Math 244 - Linear Algebra

Chapter 2: Determinants

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Definition of a matrix

Definition

For any square matrix A of size n, we associate a number, called the determinant of A and denoted $\det A$ or $|A|$, defined by

1. Consider all possible products of n entries no two of them are in the same row nor in the same colum;
2. Multiply each product by $(-1)^k$, where k is the number of switching between two columns to move the selected n entries to the main diagonal.
3. The determinant is the sum of all these products with the corresponding signs.

Definition of a matrix

- For a 2×2 matrix

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix},$$

the possible products are $a_1 b_2$ and $a_2 b_1$.
For $a_1 b_2$, $k = 0$ and for $a_2 b_1$, we switch between the two columns, so $k = 1$. Hence,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

Example

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

Definition of a matrix

- For a 3×3 matrix $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$, the possible products and their corresponding k are:

$a_1b_2c_3$	$a_1b_3c_2$	$a_2b_1c_3$	$a_2b_3c_1$	$a_3b_1c_2$	$a_3b_2c_1$
	$2 \leftrightarrow 3$	$1 \leftrightarrow 2$	$1 \leftrightarrow 2, 2 \leftrightarrow 3$	$1 \leftrightarrow 3, 2 \leftrightarrow 3$	$1 \leftrightarrow 3$
$k = 0$	$k = 1$	$k = 1$	$k = 2$	$k = 2$	$k = 1$

Hence,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1.$$

Definition of a matrix

For an $n \times n$ matrix $A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$, to find all possible products we proceed as follow:

1. We have n possible choices of an entry a_i from the first row.
2. Once a_i has been choosen, we are left with $n-1$ possible choices of an entry b_j from the second row.
3. We proceed in this way until we are left with only 1 choice of an entry from the last row.

In total, we have $n(n-1)(n-2)\cdots 1 = n!$ possible products. The following table shows how big is this number:

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$
2	6	24	120	720	5 040	40 320

Evaluation of a determinant

Here is a mnemonic technique to memorize the formula

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 .$$

This technique is called *Rule of Sarrus*. Recopy the first two columns to the right of the third one. Draw three descending diagonal arrows for the products to add and three ascending diagonal arrows for the products to substract as shown in the figure below:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

Evaluation of a determinant

- Example: Compute $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$.
- Solution: Draw in your draft the following:

$$\begin{array}{ccccc} 1 & 2 & 3 & 1 & 2 \\ 3 & 2 & 1 & 3 & 2 \\ 1 & 1 & 2 & 1 & 1 \end{array}$$

We have $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4 + 2 + 9 - 6 - 1 - 12 = -4.$

- Clearly, we cannot use this technique to compute 4×4 or higher sizes determinants. Many arrows will be missing.

Evaluation of a determinant

We can rewrite the formula as follows:

$$\begin{aligned}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}\end{aligned}$$

This method is called the cofactor expansion. It can be done along any row and along any column. One needs only to know the signs to consider in the expansion: $\begin{vmatrix} a_1^+ & a_2^- & a_3^+ \\ b_1^- & b_2^+ & b_3^- \\ c_1^+ & c_2^- & c_3^+ \end{vmatrix}$

Evaluation of a determinant

Example: Evaluate the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ by using the cofactor expansion method along the 1^{st} row and along the 2^{nd} column.

Solution:

- $$\begin{vmatrix} 1^+ & 2^- & 3^+ \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1 \times \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 3 \times \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}$$
  $$= 5 - 10 - 15 = -20.$$
- $$\begin{vmatrix} 1 & 2^- & 3 \\ 3 & 2^+ & 1 \\ 1 & -1^- & 2 \end{vmatrix} = -2 \times \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + 1 \times \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}$$
  $$= -10 - 2 - 8 = -20.$$

Evaluation of a determinant

Remark

The cofactor expansion method can be used to compute a determinant of any size. For the rule of signs, start for the entry at the top left corner with the sign $+$, and then change the sign with any horizontal or vertical step.

