[illegible]

KING SAUD UNIVERSITY,DEPARTMENT OF MATHEMATICS
MATH 204. TIME: 3H, FULL MARKS: 40, FINAL EXAM

Question 1. [5,4] **a)** Initial population of a town increases by 1% in first two years and becomes 10000 in four years. What is the initial population, if the rate of growth of the population is directly proportional to the population at that instant?.

b) Find the general solution of the differential equation

$$xy^4dx + (2+y^2)e^{-3x}dy = 0, \ y > 0.$$

Question 2. [4,5] **a)** For the differential equation

$$(-xy\sin x + 2y\cos x)dx + 2x\cos x dy = 0, \ x \neq 0, \ y \neq 0.$$

verify that $\mu(x,y) = xy$ is an integrating factor, hence solve it.

b) Use power series method to find the first four terms of the solution for the initial value problem

$$(x+1)y'' = 1, \ \ y(0) = 0, \ y'(0) = 1.$$

Question 3. [5,5] **a)** Consider the $2\pi-$periodic function $f(x) = x$, for $x \in (-\pi, \pi]$.

Sketch the graph of f on $(-3\pi, 3\pi)$, obtain the Fourier series for the function f, and deduce the value of the numerical series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \qquad (\text{Hint} : \sin\frac{n\pi}{2}\begin{cases} 0, \ n = 2k \\ (-1)^k, \ n = 2k+1 \end{cases}$$

.

b) Consider the function: $f(x) = \begin{cases} \cos x, & |x| < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}$

Sketch the graph of f, find the Fourier integral representation, and deduce tha $\int_0^{\infty} \frac{\cos(\frac{\pi\lambda}{2})}{1-\lambda^2}d\lambda = \frac{\pi}{2}$.

Question 4. [5,3,4] **a)** Find the largest interval for which the following initial value problem admits a unique solution

$$\begin{cases} \frac{x^2}{x^2+4}y'' + \frac{x+3}{(4-x)^{1/3}}y' + \frac{2y}{\sqrt{x-2}} = 0 \\ y(5) = 1, \ y'(5) = -2 \end{cases}$$

b) Determine the geneal solution of the homogeneous differential equation having the characteristic equation

$$(m^4 - 1)(m-1)^4m^4 = 0.$$

c) Solve the the differential equation $y'' + 2y' + y = e^{-x}\ln x, \ x > 0.$

4/3

Complete Solution of the final exam

First Semester 1443 (2021-2022)

Question 1:

(a) 5 (a) Let $P(t)$ be the population of the town at time t and P_0 be the initial population. Then we have

$$\frac{\frac{dP}{dt}}{P} = k \Rightarrow \frac{dP}{P} = k\,dt \Rightarrow P(t) = c\,e^{kt} \quad (1)$$

At $t=0$, $P(0) = P_0 \Rightarrow P(t) = P_0 e^{kt}$. Also, for $t=2$

$P(2) = P_0 + \frac{1}{100} P_0 = (1.01) P_0$, hence $P(2) = P_0 e^{2k} = (1.01) P_0$, that is (2)

$e^{2k} = 1.01 \Rightarrow 2k = \ln(1.01)$ or $\boxed{k = \frac{1}{2}\ln(1.01)}$.

By using $t=4$, $P(4) = 10000$, then $e^{4k} = (1.01)^2$

$P(4) = 10000 = P_0 e^{4k}$, as $e^{2k} = 1.01 \Rightarrow$

$10000 = P_0 (1.01)^2 \Rightarrow P_0 = \frac{10000}{(1.01)^2} = \frac{10000}{1.0201} \# 9802$ 2

4 (b) $x y^4 dx + (2+y^2) e^{-3x} dy = 0$, $y > 0$

$x e^{3x} dx + \frac{2+y^2}{y^4} dy = 0$, $x e^{3x} dx + (2y^{-4} + y^{-2}) dy = 0$ (2)

$\int x e^{3x} dx + \int (2y^{-4} + y^{-2}) dy = 0$ (2)

$\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} - \frac{2}{3} y^{-3} - \frac{1}{y} = c \Rightarrow \boxed{e^{3x}(3x-1) - 6y^{-3} - 9y^{-1} = c_1}$

