Chapter 2: Sampling Distribution

2.1: If e^{3t+4t^2} is the MGF of the random variable $\bar{X}$ with sample size 6, find P($-2<\bar{X}<6$).

given that, $M_{\bar{X}}(t)=e^{3t+4t^2} \Rightarrow \bar{X}\sim N(\mu_{\bar{X}},\sigma^2{}_{\bar{X}})$

know that if $X\sim N(\mu_X,\sigma^2{}_X) \Rightarrow M_X(t)=e^{\mu t+\frac{1}{2}\sigma^2 t^2}$

Then, $\mu_{\bar{X}}=3$ and $\frac{1}{2}\sigma^2{}_{\bar{X}}=4 \Rightarrow \sigma^2{}_{\bar{X}}=8$

$\Rightarrow \bar{X}\sim N(3,8)$

$P(-2<\bar{X}<6)=P\left(\frac{-2-3}{\sqrt{8}}<Z<\frac{6-3}{\sqrt{8}}\right)$

$P(-1.77<Z<1.06)$

$=P(Z<1.06)-P(Z<-1.77)$

$=0.8554-0.0384$

$=0.8170$

2.2: **Let X be the mean of a random sample of size 5 from a normal distribution with $\mu=0$ and $\sigma 2=125$. Determine c so that $(\bar{X}<c)=0.975$.**

given that $X\sim N(0,125)$, n=5 and $P(\bar{X}<c)=0.975$

$X\sim N(\mu,\sigma^2)\Rightarrow \bar{X}\sim N(\mu,\frac{\sigma^2}{n})$

Since $X\sim N(0,125)\Rightarrow \bar{X}\sim N(0,\frac{125}{5}=25)$

$(\bar{X}<c)=0.97 \quad \Rightarrow P\left(Z<\frac{c-0}{5}\right)=0.975$

$\Rightarrow \quad \frac{c-0}{5}=\frac{c}{5}=1.96$

c=9.8

2.3: **Determine the mean and variance of the mean $\overline{X}$ of a random sample of size 9 from a distribution having pdf** $f(x) = \begin{cases} 4x^3, & 0 < x < 1 \\ 0 & , else\ where \end{cases}$

We want to find $\mu_{\bar{X}}$ and $\sigma^2{}_{\bar{X}}$, we have to find first μ_X and $\sigma^2{}_X$

$$\mu_X = E(X) = \int_0^1 x\, f(x) dx = \int_0^1 x\, 4x^3 dx = \int_0^1 4x^4\, dx = \frac{4}{5}$$

$$\sigma^2{}_X = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \int_0^1 x^2\, f(x) = \int_0^1 x^2\, 4x^3 dx = \int_0^1 4x^5\, dx = \frac{4}{6}$$

$$\sigma^2{}_X = \frac{4}{6} - \left(\frac{4}{5}\right)^2 = \frac{2}{75}$$

Then, $\mu_{\bar{X}} = \mu_X = \frac{4}{5}$ $\quad \sigma^2{}_{\bar{X}} = \frac{\sigma^2{}_X}{n} = \frac{\frac{2}{75}}{9} = \frac{2}{675}$.

2.4: **Let $Z_1, Z_2, \dots, Z_{16}$ be a random sample of size 16 from the standard normal distribution $(0, 1)$. Let $X_1, X_2, \dots, X_{64}$ be a random sample of size 64 from the normal distribution $(\mu, 1)$. The two samples are independent.**

(a) $P(Z_1 < 2) = 0.9772$

(b) $P\left(\sum_{i=1}^{16} Z_i > 2\right)$

We know that if

$X_i \sim N(\mu, \sigma^2), i = 1,2,..,n \Rightarrow \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$

$Z \sim N(0,1)$ Then, $\sum_{i=1}^{16} Z_i \sim N(0,16)$

$P(\sum_{i=1}^{16} Z_i > 2) = \left(Z > \frac{2-0}{4}\right) = P(Z > 0.5) = P(Z < -0.5) = 0.3085$

> $P(Z > a) = 1 - P(Z < a)$
>
> or $P(Z > a) = P(Z < -a)$

(c) $P\left(\sum_{i=1}^{16} Z^2{}_i > 6.91\right)$

We know that if $Z_i \sim N(0,1), i = 1,2,..,n \Rightarrow Z_i{}^2 \sim \chi^2{}_1 \Rightarrow \sum_{i=1}^{n} Z^2{}_i \sim \chi^2{}_n$

Then, $P(\sum_{i=1}^{16} Z^2{}_i > 6.91) = P\left(\chi^2{}_{16} > 6.91\right) = 0.975$

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(a) find c such that $P(S^2 > c) = 0.05$

since $Z_i \sim N(0,1), i = 1,2,..,16$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} = \frac{(16-1)S^2}{1} = 15S^2 \sim \chi^2_{n-1} = \chi^2_{15}$$