Example:

$$\begin{vmatrix} 1^+ & 2^- & \color{red}{0^+} & 1^- \\ 2^- & 1^+ & \color{red}{1^-} & 0^+ \\ 0^+ & 1^- & \color{red}{0^+} & -1^- \\ 1^- & 1^+ & \color{red}{0^-} & 1^+ \end{vmatrix} = 0 - 1 \times \begin{vmatrix} 1^+ & 2 & 1 \\ 0^- & 1 & -1 \\ 1^+ & 1 & 1 \end{vmatrix} + 0 - 0$$

$$= -\left(\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - 0 + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \right) = -(2-3) = 1$$

Properties of determinants

Properties

Let A, B, C be square matrices and k a scalar.

1. *If all entries of a row or a column of A are zeros, then* $\det A = 0$.
2. $\det A^T = \det A$.
3. *If B is obtained from A by multiplying one row or one column by k then* $\det B = k \det A$.
4. *If size of A is $n \times n$, then* $\det(kA) = k^n \det A$.
5. *If the three matrices A, B, C differ at most in only one row, and if this row in C is obtained from A and B by adding the corresponding rows, then* $\det C = \det A + \det B$.
6. $\det(AB) = \det A \det B$.

Properties of determinants

Examples:

2. $$\begin{vmatrix} 3 & 6 & 18 \\ 5 & 20 & 30 \\ 2 & 2 & 6 \end{vmatrix} = 3 \times 5 \begin{vmatrix} 1 & 2 & 6 \\ 1 & 4 & 6 \\ 2 & 2 & 6 \end{vmatrix} = 2 \times 6 \times 15 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}.$$

3. $$\begin{vmatrix} 3 & 6 & 18 \\ -3 & 15 & 30 \\ 6 & 3 & 6 \end{vmatrix} = 3^3 \begin{vmatrix} 1 & 2 & 6 \\ -1 & 5 & 10 \\ 2 & 1 & 1 \end{vmatrix}.$$

4. $$\begin{vmatrix} 1 & 5 & 2 \\ 3 & 2 & 5 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 5 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 5 & 2 \\ 4 & 5 & 5 \\ 2 & 0 & 1 \end{vmatrix}.$$

Determinants of triangular matrices

In what follow, the cofactor expansion method is applied along the 1^{st} column:

$$\begin{vmatrix} 5 & 3 & 2 & 1 & 5 \\ 0 & 7 & 6 & 3 & 7 \\ 0 & 0 & 3 & 2 & 7 \\ 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 5 \begin{vmatrix} 7 & 6 & 3 & 7 \\ 0 & 3 & 2 & 7 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 5 \times 7 \begin{vmatrix} 3 & 2 & 7 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 5 \times 7 \times 3 \begin{vmatrix} 2 & 6 \\ 0 & 1 \end{vmatrix} = 5 \times 7 \times 3 \times 2 \times 1.$$

Theorem

The determinant of a triangular matrix is the product of the entries in the main diagonal.

Determinants of invertible matrices

Theorem

A matrix A is invertible if and only if $\det A \neq 0$*. In this case,* $\det\left(A^{-1}\right) = \dfrac{1}{\det A}$.

Proof.

- If A is invertible then $AA^{-1} = I$. In this case, we have
$$1 = \det I = \det\left(AA^{-1}\right) = \det A \det\left(A^{-1}\right).$$
Hence, $\det A \neq 0$ and $\det\left(A^{-1}\right) = \dfrac{1}{\det A}$.
- The converse will be proved later.

□

Determinants of elementary matrices

Theorem

Let E be an elementary matrix.

1. *If E is obtained from I by interchanging two rows then* $\det E = -1$.
2. *If E is obtained from I by multiplying a row by a scalar k then* $\det E = k$.
3. *If E is obtained from I by Adding to a row a multiple of another row, then* $\det E = 1$.

Examples:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1, \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 3, \qquad \begin{vmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Determinants and elementary row operations

The following theorem shows how the value of a determinant is affected by applying row elementary operations.

Theorem

Let A and B be two matrices.

1. *If B is obtained from A by interchanging two rows then* $\det B = -\det A$.
2. *If B is obtained from A by multiplying a row by a scalar k then* $\det B = k \det A$.
3. *If B is obtained from A by Adding to a row a multiple of another row, then* $\det B = \det A$.

To evaluate any determinant, we can reduce it to a triangular form by performing elementary row operations.