Question 2: $xy\left[(-xy\sin x + 2y\cos x)dx + 2x\cos x\, dy = 0\right]$, $x \neq 0$, $y \neq 0$

(10) (a) $\underbrace{(-x^2y^2\sin x + 2y^2x\cos x)}_{M}dx + \underbrace{2x^2y\cos x}_{N}\, dy = 0$ --- (1)

4

$\frac{\partial M}{\partial y} = -2x^2y\sin x + 4yx\cos x$, $\quad \frac{\partial N}{\partial x} = 4xy\cos x - 2x^2y\sin x$ 1

Then the D.E. (1) is an exact equation, hence there exists a function F of x and y s.t.

$\frac{\partial F}{\partial x} = M = -x^2y^2\sin x + 2y^2x\cos x$ (2)

$\frac{\partial F}{\partial y} = 2x^2y\cos x \Rightarrow F(x,y) = \int 2x^2y\cos x\, dy = x^2y^2\cos x + \phi(x)$

$$\frac{\partial F}{\partial x} = 2xy^2\cos x - x^2y^2\sin x + \phi'(x) = -x^2y^2\sin x + 2y^2x\cos x$$

$$\phi'(x) = 0 \Rightarrow \phi(x) = C$$

Then the solution of the D.E.O is 2

$$F(x,y) = \boxed{x^2y^2\cos x + C = 0}$$

(b) $(1+x)y'' = 1, \quad y(0) = 0, \quad y'(0) = 1$

5

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots, \quad x \in \mathbb{R}$$

$y(0) = 0 \Rightarrow a_0 = 0$, $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'(0) = a_1 = 1$

$$(1+x)\sum_{2}^{\infty} n(n-1)a_n x^{n-2} = 1$$ 1

$$\sum_{2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{2}^{\infty} n(n-1)a_n x^{n-1} = 1$$

$n-2 = k$, $n = k+2$ | $n-1 = k$, $n = k+1$

$$\sum_{0}^{\infty}(k+1)(k+2)a_{k+2}x^k + \sum_{1}^{\infty}(k+1)k\,a_{k+1}x^k = 1$$

$$2a_2 + \sum_{1}^{\infty}\left[(k+1)(k+2)a_{k+2} + (k+1)k\,a_{k+1}\right]x^k = 1$$

$\Rightarrow 2a_2 = 1, \quad \boxed{a_2 = \frac{1}{2}}$

$$a_{k+2} = -\frac{k(k+1)a_{k+1}}{(k+1)(k+2)}; \quad k \geq 1$$

$$\boxed{a_{k+2} = -\frac{k}{k+2}a_{k+1}, \quad k \geq 1}$$ 2

$k=1, \quad a_3 = \frac{-1}{3}a_2 = \frac{-1}{6}$

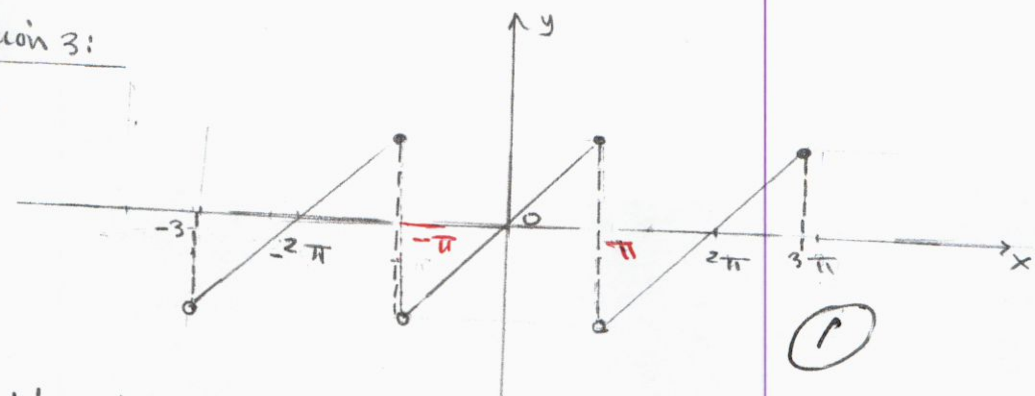
$k=2, \quad a_4 = \frac{-2}{4}a_3 = \frac{-1}{2}\left(\frac{-1}{6}\right) = \frac{1}{12}$

