Math 244 - Linear Algebra

Chapter 1: Matrices

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Definition of a matrix

Definition

- A **matrix** is a rectangular array of numbers.
  Examples:

$$\begin{pmatrix} 1 & 2 & 3 \\ \pi & \sqrt{2} & \frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 5 \end{pmatrix}, \quad \begin{pmatrix} 3 \end{pmatrix}$$

- Numbers in a matrix are called **entries** (Singular: **entry**).

Notation

- *We use capital letters to denote matrices.*
- *We use small letters to denote entries.*

Example: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Size of a matrix

Definition

- The **size** of a matrix is the number of rows " $\times$ " the number of columns. Examples:

$$\text{Size}\begin{pmatrix} 1 & 2 & 3 \\ \pi & \sqrt{2} & \frac{1}{2} \end{pmatrix} = 2 \times 3, \quad \text{Size}\begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} = 2 \times 2,$$

$$\text{Size}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2 \times 1, \quad \text{Size}\begin{pmatrix} 1 & 3 & 5 \end{pmatrix} = 1 \times 3, \quad \text{Size}\begin{pmatrix} 3 \end{pmatrix} = 1 \times 1$$

- A **row matrix/horizontal vector** is a matrix with only one row.
- A **column matrix/vertical vector** is a matrix with only one column.
- A **square matrix** is a matrix of size $n \times n$.

Operations on matrices

- Two matrices are said to be **equal** if they have the same size and the same entries at the same position.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ is equivalent to saying } \begin{cases} a = 1, & b = 2, \\ c = 3, & d = 4. \end{cases}$$

Operations on matrices

- Addition or substraction of two matrices is defined only when the two matrices are of the same size. In this case, the operation is performed between entries at the same positition:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \pm \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ is undefined.}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}.$$

- Multiplication of a matrix by a scalar is always defined. It is performed by multiplying all the entries by this scalar:

$$3 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}.$$

The product of two matrices

- For two matrices A, B of size $m \times n$ and $p \times q$, respectively, the product AB is defined if and only if $n = p$. In this case, $\text{Size}(AB) = m \times q$.

$$\underset{2\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}} \text{ is undefined.}$$

$$\underset{2\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} * & * \\ * & * \end{pmatrix}}$$

$$\underset{3\times 2}{\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}}$$

Operations on matrices

- If the condition on the sizes is satisfied, to compute any entry of the product AB, we consider the position of this entry, its row, say i, and its column, say j. We multiply each entry of the i^{th} row of the matrix A by the corresponding entry of the j^{th} column of the matrix B, the first entry by the first entry, the second entry by the second entry, and so on... Then, we sum up all these products.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 6 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 6 \\ 32 & 9 \end{pmatrix},$$

$$2 \times 5 + 3 \times 6 + 4 \times 1 = 32.$$

Properties of the operations on matrices

1. **Commutativity**:
   - The addition of matrices is commutative: $A+B=B+A$, for all matrices A, B of the same size.
   - In general, $AB \neq BA$:
     - $\underset{2\times 2}{A}\ \underset{2\times 3}{B}$ is defined while $\underset{2\times 3}{B}\ \underset{2\times 2}{A}$ is undefined,
     - $\underset{3\times 2}{A}\ \underset{2\times 3}{B}$ has size 3×3 while $\underset{2\times 3}{B}\ \underset{3\times 2}{A}$ has size 2×2,
     - Even if they have the same size:

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix} = \begin{pmatrix}5 & 8\\ 13 & 20\end{pmatrix} \neq \begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}15 & 22\\ 7 & 10\end{pmatrix}.$$

For some matrices we have $AB = BA$. For example:

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 10\\ 15 & 16\end{pmatrix} = \begin{pmatrix}1 & 10\\ 15 & 16\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = \begin{pmatrix}31 & 42\\ 63 & 94\end{pmatrix}.$$

We say in this case that the matrices A and B **commute**.

Properties of the operations on matrices

2. **Associativity**:
   - The addition of matrices is associative: $(A+B)+C = A+(B+C)$.
   - The product of matrices is associative: $(AB)C = A(BC)$.

For example:

$$\left(\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix}\right)\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} = \begin{pmatrix}5 & 8\\ 13 & 20\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix} = \begin{pmatrix}21 & 18\\ 53 & 46\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\left(\begin{pmatrix}3 & 4\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}\right) = \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}11 & 10\\ 5 & 4\end{pmatrix} = \begin{pmatrix}21 & 18\\ 53 & 46\end{pmatrix}$$

That is why, no need to write parentheses. We simply write ABC.

