Chapter 3: Point Estimation

3.1: **Suppose $X_1, X_2, \dots, X_n$ is a random sample from gamma distribution:**

$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad , x > 0 \Rightarrow X \sim \text{Gamma}\left(\alpha, \frac{1}{\beta}\right) \Rightarrow E(X) = \frac{\alpha}{\beta} \text{ and } V(X) = \frac{\alpha}{\beta^2}$

Derive the MME for parameters α and β.

MME:

$$\mu_i' = M_i, \text{ where } \mu_i' = E\left(X^i\right) \text{ and } M_i = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,i}$$

$$\mu_1' = M_1 \qquad\qquad \mu_2' = M_2$$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^1) = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,1} \qquad\qquad E(X^2) = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,2}$$

$$E(X) = \frac{1}{n}\sum_{j=1}^{n} X_j \qquad\qquad V(X) + (E(X))^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,2}$$

$$\frac{\alpha}{\beta} = \bar{X} \quad \dots(1) \qquad\qquad \frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} \quad \dots(2)$$

From (1) we have $\alpha = \bar{X}\beta \quad \dots(3)$

By putting (3), *into* (2), *we get*

$$\frac{\bar{X}\beta}{\beta^2} + \left(\frac{\bar{X}\beta}{\beta}\right)^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} \Rightarrow \frac{\bar{X}}{\beta} + \bar{X}^2 = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,2}$$

$$\Rightarrow \frac{\bar{X}}{\beta} = \frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} - \bar{X}^2$$

$$\Rightarrow \hat{\beta} = \frac{\bar{X}}{\frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} - \bar{X}^2} \quad \dots(4)$$

We put (4) in (3) we got

$$\hat{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{j=1}^{n} X_j^{\,2} - \bar{X}^2}.$$

Note:

- $\prod_{i=1}^{n} x_i^{\,a} = \left(\prod_{i=1}^{n} x_i\right)^a$
- $\prod_{i=1}^{n} e^{\frac{x_i}{\theta}} = e^{\frac{\sum_{i=1}^{n} x_i}{\theta}}$
- $\prod_{i=1}^{n} \frac{2}{\theta} = \left(\frac{2}{\theta}\right)^n$
- $\prod_{i=1}^{n} x_i! = \prod_{i=1}^{n} x_i!$
- $\prod_{i=1}^{n} e^{\theta} = e^{n\theta}$
- $\ln\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \ln(x_i)$
- $\sum_{i=1}^{n} 1 = n$

3.2: **Find the MME and the MLE for the parameter p of Bernoulli distribution:**

$$f(x) = p^x q^{1-x}, \ x = 0, 1$$

Then, determine the unbiasedness, sufficiency and consistency of the MLE

$\Rightarrow X \sim \text{Bernoulli}(p) \ \Rightarrow E(X) = p \ \ and \ \ V(X) = pq$ where, $q = 1 - p$

MME:

$$\mu_i{}' = M_i, \ where \ \ \mu_i{}' = E\left(X^i\right) \ \ and \ \ M_i = \frac{1}{n}\sum_{j=1}^{n} X_j{}^i$$

$$\mu_1{}' = M_1 \ \Rightarrow \ E(X^1) = \frac{1}{n}\sum_{j=1}^{n} X_j{}^1 \Rightarrow \ \hat{p} = \bar{X}$$

MLE:

(1) $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p^{x_i} q^{1-x_i} = p^{\sum_{i=1}^{n} x_i} q^{n-\sum_{i=1}^{n} x_i}$

(2) $logL = \sum_{i=1}^{n} x_i \, logp + (n - \sum_{i=1}^{n} x_i) \log(1-p)$

(3) $\frac{\partial}{\partial p} \ logL = 0$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{p} - \frac{n-\sum_{i=1}^{n} x_i}{1-p} = 0$$

$$\Rightarrow \frac{(1-p)\sum_{i=1}^{n} x_i - p(n-\sum_{i=1}^{n} x_i)}{p(1-p)} = 0$$

$$\Rightarrow (1-p)\sum_{i=1}^{n} x_i - p(n - \sum_{i=1}^{n} x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} x_i - np + p\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - np = 0$$

$$\Rightarrow p = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \ \hat{p} = \bar{X}$$

Unbiasedness:

$E(\hat{p}) = E(\bar{X}) = E(X) = p \ \Rightarrow \hat{p} = \bar{X} \ is \ unbiased \ estimator \ of \ p$

Sufficiency:

By using factorization theorem

We have $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p^{x_i} q^{1-x_i} = p^{\sum_{i=1}^{n} x_i} q^{n-\sum_{i=1}^{n} x_i} = p^{n\bar{X}} q^{n-n\bar{X}}$

We can see that we can write $\prod_{i=1}^{n} f(x_i) = K_1(\overline{X}, P)K_2(x_1, x_2, \dots, x_n)$,

Where $K_1(\overline{X}, P) = p^{n\overline{X}} q^{n-n\overline{X}}$ and $K_2(x_1, x_2, \dots, x_n) = 1$

$\Rightarrow \hat{p} = \overline{X}$ Sufficient of P.

Consistency:

(1) $E(\hat{p}) = \boldsymbol{E}(\overline{X}) = \boldsymbol{E}(X) = p$

$\Rightarrow \lim_{n\to\infty} E(\hat{p}) = \boldsymbol{p}$

(2) $V(\hat{p}) = \boldsymbol{V}(\overline{X}) = \frac{\boldsymbol{V}(X)}{n} = \frac{p(1-p)}{n}$

$\Rightarrow \lim_{n\to\infty} V(\hat{p}) = \lim_{n\to\infty} \frac{p(1-p)}{n} = \mathbf{0}$

Then, from (1) and (2) $\hat{p} = \overline{X}$ consistent of P.

3.3: **Let $f(x) = \theta e^{-\theta x}, x > 0$, and let T be an estimator for $\tau(\theta)$. Study if T is unbiased, consistent estimator for $\tau(\theta)$, then compute MSE in the three cases**

Given that $f(x) = \theta e^{-\theta x}, x > 0 \Rightarrow X \sim \exp\left(\frac{1}{\theta}\right) \Rightarrow E(X) = \frac{1}{\theta}$ and $V(X) = \frac{1}{\theta^2}$

(a) $T = \overline{X}$ $\quad \tau(\theta) = \frac{1}{\theta}$

Unbiasedness:

$E(T) = E(\overline{X}) = E(X) = \frac{1}{\theta} = \boldsymbol{\tau(\theta)} \quad \Rightarrow T = \overline{X}$ *is unbiased estimator of* $\frac{1}{\theta}$.

