

Chapter 2

Continuous Random Variable

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 - Probability Density Function Definition and Examples
 - Cumulative Distribution Function
 - Mean of the Random Variable
 - Variance of Random Variable

- 2 Some Continuous Probability Distributions
 - Continuous Uniform Distribution
 - Normal Distribution
 - Exponential Distribution
 - Chi-square Distribution
 - T-Distribution
 - F-Distribution

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1.1) Probability Density Function

Definition

The function $f(x)$ is a probability density function (pdf) for the continuous random variable X , defined over the set of real numbers, if

- 1 $f(x) \geq 0$, for all $x \in R$.
- 2 $\int_{-\infty}^{\infty} f(x)dx = 1$.
- 3 $P(a \leq X \leq b) = \int_a^b f(x)dx$.

Example 1 Suppose that the error in the reaction temperature, in C° , for a controlled laboratory experiment is a continuous random variable X having the probability density function

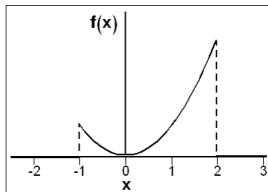
$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

- (a) Verify that $f(x)$ is a density function.
- (b) Find $P(0 \leq X \leq 1)$.
- (c) Find $P(0 < X < 1)$.

Solution

Let X = the error in the reaction temperature, in C° .

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

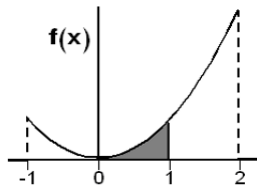


(a) $f(x) > 0$ because $f(x)$ is quadratic function.

$$\begin{aligned} \int_{-\infty}^{+\infty} f(x) dx &= \int_{-1}^2 f(x) dx \\ &= \int_{-1}^2 \frac{x^2}{3} dx \\ &= \frac{1}{3} \times \frac{1}{3} [x^3]_{-1}^2 = 1 \end{aligned}$$

(b)

$$\begin{aligned} P(0 \leq X \leq 1) &= \int_0^1 f(x) dx \\ &= \int_0^1 \frac{x^2}{3} dx \\ &= \frac{1}{3} \times \frac{1}{3} [x^3]_0^1 = \frac{1}{9} \end{aligned}$$



(c) By the same way, we have

$$P(0 < X < 1) = \frac{1}{9}$$

1.2) Cumulative Distribution Function

Definition

The cumulative distribution function $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t)dt, \quad \text{for } -\infty < x < \infty.$$

Example 2

For the density function of Example 1, find $F(x)$, and use it to evaluate $P(0 < X \leq 1)$.

Solution

By definition, we have

$$\begin{aligned} F(x) &= P(X \leq x) = \int_{-\infty}^x f(t) dt \\ &= \int_{-1}^x \frac{t^2}{3} dt = \frac{1}{3} \int_{-1}^x t^2 dt \\ &= \frac{1}{9} t^3 \Big|_{-1}^x = \frac{1}{9} x^3 + \frac{1}{9} \end{aligned}$$

Therefore,

$$\begin{aligned} P(0 < X \leq 1) &= P(X \leq 1) - P(X < 0) \\ &= F(1) - F(0) \\ &= \frac{1}{9} \end{aligned}$$

1.3) Mean of the Random Variable

Definition

Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)dx$$

Example 3

For the density function of Example 1, find $E(X)$.

Solution

$$\begin{aligned} E(X) &= \int_{-\infty}^{+\infty} xf(x)dx = \int_{-1}^2 xf(x)dx = \int_{-1}^2 x \frac{x^2}{3} dx \\ &= \frac{1}{3} \int_{-1}^2 x^3 dx = \frac{1}{12} (16 - 1) = \frac{15}{12} \end{aligned}$$

Theorem

Let X be a random variable with probability distribution $f(x)$.
The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

Example 4

For the density function of Example 1, Find the expected value of the random variable $g(X)$ where $g(X) = 2X + 1$.

Solution

$$\begin{aligned} E[g(X)] &= \int_{-\infty}^{+\infty} g(x)f(x)dx = E[g(X)] = \int_{-1}^2 (2x + 1) \frac{x^2}{3} dx \\ &= \frac{21}{6} \end{aligned}$$

1.4) Variance of Random Variable

Theorem

Let X be a random variable with probability distribution $f(x)$ and mean μ . The variance of X is

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)$$

Theorem

Let X a random variable. The variance of a random variable X is

$$\sigma^2 = E(X^2) - E(X)^2.$$

Theorems

Let X a random variable. If a and b are constants, then

- $E(aX + b) = aE(X) + b$.
- $Var(aX + b) = a^2 Var(X)$.

Theorem

The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

Example 5

Find the expected value of the random variable $g(X)$ where $g(X) = 2X + 1$ by the previous properties of the mean theorems and compare your result with the solution in Example 4. (H.W)

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Definition

The density function of the continuous uniform random variable X on the interval $[a, b]$, and denoted by $U(a, b)$, is

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise.} \end{cases}$$

Definition

The cumulative distribution function (CDF) of $U(a, b)$ is:

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x > b. \end{cases}$$

Theorem

The mean and variance of the uniform distribution $U(a, b)$ are

$$\mu = E(X) = \frac{a + b}{2} \text{ and } \sigma^2 = \frac{(b - a)^2}{12}$$

Example

A delivery company divides their packages into weight classes. Suppose packages in the 10 to 20 pound class are uniformly distributed, meaning that all weights within that class are equally likely to occur. If we are interested in studying packages within weights of 10 to 20 pound class,

- 1 Write the pdf of the random variable represents the weight of package 10 to 20 class.
- 2 Find the probability that the package weights are between 12 and 16.5 pounds
- 3 Find the probability that a package weights is at most 15 pounds
- 4 Find the probability that a package weights is at least 18 pounds
- 5 Find the probability that a package weights is exactly 17 pounds
- 6 Find the expected value and the variance
- 7 Derive the cumulative distribution function CDF of X

Solution

1. Let X represents the weight of package 10 to 20 class. Then the pdf is given by

$$f(x) = \begin{cases} \frac{1}{10}, & 10 \leq x \leq 20 \\ 0, & \text{elsewhere.} \end{cases}$$

2. $P(12 < X < 16.5) = \int_{12}^{16.5} \frac{1}{10} dt = \frac{t}{10} \Big|_{12}^{16.5} = \frac{1}{10}(16.5 - 12) = 0.45.$

3. $P(X \leq 15) = \int_{10}^{15} \frac{1}{10} dt = \frac{t}{10} \Big|_{10}^{15} = \frac{1}{10}(15 - 10) = 0.5.$

4. $P(X \geq 18) = 1 - P(X < 18) = 1 - [\int_{10}^{18} \frac{1}{10} dt] = 1 - [\frac{1}{10}(18 - 10)] = 0.2.$

Or: $P(X \geq 18) = [\int_{18}^{20} \frac{1}{10} dt] = \frac{1}{10}(20 - 18) = 0.2.$

Solution

5. $P(X = 17) = 0.$

6. $E(X) = \frac{10+20}{2} = 15$ and $\sigma^2 = \frac{(20-10)^2}{12} = 8.33.$

7. Since

$$F(x) = \int_{10}^x f(t)dt = \int_{10}^x \frac{1}{10}dt = \frac{x-10}{10}.$$

Then:

$$F(x) = \begin{cases} 0, & x < 10 \\ \frac{x-10}{10}, & 10 \leq x \leq 20 \\ 1, & x > 20. \end{cases}$$

2.1) Normal Distribution

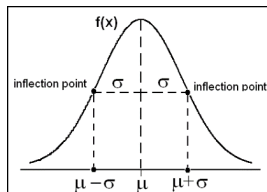
Definition

The density of the normal random variable X , with mean μ and variance σ^2 and denoted by $N(\mu, \sigma^2)$, is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,$$

where $\pi = 3.14159\dots$ and $e = 2.71828\dots$

Note: The graph of the probability density function (pdf) of a normal distribution called the normal curve. The mean and variance are μ and σ^2 .



Properties of the Normal Distribution:

- ① The curve is symmetric and bell-shaped about a vertical axis through the mean, i.e. mean = mode = median = μ .
- ② The total area under the curve and above the horizontal axis is equal to 1.
- ③ Area under the normal curve:
 - I. Approximately 68% of the values in a normally distributed population within 1 standard deviation from the mean, that is:
 $P(\mu - \sigma < X < \mu + \sigma)$
 - II. Approximately 95.5% of the values in a normally distributed population within 2 standard deviations from the mean, that is:
 $P(\mu - 2\sigma < X < \mu + 2\sigma)$
 - III. Approximately 99.7% of the values in a normally distributed population within 3 standard deviation from the mean, that is:
 $P(\mu - 3\sigma < X < \mu + 3\sigma)$

Definition

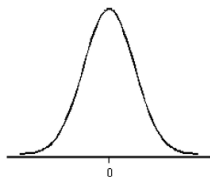
The density of the standard normal distribution Z is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad -\infty < x < \infty,$$

we write

$$Z \sim \text{Normal}(0, 1) \quad \text{or} \quad Z \sim N(0, 1)$$

Note: The graph of the probability density function (pdf) of a standard normal distribution.