The zero matrix

3. The **zero matrix** is the matrix denoted $0_{m,n}$ (or simply 0) where all entries are zeros. It satisfies

$$A + 0 = 0 + A = A, \quad 0A = 0 \text{ and } A0 = 0.$$

The size is determined by the context.

Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The identity matrix

4. The **identity matrix** is the square matrix denoted by I_n (or Id_n, or simply I or Id), where the entries of the main diagonal are all 1 and the other entries zeros. It satisfies

$$IA = AI = A.$$

The size of the matrix is determined by the context.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ldots$$

Example:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix}$$

Properties of the operations on matrices

Other basic algebraic properties:

5. **Distributivity**:
   - $A(B \pm C) = AB \pm AC$;
   - $(A \pm B)C = AC \pm BC$;
6. $(kA)B = A(kB) = k(AB)$.
7. $k(A \pm B) = kA \pm kB$;
8. $(k_1 \pm k_2)A = k_1 A \pm k_2 A$;
9. $k_1(k_2 A) = (k_1 k_2)A$.

The zero divisors

- For numbers: If $ab = 0$, then $a = 0$ or $b = 0$.
- For matrices, if $AB = 0$, it is not true that $A = 0$ or $B = 0$. We have for example:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0.$$

In this case, the matrices A and B are called zero divisors.

The cancellation law

- For numbers: If $ab = ac$ and $a \neq 0$, then $b = c$.
- For matrices, if $AB = AC$ and $A \neq 0$, it is not true that $B = C$. We have for example:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \neq \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}.$$

The cancellation law fails.

Remark. Notice that if $ab = ac$ then $a(b - c) = 0$. If moreover $a \neq 0$, this implies for numbers that $b - c = 0$ and therefore $b = c$. This argument fails for matrices.

Square roots of 1

- For numbers, the equation $a^2 = 1$ has exactly two solutions:

$$a = \pm 1.$$

- For matrices, there are infinitely many matrices A satisfying $A^2 = I$. For example, for any $t, \theta \in \mathbb{R}$, the matrices

$$A = \begin{pmatrix} t & 1-t \\ 1+t & -t \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

Remark. Notice that if $a^2 = 1$ then $(a-1)(a+1) = 0$. This implies for numbers that $a - 1 = 0$ or $a + 1 = 0$, that is $a = \pm 1$. This argument fails for matrices.

Idempotent matrices

- For numbers, if $a^2 = a$ then $a = 0$ or $a = 1$.
- We have infinitely many matrices A such that $A^2 = A$. For example

$$\begin{pmatrix} a & a \\ 1-a & 1-a \end{pmatrix}^2 = \begin{pmatrix} a & a \\ 1-a & 1-a \end{pmatrix}, \text{ for all } a \in \mathbb{R}.$$

These are called idempotent matrices.

Remark. Notice that if $a^2 = a$ then $a(a-1) = 0$. This implies for numbers that $a = 0$ or $a - 1 = 0$, that is $a = 0$ or $a = 1$. This argument fails for matrices.

Inverse of a matrix

Definition

Let A be a matrix.

1. We say that a matrix B is a left inverse matrix of A if $BA = I$.
2. We say that a matrix C is a right inverse matrix of A if $AC = I$.
3. We say that A is invertible if it has a left inverse and a right inverse matrices.

Remark. If Size $A = m \times n$, then Size $B =$ Size $C = n \times m$.

Theorem (The cancellation law)

1. *If $AX = AY$ and A has a left inverse, then $X = Y$.*
2. *If $XA = YA$ and A has a right inverse, then $X = Y$.*

Invertible matrices

Examples.

- Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. We have

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = I \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}.$$

  The matrix A has infinitely many left inverse matrices and no right inverse matrix.

- The matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has infinitely many right inverse matrices and no left inverse matrix.

Invertible matrices

Theorem

Let A, B be two matrices such that $AB = 0$

1. *If A has a left inverse matrix, then $B = 0$.*
2. *If B has a right inverse matrix, then $A = 0$.*
3. *If $A \neq 0$ and $B \neq 0$ then neither A has a left inverse matrix nor B a right inverse matrix.*

Example. Because $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$, the matrices $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ are not invertible.

Invertible matrices

Theorem

Let A be a matrix of size $m \times n$.

1. *If $m > n$, then A has no right inverse matrix and if it has a left inverse matrix then it has infinitely many left inverses.*
2. *If $m < n$, then A has no left inverse matrix and if it has a right inverse matrix then it has infinitely many right inverses.*
3. *If $m = n$ and A has a left or a right inverse matrix then A is invertible.*

Comments. This theorem means that a non square matrix is never invertible. Moreover, for a square matrix, it is enough to check in only one side if it has an inverse matrix.