Consistency:

(1) $E(T) = E(\overline{X}) = E(X) = \frac{1}{\theta}$

$\Rightarrow \lim_{n\to\infty} E(T) = \frac{1}{\theta} = \boldsymbol{\tau(\theta)}$

(2) $V(T) = V(\overline{X}) = \frac{V(X)}{n} = \frac{\frac{1}{\theta^2}}{n}$

$\Rightarrow \lim_{n\to\infty} V(T) = \lim_{n\to\infty} \frac{1}{n\theta^2} = 0$

MSE:

$$MSE(T) = V(T) + \left[\frac{1}{\theta} - E(T)\right]^2 = \frac{1}{n\theta^2} + \left[\frac{1}{\theta} - \frac{1}{\theta}\right]^2 = \frac{1}{n\theta^2}$$

Or since $T = \bar{X}$ *is unbiased estimator of* $\frac{1}{\theta}$ then

$$MSE(T) = V(T) = V(\bar{X}) = \frac{V(X)}{n} = \frac{\frac{1}{\theta^2}}{n} = \frac{1}{n\theta^2}$$

(b) $T = \frac{1}{\bar{X}} \qquad \tau(\theta) = \theta$

Since $X \sim \exp\left(\frac{1}{\theta}\right) \equiv X \sim \text{Gamma}\left(1, \frac{1}{\theta}\right) \Rightarrow y = \sum_{i=1}^{n} X_i \sim \text{Gamma}\left(n, \frac{1}{\theta}\right) \Rightarrow f(y) = \frac{\theta^n}{\Gamma(n)} y^{n-1} e^{-\theta y}$

Then, $E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y)dy = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{y} y^{n-1} e^{-\theta y} dy$

$$= \frac{\theta^n}{\Gamma(n)} \int_0^\infty y^{n-2} e^{-\theta y} dy$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} \int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} dy$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-1)}{\theta^{n-1}} = \boxed{\frac{\theta}{n-1}}$$

$$\boxed{\int_0^\infty \frac{\theta^{n-1}}{\Gamma(n-1)} y^{n-2} e^{-\theta y} dy = 1}$$

$$\boxed{\Gamma(n) = (n-1)!}$$

$E\left(\frac{1}{Y^2}\right) = \int_0^\infty \frac{1}{y^2} f(y)dy = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{y^2} y^{n-1} e^{-\theta y} dy$

$$= \frac{\theta^n}{\Gamma(n)} \int_0^\infty y^{n-3} e^{-\theta y} dy$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} \int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} y^{n-3} e^{-\theta y} dy$$

$$\boxed{\int_0^\infty \frac{\theta^{n-2}}{\Gamma(n-2)} y^{n-3} e^{-\theta y} dy = 1}$$

$$= \frac{\theta^n}{\Gamma(n)} \frac{\Gamma(n-2)}{\theta^{n-2}} = \boxed{\frac{\theta^2}{(n-1)(n-2)}}$$

Then, $V\left(\frac{1}{Y}\right) = E\left(\frac{1}{Y^2}\right) - \left(E\left(\frac{1}{Y}\right)\right)^2 = \frac{\theta^2}{(n-1)(n-2)} - \left(\frac{\theta}{n-1}\right)^2 = \frac{\theta^2}{(n-1)^2(n-2)}$

$$\boxed{E\left(\frac{1}{\bar{X}}\right) \neq \frac{E(1)}{E(\bar{X})} \qquad E\left(\frac{X}{a}\right) = \frac{E(X)}{a}}$$

Unbiasedness:

$$E(T) = E\left(\frac{1}{\bar{X}}\right) = E\left(\frac{1}{\frac{\sum_{i=1}^{n} X_i}{n}}\right) = E\left(\frac{n}{\sum_{i=1}^{n} X_i}\right) = nE\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = nE\left(\frac{1}{Y}\right) = \frac{n\theta}{\mathrm{n-1}} \neq \theta = \boldsymbol{\tau(\theta)}$$

$T = \frac{1}{\bar{X}}$ *is a biased estimator of* $\boldsymbol{\tau(\theta)} = \theta.$

Consistency:

$$(1)\ \ E(T) = E\left(\frac{1}{\bar{X}}\right) = E\left(\frac{1}{\frac{\sum_{i=1}^{n} X_i}{n}}\right) = E\left(\frac{n}{\sum_{i=1}^{n} X_i}\right) = nE\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = nE\left(\frac{1}{Y}\right) = \frac{n\theta}{\mathrm{n-1}}$$

$$\Rightarrow lim_{n\to\infty}\, E(T) = lim_{n\to\infty} \frac{n\theta}{\mathrm{n-1}} = \theta = \boldsymbol{\tau(\theta)}$$

$$(2)\ V(T) = V\left(\frac{1}{\bar{X}}\right) = V\left(\frac{1}{\frac{\sum_{i=1}^{n} X_i}{n}}\right) = V\left(\frac{n}{\sum_{i=1}^{n} X_i}\right) = n^2 V\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = n^2 V\left(\frac{1}{Y}\right) = \frac{n^2\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})}$$

$$\Rightarrow lim_{n\to\infty}\, V(T) = lim_{n\to\infty} \frac{n^2\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})} = 0$$

Then, from (1) and (2) $T = \frac{1}{\bar{X}}$ *is* consistent. *of* θ.

MSE:

$$MSE(T) = V(T) + [\theta - E(T)]^2 = \frac{n^2\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})} + \left[\theta - \frac{n\theta}{\mathrm{n-1}}\right]^2 = \frac{(n^2+n-2)\theta^2}{(\mathrm{n-1})^2(\mathrm{n-2})}$$

(c) $\boldsymbol{T = \frac{n-1}{\sum_{i=1}^{n} X_i}}$ $\quad \boldsymbol{\tau(\theta) = \theta}$

Unbiasedness:

$$E(T) = E\left(\frac{\boldsymbol{n-1}}{\sum_{i=1}^{n} X_i}\right) = (n-1)E\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = (n-1)E\left(\frac{1}{Y}\right) = (n-1)\frac{\boldsymbol{\theta}}{\mathbf{n-1}} = \theta$$

$\boldsymbol{T} = \frac{\boldsymbol{n-1}}{\sum_{i=1}^{n} X_i}$ *is an unbiased estimator of* $\boldsymbol{\tau(\theta)} = \theta.$

Consistency:

(1) $E(T) = E\left(\frac{n-1}{\sum_{i=1}^{n} X_i}\right) = (n-1)E\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = (n-1)E\left(\frac{1}{Y}\right) = \theta$

$\Rightarrow lim_{n\to\infty} E(T) = lim_{n\to\infty}\, \theta = \theta$

(2) $V(T) = V\left(\frac{n-1}{\sum_{i=1}^{n} X_i}\right) = (n-1)^2 V\left(\frac{1}{\sum_{i=1}^{n} X_i}\right) = (n-1)^2 V\left(\frac{1}{Y}\right) = \frac{(n-1)^2\theta^2}{(n-1)^2(n-2)} = \frac{\theta^2}{(n-2)}$

$\Rightarrow lim_{n\to\infty} V(T) = lim_{n\to\infty} \frac{\theta^2}{(n-2)} = 0$

Then, from (1) and (2) $T = \frac{n-1}{\sum_{i=1}^{n} X_i}$ is consistent. of θ.