Theorem

The mean and variance of standard normal distribution are 0 and 1, respectively.

Theorem

- 1 If X is normal random variable $N(\mu, \sigma^2)$, then the random variable $\frac{X - \mu}{\sigma}$ is a standard normal distribution Z with mean 0 and variance 1.
- 2 If X and Y are independent, $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ then

$$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Table: Probabilities of the standard normal distribution $Z \sim N(0, 1)$ of the form $P(Z \leq a)$ are tabulated.

Note: $P(Z = a) = 0$ for every a .

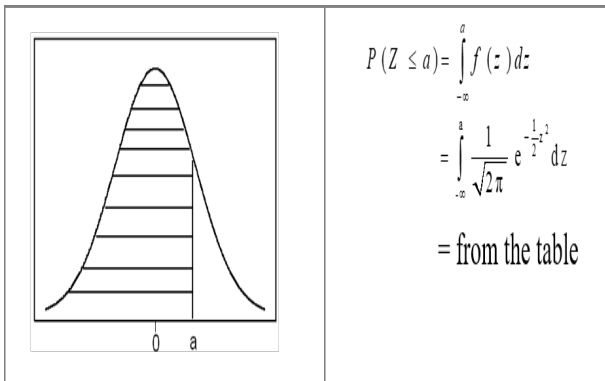
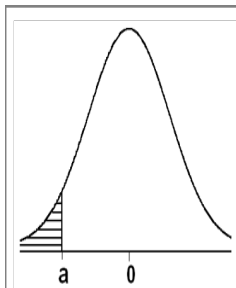
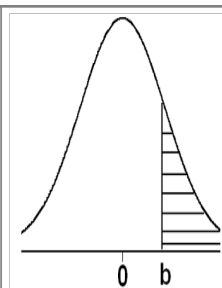


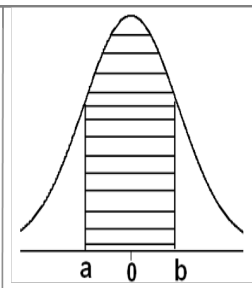
Figure: Areas under the Normal Curve



$$P(Z \leq a) = \text{From Table}$$



$$P(Z \geq b) = 1 - P(Z \leq b)$$



$$P(a \leq Z \leq b) = P(Z \leq b) - P(Z \leq a)$$

| Z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| -3.4 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0003 | 0.0002 |
| -3.3 | 0.0005 | 0.0005 | 0.0005 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0004 | 0.0003 |
| -3.2 | 0.0007 | 0.0007 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0006 | 0.0005 | 0.0005 | 0.0005 |
| -3.1 | 0.0010 | 0.0009 | 0.0009 | 0.0009 | 0.0008 | 0.0008 | 0.0008 | 0.0008 | 0.0007 | 0.0007 |
| -3.0 | 0.0013 | 0.0013 | 0.0013 | 0.0012 | 0.0012 | 0.0011 | 0.0011 | 0.0011 | 0.0010 | 0.0010 |
| -2.9 | 0.0019 | 0.0018 | 0.0018 | 0.0017 | 0.0016 | 0.0016 | 0.0015 | 0.0015 | 0.0014 | 0.0014 |
| -2.8 | 0.0026 | 0.0025 | 0.0024 | 0.0023 | 0.0023 | 0.0022 | 0.0021 | 0.0021 | 0.0020 | 0.0019 |
| -2.7 | 0.0035 | 0.0034 | 0.0033 | 0.0032 | 0.0031 | 0.0030 | 0.0029 | 0.0028 | 0.0027 | 0.0026 |
| -2.6 | 0.0047 | 0.0045 | 0.0044 | 0.0043 | 0.0041 | 0.0040 | 0.0039 | 0.0038 | 0.0037 | 0.0036 |
| -2.5 | 0.0062 | 0.0060 | 0.0059 | 0.0057 | 0.0055 | 0.0054 | 0.0052 | 0.0051 | 0.0049 | 0.0048 |
| -2.4 | 0.0082 | 0.0080 | 0.0078 | 0.0075 | 0.0073 | 0.0071 | 0.0069 | 0.0068 | 0.0066 | 0.0064 |
| -2.3 | 0.0107 | 0.0104 | 0.0102 | 0.0099 | 0.0096 | 0.0094 | 0.0091 | 0.0089 | 0.0087 | 0.0084 |
| -2.2 | 0.0139 | 0.0136 | 0.0132 | 0.0129 | 0.0125 | 0.0122 | 0.0119 | 0.0116 | 0.0113 | 0.0110 |
| -2.1 | 0.0179 | 0.0174 | 0.0170 | 0.0166 | 0.0162 | 0.0158 | 0.0154 | 0.0150 | 0.0146 | 0.0143 |
| -2.0 | 0.0228 | 0.0222 | 0.0217 | 0.0212 | 0.0207 | 0.0202 | 0.0197 | 0.0192 | 0.0188 | 0.0183 |
| -1.9 | 0.0287 | 0.0281 | 0.0274 | 0.0268 | 0.0262 | 0.0256 | 0.0250 | 0.0244 | 0.0239 | 0.0233 |
| -1.8 | 0.0359 | 0.0351 | 0.0344 | 0.0336 | 0.0329 | 0.0322 | 0.0314 | 0.0307 | 0.0301 | 0.0294 |
| -1.7 | 0.0446 | 0.0436 | 0.0427 | 0.0418 | 0.0409 | 0.0401 | 0.0392 | 0.0384 | 0.0375 | 0.0367 |
| -1.6 | 0.0548 | 0.0537 | 0.0526 | 0.0516 | 0.0505 | 0.0495 | 0.0485 | 0.0475 | 0.0465 | 0.0455 |
| -1.5 | 0.0668 | 0.0655 | 0.0643 | 0.0630 | 0.0618 | 0.0606 | 0.0594 | 0.0582 | 0.0571 | 0.0559 |
| -1.4 | 0.0808 | 0.0793 | 0.0778 | 0.0764 | 0.0749 | 0.0735 | 0.0721 | 0.0708 | 0.0694 | 0.0681 |
| -1.3 | 0.0968 | 0.0951 | 0.0934 | 0.0918 | 0.0901 | 0.0885 | 0.0869 | 0.0853 | 0.0838 | 0.0823 |
| -1.2 | 0.1151 | 0.1131 | 0.1112 | 0.1093 | 0.1075 | 0.1056 | 0.1038 | 0.1020 | 0.1003 | 0.0985 |
| -1.1 | 0.1357 | 0.1335 | 0.1314 | 0.1292 | 0.1271 | 0.1251 | 0.1230 | 0.1210 | 0.1190 | 0.1170 |
| -1.0 | 0.1587 | 0.1562 | 0.1539 | 0.1515 | 0.1492 | 0.1469 | 0.1446 | 0.1423 | 0.1401 | 0.1379 |
| -0.9 | 0.1841 | 0.1814 | 0.1788 | 0.1762 | 0.1736 | 0.1711 | 0.1685 | 0.1660 | 0.1635 | 0.1611 |
| -0.8 | 0.2119 | 0.2090 | 0.2061 | 0.2033 | 0.2005 | 0.1977 | 0.1949 | 0.1922 | 0.1894 | 0.1867 |
| -0.7 | 0.2420 | 0.2389 | 0.2358 | 0.2327 | 0.2296 | 0.2266 | 0.2236 | 0.2206 | 0.2177 | 0.2148 |
| -0.6 | 0.2743 | 0.2709 | 0.2676 | 0.2643 | 0.2611 | 0.2578 | 0.2546 | 0.2514 | 0.2483 | 0.2451 |
| -0.5 | 0.3085 | 0.3050 | 0.3015 | 0.2981 | 0.2946 | 0.2912 | 0.2877 | 0.2843 | 0.2810 | 0.2776 |
| -0.4 | 0.3446 | 0.3409 | 0.3372 | 0.3336 | 0.3300 | 0.3264 | 0.3228 | 0.3192 | 0.3156 | 0.3121 |
| -0.3 | 0.3821 | 0.3783 | 0.3745 | 0.3707 | 0.3669 | 0.3632 | 0.3594 | 0.3557 | 0.3520 | 0.3483 |
| -0.2 | 0.4207 | 0.4168 | 0.4129 | 0.4090 | 0.4052 | 0.4013 | 0.3974 | 0.3936 | 0.3897 | 0.3859 |
| -0.1 | 0.4602 | 0.4562 | 0.4522 | 0.4483 | 0.4443 | 0.4404 | 0.4364 | 0.4325 | 0.4286 | 0.4247 |
| -0.0 | 0.5000 | 0.4960 | 0.4920 | 0.4880 | 0.4840 | 0.4801 | 0.4761 | 0.4721 | 0.4681 | 0.4641 |

Figure: Areas under the Normal Curve $Z \sim \text{Normal}(0, 1)$

| | | | | | | | | | | |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |

Example

Let $Z \sim N(0, 1)$, then calculate:

- 1 the area under the normal curve up to $z = 1.58$.
- 2 the area under the normal curve to the right of to $z = 1.84$.
- 3 the area between the mean and 0.85 standard deviations below the mean (i.e between -0.85 and 0).
- 4 the area above 2.15 .