Invertible matrices

Theorem

Let A be a matrix, B a left inverse matrix of A and C a right inverse matrix of A. We have $B = C$.
In particular, if A is invertible, it has a unique (left/right) inverse matrix. It is denoted A^{-1}.

Proof.

We have $B = BI = B(AC) = (BA)C = IC = C$. □

For 2×2 matrices

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We have

$$AB = (ad - bc)I.$$

- If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

- If $ad - bc = 0$, then A is not invertible since $AB = 0$.

Examples. The matrix $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is not invertible.

The matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is invertible and $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

Elementary row and column operations

Definition

We call **elementary row operation** on matrices each of the following operations:

1. Interchanging two rows of a matrix.
2. Multiplying a row of a matrix by a nonzero constant.
3. Adding to a row of a matrix a constant times another row of this matrix.

Remark. The inverse operation of any elementary row operation is an elementary row operation.

Definition

Two matrices A and B are said to be **row equivalent** if we can obtain B from A after applying consecutive elementary operations. We denote it by $A \sim B$.

Elementary row and column operations

Definition

We say that a matrix is in a **row echelon form** if it satisfies the following conditions:

1. At each row, the first non zero entry considered from the left, if it exists, must be 1. It is called a **leading one**.
2. For any two non zero rows, the leading 1 in the lower row must be farther to the right than the leading 1 in the higher row.
3. Zero rows, if they exist, must be at the bottom of the matrix.

We say that it is in a **reduced row echelon form** if it satisfies moreover:

4. The other entries of a column with leading 1 must all be zero.

A method for finding A^{-1}

Theorem

Any matrix is row equivalent to a matrix in a reduced row echelon form.

To check if a square matrix A is invertible or not and to find its inverse, we proceed as follow:

1. Consider the matrix $(A|I)$, where I is the identity matrix of the same size as A.
2. Apply elementary row operations to obtain a matrix $(R|B)$ in a reduced row echelon form.
3. If the matrix R contains a row of zeros, then the matrix A is not invertible.
4. If the matrix R is the identity matrix, then the matrix A is invertible and B is its inverse matrix.

A method for finding A^{-1}

Example 1: Is the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ invertible?

Solution. We have

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right) \underset{\substack{R_2 - R_1 \\ R_3 - 3R_1}}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & -1 & -2 & 1 & 0 & -3 \end{array}\right) \underset{R_3 + R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{array}\right)$$

Since the obtained matrix in the left hand side has a row of zeros, we can stop here and conclude that the matrix A is not invertible.

A method for finding A^{-1}

Example 2: Is the matrix $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 2 & 3 \\ 1 & 1 & 0 \end{pmatrix}$ invertible? If yes, compute is its inverse matrix.

Solution. We have

$$\left(\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 2 & -1 & 1 & 0 & 0 \end{array}\right) \underset{\substack{R_2 - R_1 \\ R_3 - 3R_1}}{\sim}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & -1 \\ 0 & -1 & -1 & 1 & 0 & -3 \end{array}\right) \underset{R_3 + R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 & 1 & -4 \end{array}\right) \underset{\frac{1}{2} R_3}{\sim}$$

A method for finding A^{-1}

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -2 \end{array}\right) \underset{R_2-3R_3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} & 5 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -2 \end{array}\right)$$

$$\underset{R_1-R_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & -4 \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} & 5 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & -2 \end{array}\right).$$

Therefore, A is invertible and $A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 1 & -8 \\ -3 & -1 & 10 \\ 1 & 1 & -4 \end{pmatrix}$.

Elementary matrices

Definition

We call **elementary matrix** any matrix which can be obtained from an identity matrix by performing a single elementary row operation.

Examples of elementary matrices:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

However, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is not an elementary matrix since it is obtained by performing two elementary operations.

Elementary matrices

Multiplying a matrix on the left by an elementary matrix gives the same result as performing the corresponding elementary row operation directly to the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{=} \begin{pmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \underset{3R_2}{=} \begin{pmatrix} a_1 & a_2 & a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \underset{R_1 + 2R_3}{=} \begin{pmatrix} a_1 + 2c_1 & a_2 + 2c_2 & a_3 + 2c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

A method for finding A^{-1}

Theorem

Let A, B be two invertible matrices. The product AB is invertible and its inverse is $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.