MSE:

$MSE(T) = V(T) + [\theta - E(T)]^2 = \frac{\theta^2}{(n-2)} + [\theta - \theta]^2 = \frac{\theta^2}{(n-2)}$

3.4: If $X_1, X_2, \dots, X_n$ be a random sample from $(x;\theta)$. Show if the given statistic T is sufficient statistic for θ: $f(x;\theta) = e^{-(x-\theta)}$, $x > \theta$;

$T = Y_1 = \text{Minimum}(X_1, X_2, \dots, X_n)$.

we say that T is a sufficient statistic of θ if and only if $\frac{\prod_{i=1}^{n} f_X(x_i)}{f_T(t)}$ does not depend on θ

$\prod_{i=1}^{n} f_X(x_i) = \prod_{i=1}^{n} e^{-(x-\theta)}) = e^{-(\sum_{i=1}^{n} x - n\theta)} = e^{-\sum_{i=1}^{n} x + n\theta}$

$f_T(t) = f_{Y_1}(y_1) = n f_X(y_1)[1 - F_X(y_1)]^{n-1}, \quad y_1 > \theta$

$F_X(x) = P(X \le x) = \int_0^x f_X(t)\, dt = \int_0^x e^{-(t-\theta)}\, dt = 1 - e^{-(x-\theta)}, x > \theta$

$f_T(t) = f_{Y_1}(y_1) = n e^{-(y_1-\theta)}\left[1 - \left(1 - e^{-(y_1-\theta)}\right)\right]^{n-1} =$

$= n e^{-(y_1-\theta)} e^{-(y_1-\theta)^n} e^{-(y_1-\theta)^{-1}} = n e^{-n(y_1-\theta)}, \quad y_1 > \theta$

$f(X_1, X_2, \dots, X_n) = \frac{\prod_{i=1}^{n} f_X(x_i)}{f_T(t)} = \frac{e^{-\sum_{i=1}^{n} x_i + n\theta)}}{n e^{-n(y_1-\theta)}} = \frac{e^{-\sum_{i=1}^{n} x_i} e^{n\theta}}{n e^{-n y_1} e^{n\theta}} = \frac{e^{-\sum_{i=1}^{n} x_i}}{n e^{-n y_1}}$

Which does not depend on θ, then T=min$(X_1, X_2, \dots, X_n) = Y_1$ is sufficient statistic of θ

3.5: **Suppose for a given random variable T_1 and T_2 be two independents unbiased estimators for θ and with the same variance $\sigma 2$. Define two random variables as**

$$Y = \frac{3T_1+2T_2}{5} \text{ and } Z = \frac{T_1+2T_2}{3}$$

Find MSE(Y) and MSE(Z) and compare between them.

Since T_1 and T_2 are unbiased estimators of θ

Then, $E(T_1) = \theta$ and $E(T_2) = \theta$

$$\boldsymbol{MSE(Y)} = V(Y) + [\theta - E(Y)]^2$$

$$E(Y) = E\left(\frac{3T_1+2T_2}{5}\right) = \frac{3}{5}E(T_1) + \frac{2}{5}E(T_2) = \frac{3}{5}\theta + \frac{2}{5}\theta = \theta$$

$\Rightarrow Y$ is an unbiased estimators of $\theta \Rightarrow \; MSE(Y) = V(Y)$

$$V(Y) = V\left(\frac{3T_1+2T_2}{5}\right) = \frac{9}{25}V(T_1) + \frac{4}{25}V(T_2) = \frac{9}{25}\sigma^2 + \frac{4}{25}\sigma^2 = \boldsymbol{\frac{13}{25}\sigma^2}$$

$$\Rightarrow MSE(Y) = \boldsymbol{\frac{13}{25}\sigma^2}$$

$$\boldsymbol{MSE(Z)} = V(Z) + [\theta - E(Z)]^2$$

$$E(Z) = E\left(\frac{T_1+2T_2}{3}\right) = \frac{1}{3}E(T_1) + \frac{2}{3}E(T_2) = \frac{1}{3}\theta + \frac{2}{3}\theta = \theta$$

$\Rightarrow Z$ is an unbiased estimators of $\theta \Rightarrow \; MSE(Z) = V(Z)$

$$V(Z) = V\left(\frac{T_1+2T_2}{3}\right) = \frac{1}{9}V(T_1) + \frac{4}{9}V(T_2) = \frac{1}{9}\sigma^2 + \frac{4}{9}\sigma^2 = \boldsymbol{\frac{5}{9}\sigma^2}$$

$$\Rightarrow MSE(Z) = \boldsymbol{\frac{5}{9}\sigma^2}$$

Comparing $\boldsymbol{MSE(Y)}$ and $\boldsymbol{MSE(Z)}$

$$\left.\begin{aligned} MSE(Y) &= \frac{13}{25}\sigma^2 = \frac{13(9)}{25(9)}\sigma^2 = \frac{117}{225}\sigma^2 \\ MSE(Z) &= \frac{5}{9}\sigma^2 = \frac{5(25)}{9(25)}\sigma^2 = \frac{125}{225}\sigma^2 \end{aligned}\right\} \Rightarrow \frac{125}{225}\sigma^2 > \frac{117}{225}\sigma^2 \Rightarrow \text{Y is better estimator of } \theta \text{ than Z}$$

3.6: Let $(x,)=\frac{1}{\theta}$;$x\in(0,\theta)$, and let T **be an estimator for** θ. **Study if** T **is unbiased, consistent and compute MSE, then compare between their variances for the following cases:**

Given $X \sim Uniform(0, \theta)$

$$\Rightarrow f_X(x) = \frac{1}{\theta},\ E(X) = \frac{\theta}{2},\ V(X) = \frac{\theta^2}{12} \text{ and } F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{\theta}, & 0 \le x < \theta \\ 1, & x \ge \theta \end{cases}$$