Solution:

- 1 $P(Z \leq 1.58) = 0.9429$
- 2 $P(Z > 1.84) = 1 - P(Z \leq 1.84) = 0.0329$
- 3 $P(-0.85 < Z < 0) = P(Z < 0) - P(Z < -0.85) = 0.5 - 0.1977 = 0.3023$.
- 4 $P(Z > 2.15) = 1 - P(Z \leq 2.15) = 1 - 0.9842 = 0.0158$.

Example

Use the standard normal table to find the z-value for the following:

- 1 $P(Z \leq z) = 0.4090$.
- 2 $P(Z \leq z) = 0.80$.
- 3 $P(Z > z) = 0.4090$.
- 4 the value of z that leaves area of 0.10 from the right under the normal curve?

Solution

- 1 $z = -0.23$.
- 2 $z = \frac{0.84+0.85}{2} = 0.845$.
- 3 $P(Z \leq z) = 1 - P(Z > z) = 1 - 0.4090 = 0.5910$. Thus $z = 0.23$ corresponds to the area of 0.5910 to the left and 0.4090 to the right.
- 4 The value of z that leaves an area of 0.10 from the right under the normal curve can be obtained from $P(Z > z) = 0.1$, hence $P(Z \leq z) = 0.9$ and $z = 1.285$.

Example

Assume that the student's scores in the General Aptitude Tests (GAT) of the National center for Assessment in Higher Education (NCAHE) of Saudi Arabia follow normal distribution with mean $= 80$ and standard deviation $= 5$.

- ① What proportion of GAT scores falls below 75?
- ② What proportion of GAT scores falls between 76 and 82?

Solution

1

$$\begin{aligned}P(X < 75) &= P\left(\frac{X - \mu}{\sigma} < \frac{75 - \mu}{\sigma}\right) \\&= P\left(Z < \frac{75 - 80}{5}\right) \\&= P(Z < -1) \\&= 0.1587\end{aligned}$$

2

$$\begin{aligned}P(76 < X < 82) &= P\left(\frac{76 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{82 - \mu}{\sigma}\right) \\&= P\left(\frac{76 - 80}{5} < Z < \frac{82 - 80}{5}\right) \\&= P(-0.8 < Z < 0.4) \\&= P(Z < 0.4) - P(Z < -0.8) \\&= 0.6554 - 0.2119 = 0.4435\end{aligned}$$

Example

The weight of Grouper fishes is normally distributed with a mean $\mu = 25$ lb and a standard deviations $\sigma = 3$ lb. Suppose that we select a fish randomly, then

- 1 Find the probability that the fish's weight is at most 23 lb.
- 2 Find the probability that the weight is between 20 lb and 27 lb.
- 3 Find the probability that the weight is more than 29 lb.
- 4 If 50 fishes are randomly selected, about how many would you expect to weigh less than 22 lb.

Solution

1.

$$\begin{aligned}P(X \leq 23) &= P\left(\frac{X - \mu}{\sigma} \leq \frac{23 - 25}{3}\right) \\&= P(Z \leq -0.67) \\&= 0.2514\end{aligned}$$

2.

$$\begin{aligned}P(20 < X < 27) &= P\left(\frac{20 - 25}{3} < \frac{X - \mu}{\sigma} < \frac{27 - 25}{3}\right) \\&= P(-1.67 < Z < 0.67) \\&= P(Z < 0.67) - P(Z < -1.67) \\&= 0.7486 - 0.0475 = 0.7011\end{aligned}$$

Solution

3.

$$\begin{aligned}P(X > 29) &= P\left(\frac{X - \mu}{\sigma} > \frac{29 - \mu}{\sigma}\right) \\&= P(Z > 1.33) \\&= 1 - 0.9082 = 0.0918.\end{aligned}$$

4.

$$\begin{aligned}P(X < 22) &= P(Z < -1) \\&= 0.1587\end{aligned}$$

then, $50(0.1587) = 7.935 \approx 8$ fishes.

Example

On an examination, the average grade was 74 and the standard deviation was 7. If 12% of the class are given A's, and the grades are curved to follow a normal distribution, what is the lowest possible A and the highest possible B?

Solution

Let X be the examination grade $X \sim N(74, 7^2)$, and let k be the lowest possible grade A, then $P(X \geq k) = 0.12$, hence $P(X < k) = 0.88 = P(Z < \frac{k-74}{7})$, and $\frac{k-74}{7} = 1.175$. Therefore, $k = 82.225$ which is 83 marks after rounding up is the lowest possible A.

Therefore, the highest possible B is 82 marks.

Definition

The continuous random variable X has an exponential distribution, with parameter λ and denoted by $\text{exp}(\lambda)$, if its density function is given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

where $\lambda > 0$.

Definition

The cumulative distribution function (CDF) of $\text{exp}(\lambda)$ is:

$$F(x) = \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Theorem

The mean and variance of the exponential distribution are $\mu = 1/\lambda$ and $\sigma^2 = 1/\lambda^2$.

Note:

- If X is the time of arrival of the first customer and if the average time is 30 minutes, then $\lambda = 1/30$.
- This distribution is commonly used to model waiting times between occurrences of rare events, lifetimes of electrical or mechanical devices.

Example

If the life length of a refrigerator follows the exponential distribution, and let X represents the life length of a refrigerator. Suppose the average life length for this type of refrigerator is 15 years. Answer the following:

- 1 What is the probability that a refrigerator can be used for less than 6 years?
- 2 What is the probability that this refrigerator can be used for more than 18 years?
- 3 What is the variance and the standard deviation of this random variable?

Solution

- ① The random variable X has an exponential distribution with mean $\mu = 1/\lambda = 15$. Thus the corresponding pdf of the life length of these refrigerators is:

$$f(x) = \begin{cases} \frac{1}{15} e^{-\frac{1}{15}x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$P(X < 6) = 1 - e^{-6/15} \approx 0.3297.$$

②

$$P(X > 18) = e^{-18/15} \approx 0.3012.$$

- ③ $\text{Var}(X) = \sigma^2 = 1/\lambda^2 = 225$ and $\sigma = 15$

Definition

If S^2 is the variance of a random sample of size n taken from a normal population having the variance σ^2 , then the statistic

$$\chi^2 = \frac{(n-1)S^2}{\sigma^2}$$

has a chi-squared distribution with $\nu = n - 1$ degrees of freedom.

Note:

$$S^2 = \frac{1}{n-1} \left[\sum_{i=1}^n X_i^2 - n\bar{X}^2 \right]$$

Theorem

- 1 If X_1, X_2, \dots, X_n an independent random sample that have the same standard normal distribution, then $X = \sum_{i=1}^n X_i^2$ is chi-squared distribution, with degrees of freedom $\nu = n$.
- 2 The mean and variance of the chi-squared distribution χ^2 with ν degrees of freedom are $\mu = \nu$ and $\sigma^2 = 2\nu$.