We have $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.
Therefore, $(AB)^{-1} = B^{-1}A^{-1}$. □

Remark. If A, B, C, D are invertible matrices, then $ABCD$ is invertible and $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$.

Properties of the inverse

Theorem

Let A be an invertible matrix

1. *If $k \neq 0$, then kA is invertible and* $(kA)^{-1} = \frac{1}{k}A^{-1}$.
2. A^{-1} *is invertible and* $\left(A^{-1}\right)^{-1} = A$.

Remark. If A, B are invertible matrices, there is no general rule for $A \pm B$.

$$\underset{Invertible}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \underset{Invertible}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} = \underset{Not\ invertible}{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}$$

The transpose of a matrix

Definition

For any matrix A, the transpose of A is the matrix denoted A^T where the rows of A^T are the columns of A following the order.

Example: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$

Properties

1. $\left(A^T\right)^T = A;$
2. $(A \pm B)^T = A^T \pm B^T;$
3. $(kA)^T = kA^T;$
4. $(AB)^T = B^T A^T$
5. *If A is invertible, then A^T is invertible and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.*

The transpose of a matrix

Example. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We have

$$(AB)^T = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)^T = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix};$$

$$A^T B^T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 6 & 4 \end{pmatrix} \neq (AB)^T;$$

$$B^T A^T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix} = (AB)^T.$$

The power of a square matrix

Definition

Let A be a square matrix and $n \geq 1$ a positive integer. We define the power of A by $A^n = \underbrace{AA \cdots A}_{n \ times}$.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$

Properties

For A a square matrix and $m, n \geq 1$ two positive integer, we have:

1. $A^m A^n = A^n A^m = A^{m+n}$. *In particular, A and A^m commute;*
2. $A\left(A^m\right)^n = \left(A^n\right)^m = A^{mn}$.

Example. Compute $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{1445}$.

The power of a square matrix

Solution. Notice that $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^2 = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$, $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^3 = \begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}$.

We state that $\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^n = \begin{pmatrix}1 & n\\0 & 1\end{pmatrix}$ and prove it by induction on n.

The statement is true for $n = 1$. Assume it is true for n. We have

$$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^{n+1} = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}^n \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & n\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & n+1\\0 & 1\end{pmatrix}.$$

Hence, it is true for $n+1$ and this completes the proof.

Exercise. Compute

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}^{1445}, \begin{pmatrix}0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}^{1445}, \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & 0 & 0\end{pmatrix}^{1445}.$$

The polynomial of a square matrix

Definition

Let A be a square matrix and $P(X)$ a polynomial. We define the matrix $P(A)$ by replacing X by A in the formula of $P(X)$ and the constant term a_0 by $a_0 I$.

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $P(X) = X^5 - 4X^3 + 2X - 5$. Compute $P(A)$.

We have

$$\begin{aligned} P(A) &= A^5 - 4A^3 + 2A - 5I \\ &= \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -6 & -5 \\ 0 & -6 \end{pmatrix}. \end{aligned}$$

The polynomial of a square matrix

Properties

For any square matrix A and any two polynomials $P(X)$ and $Q(X)$, we have:

1. *The matrices A and $P(A)$ commute. That is $AP(A) = P(A)A$.*
2. *More general, the matrices $P(A)$ and $Q(A)$ commute.*
3. *More general, if A and B commute, then $P(A)$ and $Q(B)$ commute.*

For example, if $P(X) = X^5 - 4X^3 + 2X - 5$, we have

$$\begin{aligned} AP(A) &= A\left(A^5 - 4A^3 + 2A - 5I\right) = A^6 - 4A^4 + 2A^2 - 5A \\ &= \left(A^5 - 4A^3 + 2A - 5I\right)A = P(A)A. \end{aligned}$$

In the previous example

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -6 & -5 \\ 0 & -6 \end{pmatrix} = \begin{pmatrix} -6 & -5 \\ 0 & -6 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -6 & -11 \\ 0 & -6 \end{pmatrix}.$$

Symmetric matrices

Definition

A matrix A is said to be symmetric if it satisfies $A^T = A$.

Examples of symmetric matrices: $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.