(a) T =min$(X_1, X_2, \dots, X_n) = Y_1$

$$f_{Y_1}(y_1) = n f_X(y_1)[1 - F_X(y_1)]^{n-1}, \quad 0 \le y_1 < \theta$$

$$= n\frac{1}{\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1}$$

$$E(Y_1) = \int_0^\theta y_1 f_{Y_1}(y_1) d\,y_1 = \int_0^\theta n\frac{y_1}{\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1, \quad \text{let } u = \frac{y_1}{\theta} \Rightarrow du = \frac{1}{\theta}dy_1 \Rightarrow \theta du = dy_1$$

$$= n\theta \int_0^1 u^{2-1}[1-u]^{n-1} d\,u$$

$$= n\theta\beta(2,n) = n\theta\frac{\Gamma(2)\Gamma(n)}{\Gamma(2+n)}$$

$$= n\theta\frac{\Gamma(n)}{(n+1)n\Gamma(n)} = \frac{\theta}{(n+1)}$$

- Beta function

$$\beta(a,b) = \int_0^1 x^{a-1}[1-x]^{b-1} d\,x = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

- $\Gamma(n) = (n-1)!$

$$E(Y_1^2) = \int_0^\theta y_1^2 f_{Y_1}(y_1) d\,y_1 = \int_0^\theta n\frac{y_1^2}{\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1$$

$$= \int_0^\theta \theta\theta n\frac{y_1^2}{\theta\theta}\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1$$

$$= \int_0^\theta \theta^2 n\left(\frac{y_1}{\theta}\right)^2\left[1 - \frac{y_1}{\theta}\right]^{n-1} d\,y_1, \quad \text{let } u = \frac{y_1}{\theta} \Rightarrow du = \frac{1}{\theta}dy_1 \Rightarrow \theta du = dy_1$$

$$= n\theta^2 \int_0^1 u^{3-1}[1-u]^{n-1} d\,y_1$$

$$= n\theta^2\beta(3,n) = n\theta\frac{\Gamma(3)\Gamma(n)}{\Gamma(3+n)}$$

$$= n\theta^2\frac{\Gamma(n)}{(n+2)(n+1)n\Gamma(n)} = \frac{2\theta^2}{(n+2)(n+1)}$$

$$V(\boldsymbol{Y_1}) = E(\boldsymbol{Y_1}^2) - (E(\boldsymbol{Y_1}))^2 = \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \frac{\boldsymbol{n\theta^2}}{\boldsymbol{(n+2)(n+1)^2}}$$

Unbiasedness:

$$\boldsymbol{E}(\mathrm{T}) = \boldsymbol{E}(\boldsymbol{Y_1}) = \frac{\boldsymbol{\theta}}{n+1}$$

$\Rightarrow \; T = \boldsymbol{Y_1}$ *is a biased estimator of* θ

Consistency:

(1) $E(\mathrm{T}) = E(Y_1) = \frac{\theta}{n+1}$

$\Rightarrow lim_{n\to\infty} E(T) = lim_{n\to\infty} \frac{\theta}{n+1} = 0 \neq \theta$ (it's not asymptotically unbiased)

No need to check the other condition we can see here that $T = \boldsymbol{Y_1}$ is not a consistent estimator of θ

MSE:

$$MSE(T) = V(T) + [\theta - E(T)]^2$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)} + \left[\theta - \frac{\theta}{n+1}\right]^2$$

$$= \frac{(n+2n^2+n^3)\theta^2}{(n+1)^2(n+2)}$$

(b) $\mathrm{T} = \boldsymbol{nY_1}$

Unbiasedness:

$$E(\mathrm{T}) = E(nY_1) = \frac{n\theta}{n+1}$$

$\Rightarrow \; T = nY_1$ *is a biased estimator of* θ

Consistency:

(1) $E(\mathrm{T}) = E(nY_1) = \frac{n\theta}{n+1}$

$\Rightarrow lim_{n\to\infty} E(nY_1) = lim_{n\to\infty} \frac{n\theta}{n+1} = \theta$ (it's asymptotically unbiased)

2) $V(T) = V(nY_1\ ) = n^2 V(Y_1) = n^2 \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{n^3\theta^2}{(n+1)^2(n+2)}$

$\Rightarrow lim_{n\to\infty} V(T) = lim_{n\to\infty} \frac{n^3\theta^2}{(n+1)^2(n+2)} = \theta^2 \neq 0$

Then, we can see here that $T = nY_1$ is not a consistent estimator $of\ \ \theta$

MSE:

$MSE(T) = V(T) + [\theta - E(T)]^2$

$= \frac{n^3\theta^2}{(n+1)^2(n+2)} + \left[\theta - \frac{n\theta}{n+1}\right]^2$

$= \frac{(n^3+n+2)\theta^2}{(n+1)^2(n+2)}$

(c) T = $2\bar{X}$

Unbiasedness:

$E(\mathrm{T}) = E(2\bar{X}) = 2E(\bar{X}) = 2E(X) = 2\frac{\theta}{2} = \theta$

$\Rightarrow\ \ T = 2\bar{X}\ is\ an\ unbiased\ estimator\ of\ \ \theta$

Consistency:

(1) $E(\mathrm{T}) = E(2\bar{X}) = 2E(\bar{X}) = 2E(X) = 2\frac{\theta}{2} = \theta$

$\Rightarrow lim_{n\to\infty} E(2\bar{X}) = lim_{n\to\infty} \theta = \theta$

2) $V(T) = V(2\bar{X}) = 4V(\bar{X}) = \frac{4V(X)}{n} = \frac{4\left(\frac{\theta^2}{12}\right)}{n} = \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}}$

$\Rightarrow lim_{n\to\infty} V(T) = lim_{n\to\infty} \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}} = 0$

Then, we can see here that $T = 2\bar{X}_1$ is a consistent estimator of θ .