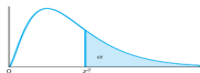


Table A.5 Critical Values of the Chi-Squared Distribution

| v | α | | | | | | | | | |
|-----|----------------------|----------------------|----------------------|----------------------|---------|--------|--------|--------|--------|--------|
| | 0.995 | 0.99 | 0.98 | 0.975 | 0.95 | 0.90 | 0.80 | 0.75 | 0.70 | 0.50 |
| 1 | 0.0 ⁴ 393 | 0.0 ³ 157 | 0.0 ³ 628 | 0.0 ³ 982 | 0.00393 | 0.0158 | 0.0642 | 0.102 | 0.148 | 0.455 |
| 2 | 0.0100 | 0.0201 | 0.0404 | 0.0506 | 0.103 | 0.211 | 0.446 | 0.575 | 0.713 | 1.386 |
| 3 | 0.0717 | 0.115 | 0.185 | 0.216 | 0.352 | 0.584 | 1.005 | 1.213 | 1.424 | 2.366 |
| 4 | 0.207 | 0.297 | 0.429 | 0.484 | 0.711 | 1.064 | 1.649 | 1.923 | 2.195 | 3.357 |
| 5 | 0.412 | 0.554 | 0.752 | 0.831 | 1.145 | 1.610 | 2.343 | 2.675 | 3.000 | 4.351 |
| 6 | 0.676 | 0.872 | 1.134 | 1.237 | 1.635 | 2.204 | 3.070 | 3.455 | 3.828 | 5.348 |
| 7 | 0.989 | 1.239 | 1.564 | 1.690 | 2.167 | 2.833 | 3.822 | 4.255 | 4.671 | 6.346 |
| 8 | 1.344 | 1.647 | 2.032 | 2.180 | 2.733 | 3.490 | 4.594 | 5.071 | 5.527 | 7.344 |
| 9 | 1.735 | 2.088 | 2.532 | 2.700 | 3.325 | 4.168 | 5.380 | 5.899 | 6.393 | 8.343 |
| 10 | 2.156 | 2.558 | 3.059 | 3.247 | 3.940 | 4.865 | 6.179 | 6.737 | 7.267 | 9.342 |
| 11 | 2.603 | 3.053 | 3.609 | 3.816 | 4.575 | 5.578 | 6.989 | 7.584 | 8.148 | 10.341 |
| 12 | 3.074 | 3.571 | 4.178 | 4.404 | 5.226 | 6.304 | 7.807 | 8.438 | 9.034 | 11.340 |
| 13 | 3.565 | 4.107 | 4.765 | 5.009 | 5.892 | 7.041 | 8.634 | 9.299 | 9.926 | 12.340 |
| 14 | 4.075 | 4.660 | 5.368 | 5.629 | 6.571 | 7.790 | 9.467 | 10.165 | 10.821 | 13.339 |
| 15 | 4.601 | 5.229 | 5.985 | 6.262 | 7.261 | 8.547 | 10.307 | 11.037 | 11.721 | 14.339 |
| 16 | 5.142 | 5.812 | 6.614 | 6.908 | 7.962 | 9.312 | 11.152 | 11.912 | 12.624 | 15.338 |
| 17 | 5.697 | 6.408 | 7.255 | 7.564 | 8.672 | 10.085 | 12.002 | 12.792 | 13.531 | 16.338 |
| 18 | 6.265 | 7.015 | 7.906 | 8.231 | 9.390 | 10.865 | 12.857 | 13.675 | 14.440 | 17.338 |
| 19 | 6.844 | 7.633 | 8.567 | 8.907 | 10.117 | 11.651 | 13.716 | 14.562 | 15.352 | 18.338 |
| 20 | 7.434 | 8.260 | 9.237 | 9.591 | 10.851 | 12.443 | 14.578 | 15.452 | 16.266 | 19.337 |
| 21 | 8.034 | 8.897 | 9.915 | 10.283 | 11.591 | 13.240 | 15.445 | 16.344 | 17.182 | 20.337 |
| 22 | 8.643 | 9.542 | 10.600 | 10.982 | 12.338 | 14.041 | 16.314 | 17.240 | 18.101 | 21.337 |
| 23 | 9.260 | 10.196 | 11.293 | 11.689 | 13.091 | 14.848 | 17.187 | 18.137 | 19.021 | 22.337 |
| 24 | 9.886 | 10.856 | 11.992 | 12.401 | 13.848 | 15.659 | 18.062 | 19.037 | 19.943 | 23.337 |
| 25 | 10.520 | 11.524 | 12.697 | 13.120 | 14.611 | 16.473 | 18.940 | 19.939 | 20.867 | 24.337 |
| 26 | 11.160 | 12.198 | 13.409 | 13.844 | 15.379 | 17.292 | 19.820 | 20.843 | 21.792 | 25.336 |
| 27 | 11.808 | 12.878 | 14.125 | 14.573 | 16.151 | 18.114 | 20.703 | 21.749 | 22.719 | 26.336 |
| 28 | 12.461 | 13.565 | 14.847 | 15.308 | 16.928 | 18.939 | 21.588 | 22.657 | 23.647 | 27.336 |
| 29 | 13.121 | 14.256 | 15.574 | 16.047 | 17.708 | 19.768 | 22.475 | 23.567 | 24.577 | 28.336 |
| 30 | 13.787 | 14.953 | 16.306 | 16.791 | 18.493 | 20.599 | 23.364 | 24.478 | 25.508 | 29.336 |
| 40 | 20.707 | 22.164 | 23.838 | 24.433 | 26.509 | 29.051 | 32.345 | 33.66 | 34.872 | 39.335 |
| 50 | 27.991 | 29.707 | 31.664 | 32.357 | 34.764 | 37.689 | 41.449 | 42.942 | 44.313 | 49.335 |
| 60 | 35.534 | 37.485 | 39.699 | 40.482 | 43.188 | 46.459 | 50.641 | 52.294 | 53.809 | 59.335 |

Figure: Table A.5 Critical Values of the Chi-Squared Distribution

Table A.5 (continued) Critical Values of the Chi-Squared Distribution

| <i>v</i> | α | | | | | | | | | |
|-----------|-------------|-------------|-------------|-------------|-------------|--------------|-------------|-------------|--------------|--------------|
| | 0.30 | 0.25 | 0.20 | 0.10 | 0.05 | 0.025 | 0.02 | 0.01 | 0.005 | 0.001 |
| 1 | 1.074 | 1.323 | 1.642 | 2.706 | 3.841 | 5.024 | 5.412 | 6.635 | 7.879 | 10.827 |
| 2 | 2.408 | 2.773 | 3.219 | 4.605 | 5.991 | 7.378 | 7.824 | 9.210 | 10.597 | 13.815 |
| 3 | 3.665 | 4.108 | 4.642 | 6.251 | 7.815 | 9.348 | 9.837 | 11.345 | 12.838 | 16.266 |
| 4 | 4.878 | 5.385 | 5.989 | 7.779 | 9.488 | 11.143 | 11.668 | 13.277 | 14.860 | 18.466 |
| 5 | 6.064 | 6.626 | 7.289 | 9.236 | 11.070 | 12.832 | 13.388 | 15.086 | 16.750 | 20.515 |
| 6 | 7.231 | 7.841 | 8.558 | 10.645 | 12.592 | 14.449 | 15.033 | 16.812 | 18.548 | 22.457 |
| 7 | 8.383 | 9.037 | 9.803 | 12.017 | 14.067 | 16.013 | 16.622 | 18.475 | 20.278 | 24.321 |
| 8 | 9.524 | 10.219 | 11.030 | 13.362 | 15.507 | 17.535 | 18.168 | 20.090 | 21.955 | 26.124 |
| 9 | 10.656 | 11.389 | 12.242 | 14.684 | 16.919 | 19.023 | 19.679 | 21.666 | 23.589 | 27.877 |
| 10 | 11.781 | 12.549 | 13.442 | 15.987 | 18.307 | 20.483 | 21.161 | 23.209 | 25.188 | 29.588 |
| 11 | 12.899 | 13.701 | 14.631 | 17.275 | 19.675 | 21.920 | 22.618 | 24.725 | 26.757 | 31.264 |
| 12 | 14.011 | 14.845 | 15.812 | 18.549 | 21.026 | 23.337 | 24.054 | 26.217 | 28.300 | 32.909 |
| 13 | 15.119 | 15.984 | 16.985 | 19.812 | 22.362 | 24.736 | 25.471 | 27.688 | 29.819 | 34.527 |
| 14 | 16.222 | 17.117 | 18.151 | 21.064 | 23.685 | 26.119 | 26.873 | 29.141 | 31.319 | 36.124 |
| 15 | 17.322 | 18.245 | 19.311 | 22.307 | 24.996 | 27.488 | 28.259 | 30.578 | 32.801 | 37.698 |
| 16 | 18.418 | 19.369 | 20.465 | 23.542 | 26.296 | 28.845 | 29.633 | 32.000 | 34.267 | 39.252 |
| 17 | 19.511 | 20.489 | 21.615 | 24.769 | 27.587 | 30.191 | 30.995 | 33.409 | 35.718 | 40.791 |
| 18 | 20.601 | 21.605 | 22.760 | 25.989 | 28.869 | 31.526 | 32.346 | 34.805 | 37.156 | 42.312 |
| 19 | 21.689 | 22.718 | 23.900 | 27.204 | 30.144 | 32.852 | 33.687 | 36.191 | 38.582 | 43.819 |
| 20 | 22.775 | 23.828 | 25.038 | 28.412 | 31.410 | 34.170 | 35.020 | 37.566 | 39.997 | 45.314 |
| 21 | 23.858 | 24.935 | 26.171 | 29.615 | 32.671 | 35.479 | 36.343 | 38.932 | 41.401 | 46.796 |
| 22 | 24.939 | 26.039 | 27.301 | 30.813 | 33.924 | 36.781 | 37.659 | 40.289 | 42.796 | 48.268 |
| 23 | 26.018 | 27.141 | 28.429 | 32.007 | 35.172 | 38.076 | 38.968 | 41.638 | 44.181 | 49.728 |
| 24 | 27.096 | 28.241 | 29.553 | 33.196 | 36.415 | 39.364 | 40.270 | 42.980 | 45.558 | 51.179 |
| 25 | 28.172 | 29.339 | 30.675 | 34.382 | 37.652 | 40.646 | 41.566 | 44.314 | 46.928 | 52.619 |
| 26 | 29.246 | 30.435 | 31.795 | 35.563 | 38.885 | 41.923 | 42.856 | 45.642 | 48.290 | 54.051 |
| 27 | 30.319 | 31.528 | 32.912 | 36.741 | 40.113 | 43.195 | 44.140 | 46.963 | 49.645 | 55.475 |
| 28 | 31.391 | 32.620 | 34.027 | 37.916 | 41.337 | 44.461 | 45.419 | 48.278 | 50.994 | 56.892 |
| 29 | 32.461 | 33.711 | 35.139 | 39.087 | 42.557 | 45.722 | 46.693 | 49.588 | 52.335 | 58.301 |
| 30 | 33.530 | 34.800 | 36.250 | 40.256 | 43.773 | 46.979 | 47.962 | 50.892 | 53.672 | 59.702 |
| 40 | 44.165 | 45.616 | 47.269 | 51.805 | 55.758 | 59.342 | 60.436 | 63.691 | 66.766 | 73.403 |
| 50 | 54.723 | 56.334 | 58.164 | 63.167 | 67.505 | 71.420 | 72.613 | 76.154 | 79.490 | 86.660 |
| 60 | 65.226 | 66.981 | 68.972 | 74.397 | 79.082 | 83.298 | 84.58 | 88.379 | 91.952 | 99.608 |

Example

For a chi-squared distribution, find

- (a) χ_{15}^2 when $\alpha = 0.025$;
- (b) χ_7^2 when $\alpha = 0.01$;
- (c) χ_{24}^2 when $\alpha = 0.05$.