$k=3, \quad a_5 = -\frac{3}{5}a_4 = \frac{-3}{5}\left(\frac{1}{12}\right) = \frac{-1}{20}, \ \cdots$

$$\boxed{y(x) = x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \cdots,} \quad x \in \mathbb{R}$$ 2

Question 3:

(5) (a)



(1)

f is odd and continuous on $\mathbb{R} \setminus \{(2k+1)\pi,\ k \in \mathbb{Z}\}$. Then $a_n = 0 ; n \geq 0$

and $b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx$, $n \geq 1$

$= \frac{2}{\pi} \left[-x \frac{\cos(nx)}{n} \right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \frac{\cos nx}{n} dx$

$b_n = \frac{2}{\pi} \left(\frac{-\pi(-1)^n}{n} \right) + \underbrace{\left[\frac{2}{\pi} \frac{\sin nx}{n^2} \right]_0^{\pi}}_{0} = \boxed{\frac{2}{n}(-1)^{n+1}}$. (2)

Hence $f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx \quad -\pi < x \leq \pi$

for $x = \pi/2$,

$f(\pi/2) = \pi/2 = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(n\pi/2)$

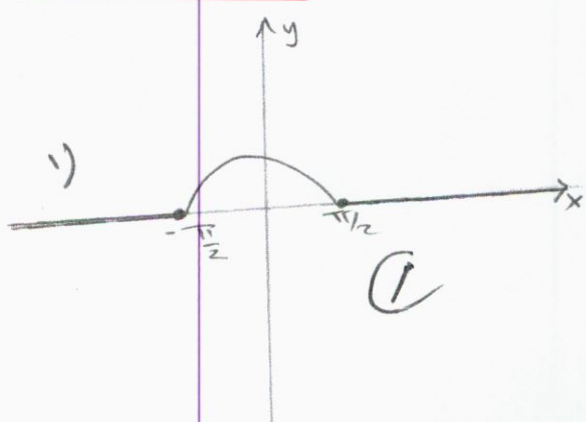
But $\sin(n\frac{\pi}{2}) = \begin{cases} 0 ; & n = 2m \\ (-1)^{m+1} ; & n = 2m-1 \end{cases}$ $m = 1, 2 \dots$ (2)

$\pi/2 = \sum_{n=1}^{\infty} \frac{2}{2n-1} (-1)^{2n-1+1} \sin\left(\frac{2n-1}{2}\pi\right)$

$\pi/2 = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{2n-1} \Rightarrow \boxed{\pi/4 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}}$

(5) (b) f is even on $\mathbb{R}$, hence $B(\lambda) = 0$

1)



(1)

2) $A(\lambda) = 2 \int_0^{\infty} f(x) \cos(\lambda x) dx = 2 \int_0^{\pi/2} \cos x . \cos \lambda x dx$

$= \int_0^{\pi/2} (\cos(1-\lambda)x + \cos(1+\lambda)x) dx$

$= \left[\frac{\sin(1-\lambda)x}{1-\lambda} + \frac{\sin(1+\lambda)x}{1+\lambda} \right]^{\pi/2}$, $\lambda \neq 1$

$=$

$$A(\lambda) = \frac{\sin\left(\frac{\pi}{2} - \lambda\frac{\pi}{2}\right)}{1-\lambda} + \frac{\sin\left(\frac{\pi}{2} + \frac{\pi}{2}\lambda\right)}{1+\lambda} = \left(\frac{1}{1-\lambda} + \frac{1}{1+\lambda}\right)\cos\left(\frac{\lambda\pi}{2}\right) \quad 2$$

$$A(\lambda) = \frac{2\cos\left(\lambda\frac{\pi}{2}\right)}{1-\lambda^2}, \quad \lambda \neq 1.$$

Then

$$f(x) = \boxed{\frac{f(x^+) + f(x^-)}{2} = \frac{2}{\pi}\int_0^\infty \frac{\cos\left(\frac{\lambda\pi}{2}\right)}{1-\lambda^2}\cos(\lambda x)\,d\lambda} \quad 1$$