MSE:

since T is an unbiased estimators of θ

$\Rightarrow\ MSE(T) = V(T)\ \ = \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}}.$

(d) $T = \frac{n+1}{n} Y_n$, where $Y_n = maximum$

$f_{Y_n}(y_n) = n f_X(y_n)[F_X(y_n)]^{n-1}, \quad 0 \le y_n < \theta$

$$= n \frac{1}{\theta}\left[\frac{y_n}{\theta}\right]^{n-1} = \frac{n}{\theta^n} {y_n}^{n-1}$$

$$E(Y_n) = \int_0^\theta y_n f_{Y_n}(y_n) d\, y_n = \int_0^\theta n \frac{y_n}{\theta}\left[\frac{y_n}{\theta}\right]^{n-1} d\, y_n$$

$$= \int_0^\theta \frac{n}{\theta^n} {y_n}^n d\, y_n = \frac{n\theta}{n+1}$$

$$E({Y_n}^2) = \int_0^\theta {y_n}^2 f_{Y_n}(y_n) d\, y_n = \int_0^\theta n \frac{{y_n}^2}{\theta}\left[\frac{y_n}{\theta}\right]^{n-1} d\, y_n$$

$$= \int_0^\theta \frac{n}{\theta^n} {y_n}^{n+1} d\, y_n = \frac{\theta^2 n}{n+2}$$

$$V(Y_n) = E({Y_n}^2) - (E(Y_n))^2 = \frac{\theta^2 n}{n+2} - \left(\frac{\theta n}{n+1}\right)^2 = \frac{n\theta^2}{(n+1)^2(n+2)}$$

Unbiasedness:

$$E(T) = E\left(\frac{n+1}{n} Y_n\right) = \frac{n+1}{n} E(Y_n) = \frac{n+1}{n}\left(\frac{\theta n}{n+1}\right) = \theta$$

$\Rightarrow \; T = \frac{n+1}{n} Y_n$ *is an unbiased estimator of* θ

Consistency:

1) $E(T) = E\left(\frac{n+1}{n} Y_n\right) = \frac{n+1}{n} E(Y_n) = \frac{n+1}{n}\left(\frac{\theta n}{n+1}\right) = \theta$

$$lim_{n\to\infty} E(T) = lim_{n\to\infty}\, \theta = \theta$$

2) $V(T) = V\left(\frac{n+1}{n} Y_n\right) = \left(\frac{n+1}{n}\right)^2 V(Y_n) = \frac{(n+1)^2}{n^2} \frac{n\theta^2}{(n+1)^2(n+2)} = \frac{\theta^2}{n(n+2)}$

$\Rightarrow lim_{n\to\infty} V(T) = lim_{n\to\infty} \frac{\theta^2}{n(n+2)} = 0$

Then, we can see here that $T = \frac{n+1}{n} Y_n$ is a consistent estimator of θ .

MSE:

since T is an unbiased estimators of θ

$$MSE(T) = V(T) \quad = \frac{\boldsymbol{\theta^2}}{n(n+2)} = \frac{\boldsymbol{\theta^2}}{n(n+2)}.$$

Comparing the MSE

We will compare (c) and (d) because they are unbiased estimators

$$MSE(2\overline{X}) = \frac{\boldsymbol{\theta^2}}{\boldsymbol{3n}} = \frac{(n+2)\theta^2}{3n(n+2)}$$

$$MSE\left(\frac{n+1}{n}Y_n\right) = \frac{\boldsymbol{\theta^2}}{\boldsymbol{n(n+2)}} = \frac{3\theta^2}{3n(n+2)}$$

Since n=1,2,3,...

$$\boldsymbol{n \geq 1 \Rightarrow n+2 \geq 1+2}$$

$$\Rightarrow n+2 \geq 3$$

$$\Rightarrow \frac{(n+2)\theta^2}{3n(n+2)} \geq \frac{3\theta^2}{3n(n+2)}$$

$\Rightarrow \frac{n+1}{n}Y_n$ is a better estimator of θ than $2\overline{X}$

3.7: **For a random sample $X1, X2, \ldots, Xn$ drawn from the following distributions, find the Fisher information, $I_X(\theta)$:**

$$\boldsymbol{I_X(\theta) = nI(\theta)}$$

$$\boldsymbol{I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2 = -E\left[\frac{d^2}{d\theta^2} lnf(x)\right]}$$

(a) $\boldsymbol{Bernoulli(\theta)}$

$$\boldsymbol{f(x) = \theta^x(1-\theta)^{1-x}, \quad x = 0, 1} \qquad \boldsymbol{E(X) = \theta} \qquad \boldsymbol{V(X) = \theta\,(1-\theta)}$$

$$lnf(x) = ln\theta^x(1-\theta)^{1-x} = xln\theta + (1-x)\ln(1-\theta)$$

$$\frac{d}{d\theta}lnf(x) = \frac{x}{\theta} - \frac{1-x}{(1-\theta)}$$

$$\frac{d^2}{d\theta^2}lnf(x) = \frac{-x}{\theta^2} - (-(-1))\frac{(1-x)}{(1-\theta)^2}$$

$$= \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}$$

$$I(\theta) = E\left[\frac{d}{d\theta}lnf(x)\right]^2 = E\left[\frac{x}{\theta} - \frac{1-x}{(1-\theta)}\right]^2$$

$$= E\left[\frac{x(1-\theta)-\theta(1-x)}{\theta(1-\theta)}\right]^2 = E\left[\frac{x-x\theta-\theta+x\theta}{\theta(1-\theta)}\right]^2$$

$$= E\left[\frac{x-\theta}{\theta(1-\theta)}\right]^2 = \frac{1}{\theta^2(1-\theta)^2}E[X-\theta]^2$$

$$= \frac{1}{\theta^2(1-\theta)^2}E[X-E(X)]^2$$

V(X)= $E[X-E(X)]^2$

$$= \frac{1}{\theta^2(1-\theta)^2}V[X] = \frac{1}{\theta^2(1-\theta)^2}\theta(1-\theta) = \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$$

Or

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2}lnf(x)\right] = -E\left[\frac{-X}{\theta^2} - \frac{(1-X)}{(1-\theta)^2}\right]$$

$$= -E\left[\frac{-X(1-\theta)^2-(1-X)\theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= -E\left[\frac{2\theta X-X-\theta^2}{\theta^2(1-\theta)^2}\right] = \frac{-1}{\theta^2(1-\theta)^2}E[2\theta X - X - \theta^2]$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[2\theta E(X) - E(X) - \theta^2]$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[2\theta\theta - \theta - \theta^2]$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[2\theta^2 - \theta - \theta^2] =$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[\theta^2-\theta] = \frac{\theta-\theta^2}{\theta^2(1-\theta)^2} = \frac{\theta(1-\theta)}{\theta^2(1-\theta)^2} = \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$$

(b) $Exponential(\frac{1}{\theta})$

$$f(x) = \theta e^{-\theta x}, \quad x > 0 \qquad E(X) = \frac{1}{\theta} \qquad V(X) = \frac{1}{\theta^2}$$

$$\ln f(x) = \ln\theta + \ln e^{-\theta x} = \ln\theta - \theta x$$

$$\frac{d}{d\theta}\ln f(x) = \frac{1}{\theta} - x$$

$$\frac{d^2}{d\theta^2}\ln f(x) = \frac{-1}{\theta^2}$$

$$I(\theta) = E\left[\frac{d}{d\theta}\ln f(x)\right]^2 = E\left[\frac{1}{\theta} - X\right]^2$$

$$= E\left[X - \frac{1}{\theta}\right]^2$$

$$= E[X - E(X)]^2$$

$$= V[X] = \frac{1}{\theta^2}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta^2}$$