Solution

- (a) 27.488.
- (b) 18.475.
- (c) 36.415.

Example

For a chi-squared distribution X , find χ_{α}^2 such that

- (a) $P(X > k) = 0.99$ when $\nu = 4$;
- (b) $P(X > k) = 0.025$ when $\nu = 19$;
- (c) $P(37.652 < X < k) = 0.045$ when $\nu = 25$.

Solution

- (a) $\chi_{\nu;\alpha}^2 = \chi_{4;0.99}^2 = 0.297$.
- (b) $\chi_{\nu;\alpha}^2 = \chi_{19;0.025}^2 = 32.852$.
- (c) $\chi_{25;0.05}^2 = 37.652$. Therefore, $\alpha = 0.05 - 0.045 = 0.005$.
Hence, $\chi_{\nu;\alpha}^2 = \chi_{25;0.005}^2 = 46.928$.

Theorem

Let Z be a standard normal random variable and V a chi-squared random variable with ν degrees of freedom. If Z and V are independent, then the distribution of the random variable is:

$$T = \frac{Z}{\sqrt{V/\nu}}$$

This is known as the t -distribution with ν degrees of freedom.

Corollary

Let X_1, X_2, \dots, X_n be independent random variables that are all normal with mean μ and standard deviation σ . Let

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad \text{and} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

Then the random variable $T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$ has a t -distribution with $\nu = n - 1$ degrees of freedom.

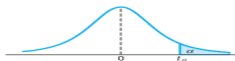


Table A.4 Critical Values of the t -Distribution

| v | α | | | | | | |
|----------|----------|-------|-------|-------|-------|-------|--------|
| | 0.40 | 0.30 | 0.20 | 0.15 | 0.10 | 0.05 | 0.025 |
| 1 | 0.325 | 0.727 | 1.376 | 1.963 | 3.078 | 6.314 | 12.706 |
| 2 | 0.289 | 0.617 | 1.061 | 1.386 | 1.886 | 2.920 | 4.303 |
| 3 | 0.277 | 0.584 | 0.978 | 1.250 | 1.638 | 2.353 | 3.182 |
| 4 | 0.271 | 0.569 | 0.941 | 1.190 | 1.533 | 2.132 | 2.776 |
| 5 | 0.267 | 0.559 | 0.920 | 1.156 | 1.476 | 2.015 | 2.571 |
| 6 | 0.265 | 0.553 | 0.906 | 1.134 | 1.440 | 1.943 | 2.447 |
| 7 | 0.263 | 0.549 | 0.896 | 1.119 | 1.415 | 1.895 | 2.365 |
| 8 | 0.262 | 0.546 | 0.889 | 1.108 | 1.397 | 1.860 | 2.306 |
| 9 | 0.261 | 0.543 | 0.883 | 1.100 | 1.383 | 1.833 | 2.262 |
| 10 | 0.260 | 0.542 | 0.879 | 1.093 | 1.372 | 1.812 | 2.228 |
| 11 | 0.260 | 0.540 | 0.876 | 1.088 | 1.363 | 1.796 | 2.201 |
| 12 | 0.259 | 0.539 | 0.873 | 1.083 | 1.356 | 1.782 | 2.179 |
| 13 | 0.259 | 0.538 | 0.870 | 1.079 | 1.350 | 1.771 | 2.160 |
| 14 | 0.258 | 0.537 | 0.868 | 1.076 | 1.345 | 1.761 | 2.145 |
| 15 | 0.258 | 0.536 | 0.866 | 1.074 | 1.341 | 1.753 | 2.131 |
| 16 | 0.258 | 0.535 | 0.865 | 1.071 | 1.337 | 1.746 | 2.120 |
| 17 | 0.257 | 0.534 | 0.863 | 1.069 | 1.333 | 1.740 | 2.110 |
| 18 | 0.257 | 0.534 | 0.862 | 1.067 | 1.330 | 1.734 | 2.101 |
| 19 | 0.257 | 0.533 | 0.861 | 1.066 | 1.328 | 1.729 | 2.093 |
| 20 | 0.257 | 0.533 | 0.860 | 1.064 | 1.325 | 1.725 | 2.086 |
| 21 | 0.257 | 0.532 | 0.859 | 1.063 | 1.323 | 1.721 | 2.080 |
| 22 | 0.256 | 0.532 | 0.858 | 1.061 | 1.321 | 1.717 | 2.074 |
| 23 | 0.256 | 0.532 | 0.858 | 1.060 | 1.319 | 1.714 | 2.069 |
| 24 | 0.256 | 0.531 | 0.857 | 1.059 | 1.318 | 1.711 | 2.064 |
| 25 | 0.256 | 0.531 | 0.856 | 1.058 | 1.316 | 1.708 | 2.060 |
| 26 | 0.256 | 0.531 | 0.856 | 1.058 | 1.315 | 1.706 | 2.056 |
| 27 | 0.256 | 0.531 | 0.855 | 1.057 | 1.314 | 1.703 | 2.052 |
| 28 | 0.256 | 0.530 | 0.855 | 1.056 | 1.313 | 1.701 | 2.048 |
| 29 | 0.256 | 0.530 | 0.854 | 1.055 | 1.311 | 1.699 | 2.045 |
| 30 | 0.256 | 0.530 | 0.854 | 1.055 | 1.310 | 1.697 | 2.042 |
| 40 | 0.255 | 0.529 | 0.851 | 1.050 | 1.303 | 1.684 | 2.021 |
| 60 | 0.254 | 0.527 | 0.848 | 1.045 | 1.296 | 1.671 | 2.000 |
| 120 | 0.254 | 0.526 | 0.845 | 1.041 | 1.289 | 1.658 | 1.980 |
| ∞ | 0.253 | 0.524 | 0.842 | 1.036 | 1.282 | 1.645 | 1.960 |

Figure: Table A.4 Critical Values of the t -Distribution

Table A.4 (continued) Critical Values of the t -Distribution

| ν | α | | | | | | |
|------------|-------------|--------------|-------------|---------------|--------------|---------------|---------------|
| | 0.02 | 0.015 | 0.01 | 0.0075 | 0.005 | 0.0025 | 0.0005 |
| 1 | 15.894 | 21.205 | 31.821 | 42.433 | 63.656 | 127.321 | 636.578 |
| 2 | 4.849 | 5.643 | 6.965 | 8.073 | 9.925 | 14.089 | 31.600 |
| 3 | 3.482 | 3.896 | 4.541 | 5.047 | 5.841 | 7.453 | 12.924 |
| 4 | 2.999 | 3.298 | 3.747 | 4.088 | 4.604 | 5.598 | 8.610 |
| 5 | 2.757 | 3.003 | 3.365 | 3.634 | 4.032 | 4.773 | 6.869 |
| 6 | 2.612 | 2.829 | 3.143 | 3.372 | 3.707 | 4.317 | 5.959 |
| 7 | 2.517 | 2.715 | 2.998 | 3.203 | 3.499 | 4.029 | 5.408 |
| 8 | 2.449 | 2.634 | 2.896 | 3.085 | 3.355 | 3.833 | 5.041 |
| 9 | 2.398 | 2.574 | 2.821 | 2.998 | 3.250 | 3.690 | 4.781 |
| 10 | 2.359 | 2.527 | 2.764 | 2.932 | 3.169 | 3.581 | 4.587 |
| 11 | 2.328 | 2.491 | 2.718 | 2.879 | 3.106 | 3.497 | 4.437 |
| 12 | 2.303 | 2.461 | 2.681 | 2.836 | 3.055 | 3.428 | 4.318 |
| 13 | 2.282 | 2.436 | 2.650 | 2.801 | 3.012 | 3.372 | 4.221 |
| 14 | 2.264 | 2.415 | 2.624 | 2.771 | 2.977 | 3.326 | 4.140 |
| 15 | 2.249 | 2.397 | 2.602 | 2.746 | 2.947 | 3.286 | 4.073 |
| 16 | 2.235 | 2.382 | 2.583 | 2.724 | 2.921 | 3.252 | 4.015 |
| 17 | 2.224 | 2.368 | 2.567 | 2.706 | 2.898 | 3.222 | 3.965 |
| 18 | 2.214 | 2.356 | 2.552 | 2.689 | 2.878 | 3.197 | 3.922 |
| 19 | 2.205 | 2.346 | 2.539 | 2.674 | 2.861 | 3.174 | 3.883 |
| 20 | 2.197 | 2.336 | 2.528 | 2.661 | 2.845 | 3.153 | 3.850 |
| 21 | 2.189 | 2.328 | 2.518 | 2.649 | 2.831 | 3.135 | 3.819 |
| 22 | 2.183 | 2.320 | 2.508 | 2.639 | 2.819 | 3.119 | 3.792 |
| 23 | 2.177 | 2.313 | 2.500 | 2.629 | 2.807 | 3.104 | 3.768 |
| 24 | 2.172 | 2.307 | 2.492 | 2.620 | 2.797 | 3.091 | 3.745 |
| 25 | 2.167 | 2.301 | 2.485 | 2.612 | 2.787 | 3.078 | 3.725 |
| 26 | 2.162 | 2.296 | 2.479 | 2.605 | 2.779 | 3.067 | 3.707 |
| 27 | 2.158 | 2.291 | 2.473 | 2.598 | 2.771 | 3.057 | 3.689 |
| 28 | 2.154 | 2.286 | 2.467 | 2.592 | 2.763 | 3.047 | 3.674 |
| 29 | 2.150 | 2.282 | 2.462 | 2.586 | 2.756 | 3.038 | 3.660 |
| 30 | 2.147 | 2.278 | 2.457 | 2.581 | 2.750 | 3.030 | 3.646 |
| 40 | 2.123 | 2.250 | 2.423 | 2.542 | 2.704 | 2.971 | 3.551 |
| 60 | 2.099 | 2.223 | 2.390 | 2.504 | 2.660 | 2.915 | 3.460 |
| 120 | 2.076 | 2.196 | 2.358 | 2.468 | 2.617 | 2.860 | 3.373 |
| ∞ | 2.054 | 2.170 | 2.326 | 2.432 | 2.576 | 2.807 | 3.290 |