Determinants and elementary row operations

Example

$$\begin{vmatrix} 2 & 7 & 4 & 9 \\ 1 & 3 & 1 & 4 \\ 3 & 11 & 9 & 18 \\ 2 & 9 & 13 & 24 \end{vmatrix} \underset{R_1 \leftrightarrow R_3}{=} - \begin{vmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 4 & 9 \\ 3 & 11 & 9 & 18 \\ 2 & 9 & 13 & 24 \end{vmatrix} \underset{\substack{R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 2R_1}}{=} - \begin{vmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 6 & 6 \\ 0 & 3 & 11 & 16 \end{vmatrix}$$

$$\underset{\substack{R_3 - 2R_2 \\ R_4 - 3R_2}}{=} - \begin{vmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 5 & 13 \end{vmatrix} \underset{\frac{1}{2}R_3}{=} -2 \begin{vmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 5 & 13 \end{vmatrix}$$

$$\underset{R_4 - 5R_3}{=} -2 \begin{vmatrix} 1 & 3 & 1 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix} = -2 \times 3 = -6.$$

Other properties of the determinant

Properties

1. *If a matrix A has two identical rows or two identical columns, then* $\det A = 0$.
2. *If a matrix A has two proportional rows or two proportional columns, then* $\det A = 0$.

Proportional means, one is obtained from the other by a multiplication by a scalar. Example

$$\begin{vmatrix} 3 & 2 & 1 & 4 & 3 & 6 \\ 2 & 5 & -1 & 6 & -3 & 1 \\ 3 & 4 & 2 & 5 & 6 & 7 \\ 6 & 5 & 3 & 2 & 9 & 5 \\ 0 & 1 & 0 & 7 & 0 & 8 \\ 1 & 5 & -2 & 8 & -6 & 9 \end{vmatrix} = 0, \text{ since } C_5 = 3C_3.$$

Adjoint of a matrix

Consider for a 3×3 matrice A, the following product:

$$\underbrace{\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}}_{A} \begin{pmatrix} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & -\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ -\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{pmatrix}$$

Depending on the colors, the product of a row by a column gives the cofactor expansion along the row of the determinant of:

- the matrix A, if the colors are the same.
- the matrix obtained from A by repeating twice the considered row, if the colors are different. This determinant is zero.

Adjoint of a matrix

The right handside matrix is called the **adjoint matrix** of A.

Theorem

Let A be a square matrix. We have

$$A\ Adj(A) = Adj(A)\ A = \det A\ I.$$

To compute the adjoint matrix of an $n \times n$ matrix A:

- For any $1 \leq i, j \leq n$, we define the **minors** $M_{i,j}$ to be the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} row and the j^{th} column.
- We define the **cofactors** by $C_{i,j} = (-1)^{i+j} M_{i,j}$, using the sign rule of the determinant. The **cofactor matrix** of A is denoted Com A.
- The **adjoint** of A is the matrix Adj $A = (\text{Com } A)^T$.

Adjoint of a matrix

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. The minors of A are

$$M_{1,1} = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3, \quad M_{1,2} = \begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix} = -4, \quad M_{1,3} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$
$$M_{2,1} = \begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = -5, \quad M_{2,2} = \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} = -4, \quad M_{2,3} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1,$$
$$M_{3,1} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = -4, \quad M_{3,2} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8, \quad M_{2,3} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4.$$

Therefore,

$$\text{Com } A = \begin{pmatrix} -3 & 4 & 1 \\ 5 & -4 & 1 \\ -4 & 8 & -4 \end{pmatrix} \text{ and Adj } A = \begin{pmatrix} -3 & 5 & -4 \\ 4 & -4 & 8 \\ 1 & 1 & -4 \end{pmatrix}.$$

Adjoint of a matrix

We have

$$A \text{ Adj } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -3 & 5 & -4 \\ 4 & -4 & 8 \\ 1 & 1 & -4 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8I.$$

We deduce from this relation the following theorem:

Theorem

Let A be a square matrix. If $\det A \neq 0$, then A is invertible and $A^{-1} = \dfrac{1}{\det A} Adj\ A$.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have Adj $A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.