Or

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2}\ln f(x)\right] = -E\left[\frac{-1}{\theta^2}\right]$$

$$= -\left[\frac{-1}{\theta^2}\right] = \frac{1}{\theta^2}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta^2}$$

(c) $Normal(\theta, \sigma^2)$ σ^2 $known$

$$f(x)=\frac{1}{\sqrt{2\pi\boldsymbol{\sigma}^2}}e^{-\frac{1}{2}\frac{(x-\boldsymbol{\theta})^2}{\boldsymbol{\sigma}^2}},\quad x\in(-\infty,\infty) \qquad E(X)=\boldsymbol{\theta} \qquad V(X)=\boldsymbol{\sigma}^2$$

$$f(x)=\frac{1}{(2\pi\boldsymbol{\sigma}^2)^{1/2}}e^{-\frac{1}{2}\frac{(x-\boldsymbol{\theta})^2}{\boldsymbol{\sigma}^2}}$$

$$lnf(x)=\ln(1)-\frac{1}{2}\ln(2\pi\boldsymbol{\sigma}^2)-\frac{1}{2}\frac{(x-\boldsymbol{\theta})^2}{\boldsymbol{\sigma}^2}$$

$$lnf(x)=-\frac{1}{2}\ln(2\pi\boldsymbol{\sigma}^2)-\frac{1}{2}\frac{(x-\boldsymbol{\theta})^2}{\boldsymbol{\sigma}^2}$$

$$\frac{d}{d\theta}lnf(x)=\frac{-1}{2\sigma^2}2(x-\boldsymbol{\theta})(-\mathbf{1})=\frac{(x-\boldsymbol{\theta})}{\boldsymbol{\sigma}^2}$$

$$\frac{d^2}{d\theta^2}lnf(x)=\frac{-1}{\sigma^2}$$

$$\boldsymbol{I(\theta)=E\left[\frac{d}{d\theta}lnf(x)\right]^2}=\boldsymbol{E}\left[\frac{X-\theta}{\sigma^2}\right]^2$$

$$=\frac{1}{\sigma^4}E[X-\boldsymbol{\theta}]^2$$

$$=\frac{1}{\sigma^4}E[X-E(X)]^2$$

$$=\frac{1}{\sigma^4}V[X]=\frac{\sigma^2}{\sigma^4}=\frac{1}{\sigma^2}$$

$$\Rightarrow \boldsymbol{I_X(\theta)=nI(\theta)}=\frac{n}{\sigma^2}$$

Or

$$\boldsymbol{I(\theta)=-E\left[\frac{d^2}{d\theta^2}lnf(x)\right]}=-\boldsymbol{E}\left[\frac{-1}{\boldsymbol{\sigma}^2}\right]$$

$$=-\left[\frac{-1}{\theta^2}\right]=\frac{1}{\sigma^2}$$

$$\Rightarrow \boldsymbol{I_X(\theta)=nI(\theta)}=\frac{n}{\sigma^2}$$

<u>3.8:</u> **Let $X_1, X_2,..., X_n$ be a random sample drawn $N(\mu,\sigma 2)$,$\sigma 2$ is known. Find:**

(a) CRLB

$$\text{CRLB}=\frac{\left((\tau(\mu)')\right)^2}{I_X(\mu)}$$

We got from (3.7) (c) $I_X(\mu) = \frac{n}{\sigma^2}$

(i) $\boldsymbol{\tau(\mu) = \mu} \Rightarrow (\tau(\mu))' = 1 \Rightarrow ((\tau(\mu)')^2 = 1$

$$\Rightarrow \text{CRLB}=\frac{((\tau(\mu)')^2}{I_X(\mu)} = \frac{1}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n}$$

(ii) $\boldsymbol{\tau(\mu) = e^{\mu}} \Rightarrow (\tau(\mu))' = e^{\mu} \Rightarrow ((\tau(\mu)')^2 = e^{2\mu}$

$$\Rightarrow \text{CRLB}=\frac{((\tau(\mu)')^2}{I_X(\mu)} = \frac{e^{2\mu}}{\frac{n}{\sigma^2}} = \frac{e^{2\mu}\sigma^2}{n}$$

(iii) $\boldsymbol{\tau(\mu)} = \frac{1}{(\mu+1)} \Rightarrow (\tau(\mu))' = \frac{-1}{(\mu+1)^2} \Rightarrow ((\tau(\mu)')^2 = \frac{1}{(\mu+1)^4}$

$$\Rightarrow \text{CRLB}=\frac{((\tau(\mu)')^2}{I_X(\mu)} = \frac{\frac{1}{(\mu+1)^4}}{\frac{n}{\sigma^2}} = \frac{\frac{1}{(\mu+1)^4}}{\frac{n}{\sigma^2}} = \frac{\sigma^2}{n(\mu+1)^4}$$

(b) MVUE of μ

We have seen that $\hat{\mu}_{MLE} = \hat{\mu}_{MME} = \boldsymbol{\bar{X}}$

$E(\bar{X}) = E(X) = \mu \Rightarrow \bar{X}$ is an unbiased estimator of μ

$$V(\bar{X}) = \frac{V(X)}{n} = \frac{\sigma^2}{n}$$

$$\boldsymbol{V(T) \geq \text{CRLB} = \frac{((\tau(\mu)')^2}{I_X(\mu)}}$$

Here we have $V(\bar{X}) = \frac{\sigma^2}{n}$ **and** CRLB= $\frac{\sigma^2}{n}$ (from 3.8 (a) (i))

$V(\bar{X})$ **is equal to the <u>lower bound</u>** $\Rightarrow \boldsymbol{\bar{X}}$ **is MVUE**

<u>3.10:</u> Let $X1,2,...,Xn$ be a random sample from a distribution with pdf

$$\boldsymbol{f(x;\theta)=\theta^2 x e^{-x\theta}\ , x > 0\ \ , \theta > 0}$$

$$\Rightarrow f(x) = \frac{\theta^2 x^{2-1} e^{-x\theta}}{\Gamma(2)} \Rightarrow X \sim Gamma(2, \tfrac{1}{\theta})$$

(a) Argue that $Y = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic for θ.