The t -value with $\nu = 14$ degrees of freedom that leaves an area of 0.025 to the left, and therefore an area of 0.975 to the right, is

$$t_{0.975} = -t_{0.025} = -2.145$$

Example

Find $P(-t_{0.025} < T < t_{0.05})$.

Solution

Since $t_{0.05}$ leaves an area of 0.05 to the right, and $-t_{0.025}$ leaves an area of 0.025 to the left, we find a total area of $1 - 0.05 - 0.025 = 0.925$ between $-t_{0.025}$ and $t_{0.05}$.

Hence

$$P(-t_{0.025} < T < t_{0.05}) = 0.925.$$

Example

Find k such that $P(k < T < -1.761) = 0.045$ for a random sample of size 15 selected from a normal distribution with

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}.$$

Solution

From Table A.4 we note that 1.761 corresponds to $t_{0.05}$ when $\nu = 14$. Therefore, $-t_{0.05} = -1.761$. Since k in the original probability statement is to the left of $-t_{0.05} = -1.761$, let $k = -t_{\alpha}$. Then, by using figure, we have

$$0.045 = 0.05 - \alpha, \text{ or } \alpha = 0.005.$$

Hence, from Table A.4 with $\nu = 14$,
 $k = -t_{0.005} = -2.977$ and $P(-2.977 < T < -1.761) = 0.045$.

Definition

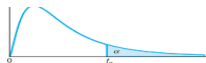
The statistic F is defined to be the ratio of two independent chi-squared random variables, each divided by its number of degrees of freedom.

Theorem 31

The random variable

$$F = \frac{U/\nu_1}{V/\nu_2}$$

where U and V are independent random variables having chi-squared distributions with ν_1 and ν_2 degrees of freedom, respectively, is the F -distribution with ν_1 and ν_2 degrees of freedom (d.f.).

Table A.6 Critical Values of the F -Distribution

| v_2 | $f_{0.05}(v_1, v_2)$ | | | | | | | | |
|----------|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|
| | v_1 | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 161.45 | 199.50 | 215.71 | 224.58 | 230.16 | 233.99 | 236.77 | 238.88 | 240.54 |
| 2 | 18.51 | 19.00 | 19.16 | 19.25 | 19.30 | 19.33 | 19.35 | 19.37 | 19.38 |
| 3 | 10.13 | 9.55 | 9.28 | 9.12 | 9.01 | 8.94 | 8.89 | 8.85 | 8.81 |
| 4 | 7.71 | 6.94 | 6.59 | 6.39 | 6.26 | 6.16 | 6.09 | 6.04 | 6.00 |
| 5 | 6.61 | 5.79 | 5.41 | 5.19 | 5.05 | 4.95 | 4.88 | 4.82 | 4.77 |
| 6 | 5.99 | 5.14 | 4.76 | 4.53 | 4.39 | 4.28 | 4.21 | 4.15 | 4.10 |
| 7 | 5.59 | 4.74 | 4.35 | 4.12 | 3.97 | 3.87 | 3.79 | 3.73 | 3.68 |
| 8 | 5.32 | 4.46 | 4.07 | 3.84 | 3.69 | 3.58 | 3.50 | 3.44 | 3.39 |
| 9 | 5.12 | 4.26 | 3.86 | 3.63 | 3.48 | 3.37 | 3.29 | 3.23 | 3.18 |
| 10 | 4.96 | 4.10 | 3.71 | 3.48 | 3.33 | 3.22 | 3.14 | 3.07 | 3.02 |
| 11 | 4.84 | 3.98 | 3.59 | 3.36 | 3.20 | 3.09 | 3.01 | 2.95 | 2.90 |
| 12 | 4.75 | 3.89 | 3.49 | 3.26 | 3.11 | 3.00 | 2.91 | 2.85 | 2.80 |
| 13 | 4.67 | 3.81 | 3.41 | 3.18 | 3.03 | 2.92 | 2.83 | 2.77 | 2.71 |
| 14 | 4.60 | 3.74 | 3.34 | 3.11 | 2.96 | 2.85 | 2.76 | 2.70 | 2.65 |
| 15 | 4.54 | 3.68 | 3.29 | 3.06 | 2.90 | 2.79 | 2.71 | 2.64 | 2.59 |
| 16 | 4.49 | 3.63 | 3.24 | 3.01 | 2.85 | 2.74 | 2.66 | 2.59 | 2.54 |
| 17 | 4.45 | 3.59 | 3.20 | 2.96 | 2.81 | 2.70 | 2.61 | 2.55 | 2.49 |
| 18 | 4.41 | 3.55 | 3.16 | 2.93 | 2.77 | 2.66 | 2.58 | 2.51 | 2.46 |
| 19 | 4.38 | 3.52 | 3.13 | 2.90 | 2.74 | 2.63 | 2.54 | 2.48 | 2.42 |
| 20 | 4.35 | 3.49 | 3.10 | 2.87 | 2.71 | 2.60 | 2.51 | 2.45 | 2.39 |
| 21 | 4.32 | 3.47 | 3.07 | 2.84 | 2.68 | 2.57 | 2.49 | 2.42 | 2.37 |
| 22 | 4.30 | 3.44 | 3.05 | 2.82 | 2.66 | 2.55 | 2.46 | 2.40 | 2.34 |
| 23 | 4.28 | 3.42 | 3.03 | 2.80 | 2.64 | 2.53 | 2.44 | 2.37 | 2.32 |
| 24 | 4.26 | 3.40 | 3.01 | 2.78 | 2.62 | 2.51 | 2.42 | 2.36 | 2.30 |
| 25 | 4.24 | 3.39 | 2.99 | 2.76 | 2.60 | 2.49 | 2.40 | 2.34 | 2.28 |
| 26 | 4.23 | 3.37 | 2.98 | 2.74 | 2.59 | 2.47 | 2.39 | 2.32 | 2.27 |
| 27 | 4.21 | 3.35 | 2.96 | 2.73 | 2.57 | 2.46 | 2.37 | 2.31 | 2.25 |
| 28 | 4.20 | 3.34 | 2.95 | 2.71 | 2.56 | 2.45 | 2.36 | 2.29 | 2.24 |
| 29 | 4.18 | 3.33 | 2.93 | 2.70 | 2.55 | 2.43 | 2.35 | 2.28 | 2.22 |
| 30 | 4.17 | 3.32 | 2.92 | 2.69 | 2.53 | 2.42 | 2.33 | 2.27 | 2.21 |
| 40 | 4.08 | 3.23 | 2.84 | 2.61 | 2.45 | 2.34 | 2.25 | 2.18 | 2.12 |
| 60 | 4.00 | 3.15 | 2.76 | 2.53 | 2.37 | 2.25 | 2.17 | 2.10 | 2.04 |
| 120 | 3.92 | 3.07 | 2.68 | 2.45 | 2.29 | 2.18 | 2.09 | 2.02 | 1.96 |
| ∞ | 3.84 | 3.00 | 2.60 | 2.37 | 2.21 | 2.10 | 2.01 | 1.94 | 1.88 |

