12, is the MGF of the random variable X, find $P(X<5.23)$.

given that $M_X(t) = (1 - 2t)^{-6}$,

We know that if $X \sim \chi^2_v \Rightarrow M_X(t) = (1 - 2t)^{-\frac{v}{2}}$,

$\Rightarrow \frac{v}{2} = 6 \Rightarrow v = 12$

$P(X < 5.23) = P(\chi^2_{12} < 5.23) = 1 - P(\chi^2_{12} > 5.23) = 1 - 0.950 = 0.05$

1.11: **Find:**

(a) $t_{0.95,28} = 1.701$

(b) $t_{0.005,16} = -t_{0.995,16} = -2.921$

(c) $-t_{0.01,4} = -(-t_{0.99,4}) = 0.747$

(d) $P(T_{24} > 1.318) = 1 - P(T_{24} < 1.318) = 1 - 0.9 = 0.1$

(e) $P(-1.356 < T_{12} < 2.179) = P(T_{12} < 2.179) - P(T_{12} < -1.356)$

$= P(T_{12} < 2.179) - P(T_{12} > 1.356)$

$= 0.975 - 0.1$

$= 0.875$

1.12: If $f(x) = \theta x^{\theta-1},\ 0 < x < 1$, find the distribution of $Y = -lnX$.

Since $0 < x < 1$

$\Rightarrow ln0 < lnx < ln1$

$\Rightarrow -\infty < lnx < 0$

$\Rightarrow \infty > -lnx > 0$

$\Rightarrow 0 < y < \infty$

- ***by using one to one transformaion method***:

$y = -lnx \quad \Rightarrow -y = lnx \quad \Rightarrow \boldsymbol{x = e^{-y}}$

$\frac{d}{dy}x = -e^{-y} \quad \Rightarrow \left|\frac{d}{dy}x\right| = e^{-y}$

Then, $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = f_X(e^{-y})\left|\frac{d}{dy}x\right| = \theta(e^{-y})^{\theta-1}e^{-y} = \theta e^{-\theta y}$

$\Rightarrow \quad Y \sim \exp\left(\frac{1}{\theta}\right)$

- ***by using CDF method***:

$F_X(x) = P(X < x) = \int_0^x f(t)dt = \int_0^x \theta t^{\theta-1}dt = x^\theta$

Then, $F_Y(y) = P(Y < y) = P(-lnX < y)$

$= P(lnX > -y)$

$= P(X > e^{-y})$

$= 1 - P(X < e^{-y})$

$= 1 - F_X(e^{-y})$

$= 1 - (e^{-y})^\theta = 1 - e^{-\theta y}$

We know that $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(1 - e^{-\theta y}) = \theta e^{-\theta y}$

Which is the PDF of $Y \sim \exp\left(\frac{1}{\theta}\right)$.

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<u>1.13:</u> **If $f(x) = 1,\ 0 < x < 1$. Find the pdf of $Y=\sqrt{x}$.**

Since $0 < x < 1$

$\Rightarrow 0 < \sqrt{x} < 1$

$\Rightarrow 0 < y < 1$

- ***by one to one transformaion method***:

$y = \sqrt{x} \Rightarrow \mathrm{x} = y^2$

$\frac{d}{dy}x = 2y \Rightarrow \left|\frac{d}{dy}x\right| = 2y$

Then, $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = f_X(y^2)\left|\frac{d}{dy}x\right| = 2y$

- ***by CDF method***:

$F_X(x) = \int_0^x f(t)dt = \int_0^x 1dt = x$

Then, $F_Y(y) = P(Y < y) = P\left(\sqrt{X} < y\right)$

$= P(X < y^2)$

$= F_X(y^2)$

$= y^2$

We know that $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}y^2 = 2y$

1.14: If $X \sim U(0,1)$, find the pdf of $Y = -2lnX$.
Name the distribution and its parameter values.

$f(x) = 1, \ 0 < x < 1$

Since $0 < x < 1$

$ln0 < lnx < ln1$

$-\infty < lnx < 0$

$\infty > -2lnx > 0$

$0 < y < \infty$

- ***by one to one transformaion method:***

$$y = -2lnx \ \Rightarrow -\frac{y}{2} = lnx \ \Rightarrow x = e^{-\frac{y}{2}}$$

$$\frac{d}{dy}x = -\frac{1}{2}e^{-\frac{y}{2}} \ \Rightarrow \left|\frac{d}{dy}x\right| = \frac{1}{2}e^{-\frac{y}{2}}$$

Then, $f_Y(y) = f_X(x)\left|\frac{d}{dy}x\right| = f_X\left(e^{-\frac{y}{2}}\right)\left|\frac{d}{dy}x\right| = \frac{1}{2}e^{-\frac{y}{2}}$

$$\Rightarrow \quad Y \sim \exp\left(\frac{1}{\theta} = \frac{1}{\frac{1}{2}} = 2\right)$$

- ***by CDF method:***

$$F_X(x) = \int_0^x f(t)dt = \int_0^x 1dt = x$$

Then, $F_Y(y) = P(Y < y) = P(-2lnX < y)$

$$= P\left(lnX > -\frac{y}{2}\right)$$

$$= P\left(X > e^{-\frac{y}{2}}\right)$$

$$= 1 - P\left(X < e^{-\frac{y}{2}}\right)$$

$$= 1 - F_X\left(e^{-\frac{y}{2}}\right)$$

$= 1 - e^{-\frac{y}{2}}$

We know that $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(1 - e^{-\frac{y}{2}}) = \frac{1}{2} e^{-\frac{y}{2}}$

Which is the PDF of $Y \sim \exp\left(\frac{1}{\theta} = \frac{1}{\frac{1}{2}} = 2\right)$.

1.15: **Suppose independent random variables X and Y are such that**

$M_{X+Y}(t) = \frac{e^{2t}-1}{2t-t^2}$.

If And $f_X(x) = 2e^{-2x},\ x > 0$, what is the distribution of Y.

Given that $f_X(x) = 2e^{-2x},\ x > 0$

Which is the PDF of $x \sim \exp\left(\frac{1}{2}\right)$. $\Rightarrow\quad M_X(t) = \frac{\theta}{\theta - t} = \frac{2}{2-t}$

As X and Y independent $\Rightarrow M_{X+Y}(t) = M_X(t) M_Y(t)$

$M_Y(t) = \frac{M_{X+Y}(t)}{M_X(t)} = \frac{e^{2t}-1}{2t-t^2} / \frac{2}{2-t} \quad \frac{e^{2t}-1}{t(2-t)} \cdot \frac{2-t}{2} = \frac{e^{2t}-1}{2t} = \frac{e^{2t}-e^{0t}}{t(2-0)}$

the MGF of $x \sim$ uniform $(a = 0, b = 2)$

1.16: **If $X_1 \sim \chi^2{}_n$ and $X_2 \sim \chi^2{}_m$ are independent random variables. Find the distribution of**

Y=$X_1 + X_2$.

We know that if $X \sim \chi^2{}_v \quad \Rightarrow \quad M_X(t) = (1-2t)^{-\frac{v}{2}}$

$M_Y(t) = M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$

$= (1-2t)^{-\frac{n}{2}} (1-2t)^{-\frac{m}{2}}$

$= (1-2t)^{-\frac{n+m}{2}}$

Which is the MGF of $y \sim \chi^2{}_{n+m}$.