$f(x) = \theta^2 x e^{-x\theta}$ is a member of exponential family

$a(\theta) = \theta^2$, $b(x) = x$, $c(\theta) = -\theta$, $d(x) = x$

Then, $Y = \sum_{i=1}^{n} d(x_i) = \sum_{i=1}^{n} x_i$ is a complete sufficient for θ

$f(x)$ is a member of exponential family if we can write it as

$$f(x) = a(\theta)\, b(x) e^{c(\theta) d(x)}$$

(b) Compute $E\left(\frac{1}{Y}\right)$ and find the function of Y which is the unique MVUE of θ.

$$E\left(\frac{1}{Y}\right) = E\left(\frac{1}{\sum_{i=1}^{n} x_i}\right)$$

We have $X \sim Gamma(2, \frac{1}{\theta}) \Rightarrow y = \sum_{i=1}^{n} x_i \sim \text{Gamma}(2n, \frac{1}{\theta})$

$$f(y) = \frac{\theta^{2n} y^{2n-1} e^{-y\theta}}{\Gamma(2n)}$$

$$E\left(\frac{1}{Y}\right) = \int_0^\infty \frac{1}{y} f(y) dy = \int_0^\infty \frac{1}{y} \frac{\theta^{2n} y^{2n-1} e^{-y\theta}}{\Gamma(2n)} dy = \int_0^\infty y^{-1} \frac{\theta^{2n} y^{2n-1} e^{-y\theta}}{\Gamma(2n)} dy$$

$$= \int_0^\infty \frac{\theta^{2n} y^{2n-2} e^{-y\theta}}{\Gamma(2n)} dy = \frac{\theta^{2n}}{\Gamma(2n)} \int_0^\infty y^{(2n-1)-1} e^{-y\theta} dy$$

$$= \frac{\theta^{2n}}{\Gamma(2n)} \frac{\Gamma(2n-1)}{\theta^{2n-1}} \int_0^\infty \frac{\theta^{2n-1}}{\Gamma(2n-1)} y^{(2n-1)-1} e^{-y\theta} dy$$

$$\int_0^\infty \frac{\theta^{2n-1}}{\Gamma(2n-1)} y^{(2n-1)-1} e^{-y\theta} dy = 1$$

$$= \frac{\theta^{2n}}{\Gamma(2n)} \frac{\Gamma(2n-1)}{\theta^{2n-1}}$$

$$= \frac{\theta^{2n}}{(2n-1)\Gamma(2n-1)} \frac{\Gamma(2n-1)}{\theta^{2n-1}}$$

$$= \frac{\theta}{2n-1}$$

By using (Lehman Scheffe Theorem) THEOREM 3.13

From (a) we got that $Y = \sum_{i=1}^{n} x_i$ is a complete sufficient for θ

We have to find a function of Y which is unbiased estimators $E(\tau(Y)) = \theta$

$$\Rightarrow E\left(\frac{1}{Y}\right) = \frac{\theta}{2n-1}$$

$$\Rightarrow E\left(\frac{2n-1}{Y}\right) = (2n-1)E\left(\frac{1}{Y}\right) = (2n-1)\frac{\theta}{2n-1} = \theta$$

Then $\frac{2n-1}{Y}$ is MVUE of θ

(c) Drive the MLE of θ and find the approximate distribution of it.

$$f(x;\theta) = \theta^2 x e^{-x\theta}$$

(1) $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta^2 x_i e^{-x_i\theta} = \theta^{2n} \prod_{i=1}^{n} x_i\, e^{-\theta \sum_{i=1}^{n} x_i}$

(2) $logL = 2nlog\theta + \sum_{i=1}^{n} \log(x_i) - \theta \sum_{i=1}^{n} x_i$

(3) $\frac{\partial}{\partial p}\, logL = 0$

$$\Rightarrow \frac{2n}{\theta} - \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{2n}{\sum_{i=1}^{n} x_i} = \frac{2}{\bar{X}}$$

Suppose that $n \to \infty$ if $\hat{\tau}(\theta)$ be the MLE of $\tau(\theta)$, then $\hat{\tau}(\theta)$ has distribution as $\sqrt{n}(\hat{\tau}(\theta) - \tau(\theta)) \to N\left(0, \frac{((\tau(\theta)')^2}{I_X(\theta)}\right)$ ***or*** $\hat{\tau}(\theta) \to N\left(\tau(\theta), \frac{((\tau(\theta)')^2}{nI_X(\theta)}\right)$.

Then $\tau(\theta) = \theta$ ***and*** $\hat{\theta} = \frac{2}{\bar{X}}$ $\quad \tau'(\theta) = 1$

$$f(x) = \theta^2 x e^{-x\theta}$$

$$lnf(x) = 2\ln(\theta) + \ln(x) - x\theta \ln e$$

$$lnf(x) = 2\ln(\theta) + \ln(x) - x$$

$$\frac{d}{d\theta} lnf(x) = \frac{2}{\theta} - x$$

$$\frac{d^2}{d\theta^2} lnf(x) = \frac{-2}{\theta^2}$$

$$I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2 = E\left[\frac{2}{\theta} - X\right]^2$$

$$= E\left[X - \frac{2}{\theta}\right]^2$$

$$= E[X - E(X)]^2$$

$$=V[X]=\frac{2}{\theta^2}$$

$$\Rightarrow I_X(\theta)=nI(\theta)=\frac{2n}{\theta^2}$$

Or

$$I(\theta)=-E\left[\frac{d^2}{d\theta^2}lnf(x)\right]=-E\left[\frac{-2}{\theta^2}\right]$$

$$=-\left[\frac{-2}{\theta^2}\right]=\frac{2}{\theta^2}$$

$$\Rightarrow I_X(\theta)=nI(\theta)=\frac{2n}{\theta^2}$$

$$\hat{\tau}(\theta)\to N\left(\tau(\theta),\frac{\left((\tau(\theta)'\right)^2}{nI_X(\theta)}\right)$$

$$\frac{2}{\bar{X}}\to N\left(\theta,\frac{1}{\frac{2n}{\theta^2}}\right).$$

$$\frac{2}{\bar{X}}\to N\left(\theta,\frac{\theta^2}{2n}\right)$$

3.11: **Let $X_1, X_2,\dots, X_n$ $n>2$, be a random sample from the binomial distribution $Bino(1,\theta)$.**

$X_i\sim Binomial(1,\theta)\Rightarrow f(x)=\theta^x(1-\theta)^{1-x}\quad x=0,1$

(a) Show that $T=X_1+X_2+\dots+X_n=\sum_{i=1}^{n}X_i$ is a complete sufficient statistic for θ.