Figure: Table A.6 Critical Values of the F -Distribution

Table A.6 (continued) Critical Values of the F -Distribution

| v_2 | $f_{0.05}(v_1, v_2)$ | | | | | | | | | |
|----------|----------------------|--------|--------|--------|--------|--------|--------|--------|--------|----------|
| | v_1 | | | | | | | | | |
| | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
| 1 | 241.88 | 243.91 | 245.95 | 248.01 | 249.05 | 250.10 | 251.14 | 252.20 | 253.25 | 254.31 |
| 2 | 19.40 | 19.41 | 19.43 | 19.45 | 19.45 | 19.46 | 19.47 | 19.48 | 19.49 | 19.50 |
| 3 | 8.79 | 8.74 | 8.70 | 8.66 | 8.64 | 8.62 | 8.59 | 8.57 | 8.55 | 8.53 |
| 4 | 5.96 | 5.91 | 5.86 | 5.80 | 5.77 | 5.75 | 5.72 | 5.69 | 5.66 | 5.63 |
| 5 | 4.74 | 4.68 | 4.62 | 4.56 | 4.53 | 4.50 | 4.46 | 4.43 | 4.40 | 4.36 |
| 6 | 4.06 | 4.00 | 3.94 | 3.87 | 3.84 | 3.81 | 3.77 | 3.74 | 3.70 | 3.67 |
| 7 | 3.64 | 3.57 | 3.51 | 3.44 | 3.41 | 3.38 | 3.34 | 3.30 | 3.27 | 3.23 |
| 8 | 3.35 | 3.28 | 3.22 | 3.15 | 3.12 | 3.08 | 3.04 | 3.01 | 2.97 | 2.93 |
| 9 | 3.14 | 3.07 | 3.01 | 2.94 | 2.90 | 2.86 | 2.83 | 2.79 | 2.75 | 2.71 |
| 10 | 2.98 | 2.91 | 2.85 | 2.77 | 2.74 | 2.70 | 2.66 | 2.62 | 2.58 | 2.54 |
| 11 | 2.85 | 2.79 | 2.72 | 2.65 | 2.61 | 2.57 | 2.53 | 2.49 | 2.45 | 2.40 |
| 12 | 2.75 | 2.69 | 2.62 | 2.54 | 2.51 | 2.47 | 2.43 | 2.38 | 2.34 | 2.30 |
| 13 | 2.67 | 2.60 | 2.53 | 2.46 | 2.42 | 2.38 | 2.34 | 2.30 | 2.25 | 2.21 |
| 14 | 2.60 | 2.53 | 2.46 | 2.39 | 2.35 | 2.31 | 2.27 | 2.22 | 2.18 | 2.13 |
| 15 | 2.54 | 2.48 | 2.40 | 2.33 | 2.29 | 2.25 | 2.20 | 2.16 | 2.11 | 2.07 |
| 16 | 2.49 | 2.42 | 2.35 | 2.28 | 2.24 | 2.19 | 2.15 | 2.11 | 2.06 | 2.01 |
| 17 | 2.45 | 2.38 | 2.31 | 2.23 | 2.19 | 2.15 | 2.10 | 2.06 | 2.01 | 1.96 |
| 18 | 2.41 | 2.34 | 2.27 | 2.19 | 2.15 | 2.11 | 2.06 | 2.02 | 1.97 | 1.92 |
| 19 | 2.38 | 2.31 | 2.23 | 2.16 | 2.11 | 2.07 | 2.03 | 1.98 | 1.93 | 1.88 |
| 20 | 2.35 | 2.28 | 2.20 | 2.12 | 2.08 | 2.04 | 1.99 | 1.95 | 1.90 | 1.84 |
| 21 | 2.32 | 2.25 | 2.18 | 2.10 | 2.05 | 2.01 | 1.96 | 1.92 | 1.87 | 1.81 |
| 22 | 2.30 | 2.23 | 2.15 | 2.07 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.78 |
| 23 | 2.27 | 2.20 | 2.13 | 2.05 | 2.01 | 1.96 | 1.91 | 1.86 | 1.81 | 1.76 |
| 24 | 2.25 | 2.18 | 2.11 | 2.03 | 1.98 | 1.94 | 1.89 | 1.84 | 1.79 | 1.73 |
| 25 | 2.24 | 2.16 | 2.09 | 2.01 | 1.96 | 1.92 | 1.87 | 1.82 | 1.77 | 1.71 |
| 26 | 2.22 | 2.15 | 2.07 | 1.99 | 1.95 | 1.90 | 1.85 | 1.80 | 1.75 | 1.69 |
| 27 | 2.20 | 2.13 | 2.06 | 1.97 | 1.93 | 1.88 | 1.84 | 1.79 | 1.73 | 1.67 |
| 28 | 2.19 | 2.12 | 2.04 | 1.96 | 1.91 | 1.87 | 1.82 | 1.77 | 1.71 | 1.65 |
| 29 | 2.18 | 2.10 | 2.03 | 1.94 | 1.90 | 1.85 | 1.81 | 1.75 | 1.70 | 1.64 |
| 30 | 2.16 | 2.09 | 2.01 | 1.93 | 1.89 | 1.84 | 1.79 | 1.74 | 1.68 | 1.62 |
| 40 | 2.08 | 2.00 | 1.92 | 1.84 | 1.79 | 1.74 | 1.69 | 1.64 | 1.58 | 1.51 |
| 60 | 1.99 | 1.92 | 1.84 | 1.75 | 1.70 | 1.65 | 1.59 | 1.53 | 1.47 | 1.39 |
| 120 | 1.91 | 1.83 | 1.75 | 1.66 | 1.61 | 1.55 | 1.50 | 1.43 | 1.35 | 1.25 |
| ∞ | 1.83 | 1.75 | 1.67 | 1.57 | 1.52 | 1.46 | 1.39 | 1.32 | 1.22 | 1.00 |

Table A.6 (continued) Critical Values of the F -Distribution

| v_2 | $f_{0.01}(v_1, v_2)$ | | | | | | | | |
|----------|----------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| | v_1 | | | | | | | | |
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 4052.18 | 4999.50 | 5403.35 | 5624.58 | 5763.65 | 5858.99 | 5928.36 | 5981.07 | 6022.47 |
| 2 | 98.50 | 99.00 | 99.17 | 99.25 | 99.30 | 99.33 | 99.36 | 99.37 | 99.39 |
| 3 | 34.12 | 30.82 | 29.46 | 28.71 | 28.24 | 27.91 | 27.67 | 27.49 | 27.35 |
| 4 | 21.20 | 18.00 | 16.69 | 15.98 | 15.52 | 15.21 | 14.98 | 14.80 | 14.66 |
| 5 | 16.26 | 13.27 | 12.06 | 11.39 | 10.97 | 10.67 | 10.46 | 10.29 | 10.16 |
| 6 | 13.75 | 10.92 | 9.78 | 9.15 | 8.75 | 8.47 | 8.26 | 8.10 | 7.98 |
| 7 | 12.25 | 9.55 | 8.45 | 7.85 | 7.46 | 7.19 | 6.99 | 6.84 | 6.72 |
| 8 | 11.26 | 8.65 | 7.59 | 7.01 | 6.63 | 6.37 | 6.18 | 6.03 | 5.91 |
| 9 | 10.56 | 8.02 | 6.99 | 6.42 | 6.06 | 5.80 | 5.61 | 5.47 | 5.35 |
| 10 | 10.04 | 7.56 | 6.55 | 5.99 | 5.64 | 5.39 | 5.20 | 5.06 | 4.94 |
| 11 | 9.65 | 7.21 | 6.22 | 5.67 | 5.32 | 5.07 | 4.89 | 4.74 | 4.63 |
| 12 | 9.33 | 6.93 | 5.95 | 5.41 | 5.06 | 4.82 | 4.64 | 4.50 | 4.39 |
| 13 | 9.07 | 6.70 | 5.74 | 5.21 | 4.86 | 4.62 | 4.44 | 4.30 | 4.19 |
| 14 | 8.86 | 6.51 | 5.56 | 5.04 | 4.69 | 4.46 | 4.28 | 4.14 | 4.03 |
| 15 | 8.68 | 6.36 | 5.42 | 4.89 | 4.56 | 4.32 | 4.14 | 4.00 | 3.89 |
| 16 | 8.53 | 6.23 | 5.29 | 4.77 | 4.44 | 4.20 | 4.03 | 3.89 | 3.78 |
| 17 | 8.40 | 6.11 | 5.18 | 4.67 | 4.34 | 4.10 | 3.93 | 3.79 | 3.68 |
| 18 | 8.29 | 6.01 | 5.09 | 4.58 | 4.25 | 4.01 | 3.84 | 3.71 | 3.60 |
| 19 | 8.18 | 5.93 | 5.01 | 4.50 | 4.17 | 3.94 | 3.77 | 3.63 | 3.52 |
| 20 | 8.10 | 5.85 | 4.94 | 4.43 | 4.10 | 3.87 | 3.70 | 3.56 | 3.46 |
| 21 | 8.02 | 5.78 | 4.87 | 4.37 | 4.04 | 3.81 | 3.64 | 3.51 | 3.40 |
| 22 | 7.95 | 5.72 | 4.82 | 4.31 | 3.99 | 3.76 | 3.59 | 3.45 | 3.35 |
| 23 | 7.88 | 5.66 | 4.76 | 4.26 | 3.94 | 3.71 | 3.54 | 3.41 | 3.30 |
| 24 | 7.82 | 5.61 | 4.72 | 4.22 | 3.90 | 3.67 | 3.50 | 3.36 | 3.26 |
| 25 | 7.77 | 5.57 | 4.68 | 4.18 | 3.85 | 3.63 | 3.46 | 3.32 | 3.22 |
| 26 | 7.72 | 5.53 | 4.64 | 4.14 | 3.82 | 3.59 | 3.42 | 3.29 | 3.18 |
| 27 | 7.68 | 5.49 | 4.60 | 4.11 | 3.78 | 3.56 | 3.39 | 3.26 | 3.15 |
| 28 | 7.64 | 5.45 | 4.57 | 4.07 | 3.75 | 3.53 | 3.36 | 3.23 | 3.12 |
| 29 | 7.60 | 5.42 | 4.54 | 4.04 | 3.73 | 3.50 | 3.33 | 3.20 | 3.09 |
| 30 | 7.56 | 5.39 | 4.51 | 4.02 | 3.70 | 3.47 | 3.30 | 3.17 | 3.07 |
| 40 | 7.31 | 5.18 | 4.31 | 3.83 | 3.51 | 3.29 | 3.12 | 2.99 | 2.89 |
| 60 | 7.08 | 4.98 | 4.13 | 3.65 | 3.34 | 3.12 | 2.95 | 2.82 | 2.72 |
| 120 | 6.85 | 4.79 | 3.95 | 3.48 | 3.17 | 2.96 | 2.79 | 2.66 | 2.56 |
| ∞ | 6.63 | 4.61 | 3.78 | 3.32 | 3.02 | 2.80 | 2.64 | 2.51 | 2.41 |