Then, $P(S^2 > c) = 0.05 \Rightarrow P(15S^2 > 15c) = 0.05 \Rightarrow P(\chi^2_{15} > 15c) = 0.05$

$$\Rightarrow 15c = 25 \Rightarrow c = \frac{25}{15} = \frac{5}{3}$$

(b) **What is the distribution** of $Y = \sum_{i=1}^{16} Z^2{}_i + \sum_{i=1}^{64}(X_i - \mu)^2$

We know from (c) that $\sum_{i=1}^{16} Z^2{}_i \sim \chi^2_{16}$

Since $X_i \sim N(\mu, 1) \Rightarrow \frac{X_i - \mu}{1} \sim N(0,1) \Rightarrow (X_i - \mu)^2 \sim \chi^2_1 \Rightarrow \sum_{i=1}^{64}(X_i - \mu)^2 \sim \chi^2_{64}$

And we know that if $X \sim \chi^2_n$ and $Y \sim \chi^2_m \Rightarrow X + Y \sim \chi^2_{n+m}$.

Then, $\boldsymbol{Y} = \sum_{i=1}^{16} Z^2{}_i + \sum_{i=1}^{64}(X_i - \mu)^2 \sim \boldsymbol{\chi^2}_{\mathbf{16+64=80}}$

(c) **E(Y)=**80

(d) **Var(Y)=**2(80)=160

(e) $\boldsymbol{P(Y > 105)} = P(\chi^2_{80} > 105) = 0.025$

(f) **Find c such that** $c\frac{\sum_{i=1}^{16} Z^2{}_i}{Y} \sim F_{16,80}$

we know that $\sum_{i=1}^{16} Z^2{}_i \sim \chi^2_{16}$ and $Y \sim \chi^2_{80}$

$$\Rightarrow \frac{\sum_{i=1}^{16} Z^2{}_i/16}{Y/80} \sim F_{16,80}$$

Then, $c = \frac{\frac{1}{16}}{\frac{1}{80}} = \frac{80}{16} = 5$

(g) **Let** $Q \sim \chi^2_{60}$ **find c such that** $P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95$

We know that $Z_1 \sim N(0,1)$ and given that $Q \sim \chi^2_{60}$ then $\frac{Z_1}{\sqrt{\frac{Q}{60}}} \sim t_{60}$

$$P\left(\frac{Z_1}{\sqrt{Q}} < c\right) = 0.95 \Rightarrow P\left(\frac{Z_1}{\sqrt{\frac{Q}{60}}} < \frac{c}{\sqrt{\frac{1}{60}}}\right) = 0.95 \Rightarrow P\left(t_{60} < \frac{c}{\frac{1}{\sqrt{60}}}\right) = 0.95$$

$\Rightarrow P(t_{60} < \sqrt{60}c) = 0.95$

$\Rightarrow \sqrt{60}c = 1.671$

$\Rightarrow c = \frac{1.671}{\sqrt{60}} = 0.2157$

(h) Find c such that $P(F_{60,20} > c) = 0.99$

$$c = F_{0.99,60,20} = \frac{1}{F_{0.01,20,60}} = \frac{1}{2.20} = 0.4545$$

2.5: Let $X \sim N(5,10)$ find $P(0.04 < (X-5)^2 < 38.4)$

Since $X \sim N(5,10) \Rightarrow \frac{X-5}{\sqrt{10}} \sim N(0,1) \Rightarrow \left(\frac{X-5}{\sqrt{10}}\right)^2 \sim \chi^2{}_1$

Then, $P(0.04 < (X-5)^2 < 38.4) = P\left(\frac{0.04}{(\sqrt{10})^2} < \left(\frac{X-5}{\sqrt{10}}\right)^2 < \frac{38.4}{(\sqrt{10})^2}\right)$

$= P(0.004 < \chi^2{}_1 < 3.84)$

$= P(\chi^2{}_1 < 3.84) - P(\chi^2{}_1 < 0.004)$

$= \left(1 - P(\chi^2{}_1 > 3.84)\right) - \left(1 - P(\chi^2{}_1 > 0.004)\right)$

$= 1 - P(\chi^2{}_1 > 3.84) - 1 + P(\chi^2{}_1 > 0.004)$

$= P(\chi^2{}_1 > 0.004) - P(\chi^2{}_1 > 3.84) = 0.95 - 0.05 = 0.90$

2.6: Let S^2 be the variance of a random sample of size 6 from the normal distribution $(\mu, 12)$. Find

Given that $X \sim N(\mu, 12) \quad n = 6$

(a) $E(S^2) = \sigma^2 = 12$

$$V(S^2) = \frac{2\sigma^4}{n-1} = \frac{2(\sigma^2)^2}{6-1} = \frac{2(12)^2}{5} = 57.6$$