Properties

Symmetric matrices are square matrices. Let A, B be symmetric matrices, k a scalar and $P(X)$ a polynomial. We have

1. $A \pm B$ *and* kA *are symmetric;*
2. AB *is symmetric if and only if* $AB = BA$*;*
3. $P(A)$ *is symmetric;*
4. *If A is invertible then* A^{-1} *is symmetric.*

Symmetric matrices

For 2: We have $(AB)^T = B^T A^T = BA$. Hence, $(AB)^T = AB$ is equivalent to $AB = BA$. Examples

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 12 & 7 \end{pmatrix} \neq \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 4 & 7 \end{pmatrix};$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 8 \\ 8 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 16 & 24 \\ 24 & 43 \end{pmatrix}.$$

For 3 Example. For $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ and $P(X) = X^3 - 2X + 5$, we have $P(A) = \begin{pmatrix} 24 & 30 \\ 30 & 54 \end{pmatrix}$.

Diagonal matrices

Definition

A diagonal matrix is a square matrix with all the entries outside the main diagonal are zeros.

Examples: $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{pmatrix}$.

Properties

1. $$\begin{pmatrix} d_1 & & (0) \\ & \ddots & \\ (0) & & d_n \end{pmatrix} \begin{pmatrix} d_1' & & (0) \\ & \ddots & \\ (0) & & d_n' \end{pmatrix} = \begin{pmatrix} d_1 d_1' & & (0) \\ & \ddots & \\ (0) & & d_n d_n' \end{pmatrix};$$

Diagonal matrices

Properties

2. $P\begin{pmatrix} d_1 & & (0) \\ & \ddots & \\ (0) & & d_n \end{pmatrix} = \begin{pmatrix} P(d_1) & & (0) \\ & \ddots & \\ (0) & & P(d_n) \end{pmatrix}$, *for any polynomial $P(X)$.*
3. *A diagonal matrix is invertible iff it has no zero on the diagonal. In this case*
$$\begin{pmatrix} d_1 & & (0) \\ & \ddots & \\ (0) & & d_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{d_1} & & (0) \\ & \ddots & \\ (0) & & \frac{1}{d_n} \end{pmatrix}.$$

Diagonal matrices

Example: For $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P(X) = X^3 - 2X + 1$,

We have

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$P(A) = \begin{pmatrix} P(2) & 0 & 0 \\ 0 & P(3) & 0 \\ 0 & 0 & P(1) \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Triangular matrices

Definition

- An upper triangular matrix is a square matrix with all entries below the main diagonal are zeros.
- A lower triangular matrix is a square matrix with all entries above the main diagonal are zeros.

Upper triangular matrices:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 3 & 4 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix},$$

Lower triangular matrices:

$$\begin{pmatrix} 1 & 0 \\ 4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 7 & 5 & 0 \end{pmatrix}.$$

1. The sum/difference/scalar multiplication/product of upper triangular matrices is upper triangular.
2. The sum/difference/scalar multiplication/product of lower triangular matrices is lower triangular.

Triangular matrices

Examples:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 4 & 5 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 17 \\ 0 & 18 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 23 & 18 \end{pmatrix}.$$

However,

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 9 & 12 \end{pmatrix} \text{ is not triangular.}$$

Properties

3. $$\begin{pmatrix} d_1 & & (*) \\ & \ddots & \\ (0) & & d_n \end{pmatrix} \begin{pmatrix} d_1' & & (*) \\ & \ddots & \\ (0) & & d_n' \end{pmatrix} = \begin{pmatrix} d_1 d_1' & & (*) \\ & \ddots & \\ (0) & & d_n d_n' \end{pmatrix};$$
4. *A similar relation for lower triangular matrices.*

Triangular matrices

Properties

5. $P\begin{pmatrix} d_1 & & (*) \\ & \ddots & \\ (0) & & d_n \end{pmatrix} = \begin{pmatrix} P(d_1) & & (*) \\ & \ddots & \\ (0) & & P(d_n) \end{pmatrix}$, *for any polynomial* $P(X)$.
6. *A triangular matrix is invertible iff it has no zero on the main diagonal. In this case*
$$\begin{pmatrix} d_1 & & (*) \\ & \ddots & \\ (0) & & d_n \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{d_1} & & (*) \\ & \ddots & \\ (0) & & \frac{1}{d_n} \end{pmatrix}.$$
7. *Similar relations for lower triangular matrices.*

Triangular matrices

Example: For $A = \begin{pmatrix} 2 & * & * \\ 0 & 3 & * \\ 0 & 0 & 1 \end{pmatrix}$ and $P(X) = X^3 - 2X + 1$,

We have

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & * & * \\ 0 & \frac{1}{3} & * \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$P(A) = \begin{pmatrix} P(2) & * & * \\ 0 & P(3) & * \\ 0 & 0 & P(1) \end{pmatrix} = \begin{pmatrix} 5 & * & * \\ 0 & 4 & * \\ 0 & 0 & 0 \end{pmatrix}.$$