Table A.6 (continued) Critical Values of the F -Distribution

| v_2 | $f_{0.01}(v_1, v_2)$ | | | | | | | | | |
|----------|----------------------|---------|---------|---------|---------|---------|---------|---------|---------|----------|
| | v_1 | | | | | | | | | |
| | 10 | 12 | 15 | 20 | 24 | 30 | 40 | 60 | 120 | ∞ |
| 1 | 6055.85 | 6106.32 | 6157.28 | 6208.73 | 6234.63 | 6260.65 | 6286.78 | 6313.03 | 6339.39 | 6365.86 |
| 2 | 99.40 | 99.42 | 99.43 | 99.45 | 99.46 | 99.47 | 99.47 | 99.48 | 99.49 | 99.50 |
| 3 | 27.23 | 27.05 | 26.87 | 26.69 | 26.60 | 26.50 | 26.41 | 26.32 | 26.22 | 26.13 |
| 4 | 14.55 | 14.37 | 14.20 | 14.02 | 13.93 | 13.84 | 13.75 | 13.65 | 13.56 | 13.46 |
| 5 | 10.05 | 9.89 | 9.72 | 9.55 | 9.47 | 9.38 | 9.29 | 9.20 | 9.11 | 9.02 |
| 6 | 7.87 | 7.72 | 7.56 | 7.40 | 7.31 | 7.23 | 7.14 | 7.06 | 6.97 | 6.88 |
| 7 | 6.62 | 6.47 | 6.31 | 6.16 | 6.07 | 5.99 | 5.91 | 5.82 | 5.74 | 5.65 |
| 8 | 5.81 | 5.67 | 5.52 | 5.36 | 5.28 | 5.20 | 5.12 | 5.03 | 4.95 | 4.86 |
| 9 | 5.26 | 5.11 | 4.96 | 4.81 | 4.73 | 4.65 | 4.57 | 4.48 | 4.40 | 4.31 |
| 10 | 4.85 | 4.71 | 4.56 | 4.41 | 4.33 | 4.25 | 4.17 | 4.08 | 4.00 | 3.91 |
| 11 | 4.54 | 4.40 | 4.25 | 4.10 | 4.02 | 3.94 | 3.86 | 3.78 | 3.69 | 3.60 |
| 12 | 4.30 | 4.16 | 4.01 | 3.86 | 3.78 | 3.70 | 3.62 | 3.54 | 3.45 | 3.36 |
| 13 | 4.10 | 3.96 | 3.82 | 3.66 | 3.59 | 3.51 | 3.43 | 3.34 | 3.25 | 3.17 |
| 14 | 3.94 | 3.80 | 3.66 | 3.51 | 3.43 | 3.35 | 3.27 | 3.18 | 3.09 | 3.00 |
| 15 | 3.80 | 3.67 | 3.52 | 3.37 | 3.29 | 3.21 | 3.13 | 3.05 | 2.96 | 2.87 |
| 16 | 3.69 | 3.55 | 3.41 | 3.26 | 3.18 | 3.10 | 3.02 | 2.93 | 2.84 | 2.75 |
| 17 | 3.59 | 3.46 | 3.31 | 3.16 | 3.08 | 3.00 | 2.92 | 2.83 | 2.75 | 2.65 |
| 18 | 3.51 | 3.37 | 3.23 | 3.08 | 3.00 | 2.92 | 2.84 | 2.75 | 2.66 | 2.57 |
| 19 | 3.43 | 3.30 | 3.15 | 3.00 | 2.92 | 2.84 | 2.76 | 2.67 | 2.58 | 2.49 |
| 20 | 3.37 | 3.23 | 3.09 | 2.94 | 2.86 | 2.78 | 2.69 | 2.61 | 2.52 | 2.42 |
| 21 | 3.31 | 3.17 | 3.03 | 2.88 | 2.80 | 2.72 | 2.64 | 2.55 | 2.46 | 2.36 |
| 22 | 3.26 | 3.12 | 2.98 | 2.83 | 2.75 | 2.67 | 2.58 | 2.50 | 2.40 | 2.31 |
| 23 | 3.21 | 3.07 | 2.93 | 2.78 | 2.70 | 2.62 | 2.54 | 2.45 | 2.35 | 2.26 |
| 24 | 3.17 | 3.03 | 2.89 | 2.74 | 2.66 | 2.58 | 2.49 | 2.40 | 2.31 | 2.21 |
| 25 | 3.13 | 2.99 | 2.85 | 2.70 | 2.62 | 2.54 | 2.45 | 2.36 | 2.27 | 2.17 |
| 26 | 3.09 | 2.96 | 2.81 | 2.66 | 2.58 | 2.50 | 2.42 | 2.33 | 2.23 | 2.13 |
| 27 | 3.06 | 2.93 | 2.78 | 2.63 | 2.55 | 2.47 | 2.38 | 2.29 | 2.20 | 2.10 |
| 28 | 3.03 | 2.90 | 2.75 | 2.60 | 2.52 | 2.44 | 2.35 | 2.26 | 2.17 | 2.06 |
| 29 | 3.00 | 2.87 | 2.73 | 2.57 | 2.49 | 2.41 | 2.33 | 2.23 | 2.14 | 2.03 |
| 30 | 2.98 | 2.84 | 2.70 | 2.55 | 2.47 | 2.39 | 2.30 | 2.21 | 2.11 | 2.01 |
| 40 | 2.80 | 2.66 | 2.52 | 2.37 | 2.29 | 2.20 | 2.11 | 2.02 | 1.92 | 1.80 |
| 60 | 2.63 | 2.50 | 2.35 | 2.20 | 2.12 | 2.03 | 1.94 | 1.84 | 1.73 | 1.60 |
| 120 | 2.47 | 2.34 | 2.19 | 2.03 | 1.95 | 1.86 | 1.76 | 1.66 | 1.53 | 1.38 |
| ∞ | 2.32 | 2.18 | 2.04 | 1.88 | 1.79 | 1.70 | 1.59 | 1.47 | 1.32 | 1.00 |

Theorem

Writing $f_\alpha(\nu_1, \nu_2)$ for f_α with ν_1 and ν_2 degrees of freedom, we have

$$f_{1-\alpha}(\nu_1, \nu_2) = \frac{1}{f_\alpha(\nu_2, \nu_1)}$$

Thus, the f -value with 6 and 10 degrees of freedom, leaving an area of 0.95 to the right, is $f_{0.95}(6, 10) = \frac{1}{f_{0.05}(10,6)} = \frac{1}{4.06} = 0.246$.

Example

For an F -distribution, find

- (a) $f_{0.05}$ with $\nu_1 = 7$ and $\nu_2 = 15$;
- (b) $f_{0.05}$ with $\nu_1 = 15$ and $\nu_2 = 7$;
- (c) $f_{0.01}$ with $\nu_1 = 24$ and $\nu_2 = 19$;
- (d) $f_{0.95}$ with $\nu_1 = 19$ and $\nu_2 = 24$;
- (e) $f_{0.99}$ with $\nu_1 = 28$ and $\nu_2 = 12$.

Solution

(a) 2.71. (b) 3.51. (c) 2.92. (d) $1/2.11 = 0.47$. (e) $1/2.90 = 0.34$.