(b) Distribution of S^2

We know that $X \sim N(\mu, 12) \Rightarrow \frac{(n-1)S^2}{\sigma^2} \sim \chi^2{}_{n-1}$

$\Rightarrow \frac{(6-1)S^2}{12} \sim \chi^2{}_{n-1} \Rightarrow \frac{5S^2}{12} \sim \chi^2{}_5$

(c) $P(2.30 < S^2 < 22.2)$

We have from (b) $\frac{5S^2}{12} \sim \chi^2{}_5$

$(\frac{5}{12}(2.30) < \frac{5}{12}S^2 < \frac{5}{12}(22.2)) = P(\frac{5}{12}(2.30) < \chi^2{}_5 < \frac{5}{12}(22.2))$

$= P(0.96 < \chi^2{}_5 < 9.25)$

$= P(\chi^2{}_5 > 0.96) - P(\chi^2{}_5 > 9.25)$

$=0.975-0.1$

$=0.875$

2.7: **Let X_1, X_2 and X_3 be iid random variable, each with pdf $f(x) = e^{-x}$, $0 < x < \infty$; and let $Y_1 < Y_2 < Y_3$ be the order statistics of the random variables. Find:**

(a) the distribution of $Y_1 = \min(X_1, X_2, X_3)$

Since the pdf is $f_X(x) = e^{-x} \Rightarrow X \sim exp(1) \Rightarrow F_X(x) = 1 - e^{-x}$

$f_{Y_1}(y_1) = n\, f_X(y_1)[1 - F_X(y_1)]^{n-1}, \quad 0 < y_1 < \infty$

$= 3e^{-y_1}[1 - (1 - e^{-y_1})]^{3-1}$

$= 3e^{-y_1}e^{-2y_1} = 3e^{-3y_1}$

$f_{Y_1}(y_1) = 3e^{-3y_1} \qquad 0 < y_1 < \infty$

$Y_1 \sim exp\left(\frac{1}{3}\right) \Rightarrow F_{Y_1}(y_1) = P(Y_1 \le y_1) = 1 - e^{-3y_1}$

(b) $P(3 \le Y_1) = P(Y_1 \ge 3) = 1 - P(Y_1 < 3) = \left[1 - \left(1 - e^{-3(3)}\right)\right] = e^{-9} = 0.00012$

(c) The joint pdf of Y_2 and Y_3

$$f_{r,k}(y_r, y_k) = \frac{n!}{(r-1)!\,(k-r-1)!\,(n-k)!}[F_X(y_r)]^{r-1}[F_X(y_k) - F_X(y_r)]^{k-r-1}[1 - F_X(y_k)]^{n-k} f_X(y_k) f_X(y_r)$$

We have here $r = 2$, $k = 3$ and n=3

$$f_{2,3}(y_2, y_3) = \frac{3!}{(2-1)!\,(3-2-1)!\,(3-3)!}[F(y_2)]^{2-1}[F(y_3) - F(y_2)]^{3-2-1}[1 - F(y_3)]^{n-k} f(y_3) f(y_2)$$

$= 6[1 - e^{-y_2}][(1 - e^{-y_3}) - (1 - e^{-y_2})]^0[1 - (1 - e^{-y_3})]^0 e^{-y_3}e^{-y_2}$

$= 6[1 - e^{-y_2}]\, e^{-(y_2+y_3)}, \quad 0 < y_2 < y_3 < \infty.$

2.8: **Let $Y_1 < Y_2 < \cdots < Y_n$ be the order statistics from a Weibull distribution. Find the distribution function and pdf of Y_1.**

If $X \sim Weibull\left(\alpha, \frac{1}{\beta}\right)$

$\Rightarrow f_X(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}$

$\Rightarrow F_X(x) = 1 - e^{-\alpha x^\beta}, \quad x \geq 0, \alpha > 0, \beta > 0$

$$f_{Y_1}(y_1) = n f_X(y_1)[1 - F_X(y_1)]^{n-1}, \quad 0 < y_1 < \infty$$

$$= n\,\alpha\beta {y_1}^{\beta-1} e^{-\alpha {y_1}^\beta} \left[1 - (1 - e^{-\alpha {y_1}^\beta})\right]^{n-1}$$

$$= n\alpha\beta {y_1}^{\beta-1} e^{-\alpha {y_1}^\beta} \left[e^{-\alpha {y_1}^\beta}\right]^{n-1}$$

$$= n\alpha\beta {y_1}^{\beta-1} \left[e^{-\alpha {y_1}^\beta}\right]^{n}$$

$$= n\alpha\beta {y_1}^{\beta-1} e^{-n\alpha {y_1}^\beta}$$

$$\Rightarrow Y_1 \sim Weibull\left(n\alpha, \frac{1}{\beta}\right)$$

Then, $F_{Y_1}(y_1) = 1 - e^{-n\alpha x^\beta}$.

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