$f(x)=\theta^x(1-\theta)^{1-x}$ is a member of exponential family

$$f(x)=\theta^x(1-\theta)^{1-x}=\theta^x(1-\theta)(1-\theta)^{-x}$$

$$=\frac{\theta^x}{(1-\theta)^x}(1-\theta)=\left(\frac{\theta}{1-\theta}\right)^x(1-\theta)=(1-\theta)e^{xln\left(\frac{\theta}{1-\theta}\right)}$$

$a(\theta)=(1-\theta)$, $\quad b(x)=1$, $\quad c(\theta)=ln\left(\frac{\theta}{1-\theta}\right)$, $\quad d(x)=x$

Then, $T=\sum_{i=1}^{n}d(x_i)=\sum_{i=1}^{n}x_i$ is a complete sufficient statistic for θ.

(b) Find the MVUE of θ.

$T = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i \sim Binomial(n, \theta) \Rightarrow E(T) = n\theta$

From (a) , $T = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic for θ

We have to find function of T which is unbiased estimator of θ , $E(\tau(T)) = \theta$

Since we know that $E(T) = n\theta \Rightarrow E(\frac{T}{n}) = \theta =$

$\Rightarrow \tau(T) = \frac{T}{n}$ is MVUE of θ

(c) **Let $T_2 = \frac{X_1+X_2}{2}$ and prove that T_2 is an unbiased estimator for θ.**

$$E(T_2) = E\left(\frac{X_1+X_2}{2}\right) = \frac{E(X_1)+E(X_2)}{2} = \frac{\theta+\theta}{2} = \theta.$$

(d) Find the approximate distribution of the MLE of θ.

MLE:

(1) $L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \theta.^{x_i} (1-\theta.).^{1-x_i} = \theta.^{\sum_{i=1}^{n} x_i} (1-\theta.)^{n-\sum_{i=1}^{n} x_i}$

(2) $logL = \sum_{i=1}^{n} x_i \, log\theta. + (n - \sum_{i=1}^{n} x_i) \log(1-\theta.)$

(3) $\frac{\partial}{\partial \theta.} logL = 0$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{\theta.} - \frac{n-\sum_{i=1}^{n} x_i}{1-\theta.} = 0$$

$$\Rightarrow \Rightarrow \frac{(1-\theta.)\sum_{i=1}^{n} x_i - \theta.(n-\sum_{i=1}^{n} x_i)}{\theta.(1-\theta.)} = 0$$

$$\Rightarrow (1-\theta.)\sum_{i=1}^{n} x_i - \theta.(n - \sum_{i=1}^{n} x_i) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - \theta.\sum_{i=1}^{n} x_i - n\theta. + \theta.\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\theta. = 0$$

$$\Rightarrow \theta. = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \hat{\boldsymbol{\theta}} = \overline{\boldsymbol{X}}$$

Suppose that $n \rightarrow \infty$ if $\hat{\tau}(\theta)$ be the MLE of $\tau(\theta)$, then $\hat{\tau}(\theta)$ has distribution as $\sqrt{n}(\hat{\tau}(\theta) - \tau(\theta)) \rightarrow N\left(0, \frac{((\tau(\theta)')^2}{I_X(\theta)}\right)$ ***or*** $\hat{\tau}(\theta) \rightarrow N\left(\tau(\theta), \frac{((\tau(\theta)')^2}{nI_X(\theta)}\right)$.

Then $\tau(\theta)=\theta$ ***and*** $\widehat{\tau(\theta)} = \hat{\theta} = \bar{X}$ $\tau'(\theta)=1$

$$f(x) = \theta^x(1-\theta)^{1-x}, \quad x = 0, 1 \qquad E(X) = \theta \qquad V(X) = \theta\,(1-\theta)$$

$$lnf(x) = ln\theta^x(1-\theta)^{1-x} = xln\theta + (1-x)\ln(1-\theta)$$

$$\frac{d}{d\theta} lnf(x) = \frac{x}{\theta} - \frac{1-x}{(1-\theta)}$$

$$\frac{d^2}{d\theta^2} lnf(x) = \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}(-(-1))$$

$$= \frac{-x}{\theta^2} - \frac{(1-x)}{(1-\theta)^2}$$

$$I(\theta) = E\left[\frac{d}{d\theta} lnf(x)\right]^2 = E\left[\frac{x}{\theta} - \frac{1-x}{(1-\theta)}\right]^2$$

$$= E\left[\frac{x(1-\theta)-\theta(1-x)}{\theta(1-\theta)}\right]^2 = E\left[\frac{x-x\theta-\theta+x\theta)}{\theta(1-\theta)}\right]^2$$

$$= E\left[\frac{x-\theta}{\theta(1-\theta)}\right]^2 = \frac{1}{\theta^2(1-\theta)^2} E[X-\theta]^2$$

$$= \frac{1}{\theta^2(1-\theta)^2} E[X-E(X)]^2$$

V(X)= $E[X-E(X)]^2$

$$= \frac{1}{\theta^2(1-\theta)^2} V[X] = \frac{1}{\theta^2(1-\theta)^2}\theta(1-\theta) = \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$$

Or

$$I(\theta) = -E\left[\frac{d^2}{d\theta^2} lnf(x)\right] = -E\left[\frac{-X}{\theta^2} - \frac{(1-X)}{(1-\theta)^2}\right]$$

$$= -E\left[\frac{-X(1-\theta)^2 - (1-X)\theta^2}{\theta^2(1-\theta)^2}\right]$$

$$= -E\left[\frac{2\theta X - X - \theta^2}{\theta^2(1-\theta)^2}\right] = \frac{-1}{\theta^2(1-\theta)^2} E[2\theta X - X - \theta^2]$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[2\theta E(X) - E(X) - \theta^2]$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[2\theta\theta - \theta - \theta^2]$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[2\theta^2 - \theta - \theta^2] =$$

$$= \frac{-1}{\theta^2(1-\theta)^2}[\theta^2 - \theta] = \frac{\theta - \theta^2}{\theta^2(1-\theta)^2} = \frac{\theta(1-\theta)}{\theta^2(1-\theta)^2} = \frac{1}{\theta(1-\theta)}$$

$$\Rightarrow I_X(\theta) = nI(\theta) = \frac{n}{\theta(1-\theta)}$$

$$\hat{\tau}(\theta) \to N\left(\tau(\theta), \frac{\left((\tau(\theta)')\right)^2}{nI_X(\theta)}\right)$$

$$\frac{2}{\bar{X}} \to N\left(\theta, \frac{1}{\frac{n}{\theta(1-\theta)}}\right)$$

$$\frac{2}{\bar{X}} \to N\left(\theta, \frac{\theta(1-\theta)}{n}\right)$$