3) At $x=0$, we have

$$f(0) = 1 = \frac{2}{\pi}\int_0^\infty \frac{\cos(\lambda\pi/2)}{1-\lambda^2}\,d\lambda \quad 1$$

$$\boxed{\int_0^\infty \frac{\cos\left(\lambda\frac{\pi}{2}\right)}{1-\lambda^2}\,d\lambda = \pi/2}, \quad \lambda \neq 1$$

Question 4: a) 5

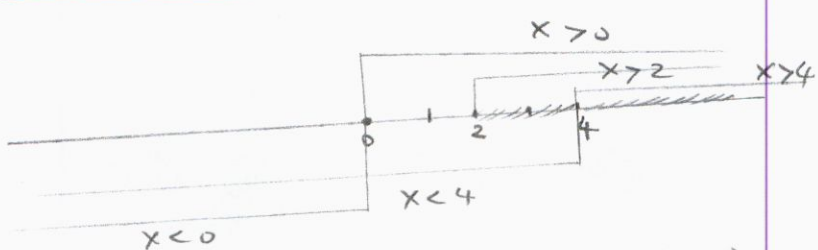
$$\begin{cases} \frac{x^2}{x^2+4}y'' + \frac{x+3}{(4-x)^{1/3}}y' + \frac{2}{\sqrt{x-2}}y = 0 \\ y(5) = 1, \quad y'(5) = -2 \end{cases}$$

1

$a_2 = \frac{x^2}{x^2+4}$ is continuous on $\mathbb{R}$ and $a_2(x) \neq 0$ if $x \neq 0$ $(x>0$ or $x<0)$

$a_1 = \frac{x+3}{(4-x)^{1/3}}$ is continuous on $\mathbb{R}\setminus\{4\}$ i.e $(x<4$ or $x>4)$ 3

$a_0 = \frac{2}{\sqrt{x-2}}$ is continuous on $\{x \in \mathbb{R};\ x>2\}$ 2



Then a_1, a_2 and a_0 are continuous on $(2,4)\cup(4,\infty)$.

But $5 \in (4,\infty)$, hence the largest interval for which the IVP has a unique solution is $\boxed{I = (4,\infty)}$ 2

(b) (i) $m^2(m-2)^3(m^2-2m+5)(m-3)^5=0$

× $m=0,0,\quad m=2,2,2,\quad m=3,3,3,3,3$

$(m-1)^2+4=0 \qquad m=1\mp 2i$

(i) or (ii) $y_c = c_1+c_2x+c_3e^{2x}+c_4xe^{2x}+c_5x^2e^{2x}+$
$c_6e^{3x}+c_7xe^{3x}+c_8x^2e^{3x}+c_9x^3e^{3x}+c_{10}x^4e^{3x}+c_{11}xe^{x}\cos 2x+$
$c_{12}xe^{x}\sin(2x)$

---

3 ✓ (ii) $(m^4-1)(m-1)^4m^4=0$

$(m-1)(m+1)(m^2+1)(m-1)^4m^4=0$ (3)

$(m-1)^5(m+1)(m^2+1)(m^4)=0$

$m=1,1,1,1,1,\quad m=-1,\quad m=0,0,0,0,\quad m=\mp i$

$y_c = \underline{c_1+c_2x+c_3x^2+c_4x^3} + \underline{c_5e^x+c_6xe^x+c_7x^2e^x+c_8x^3e^x+c_9x^4e^x}$ (1) (1)
$\underline{+c_{10}e^{-x}+c_{11}\cos x+c_{12}\sin x}$ (1)

---

4 (c) $y''+2y'+y=e^{-x}\ln x,\quad x>0$

1) $y''+2y'+y=0,\quad (m+1)^2=0,\quad m=-1,-1 \qquad y_c=c_1e^{-x}+c_2xe^{-x}$ (1)

Let $y_1=e^{-x},\ y_2=xe^{-x}$, then

$$W=\begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x}-xe^{-x}\end{vmatrix}=e^{-2x},\quad W_1=\begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & e^{-x}-xe^{-x}\end{vmatrix}=-xe^{-2x}\ln x$$

$$W_2=\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln x\end{vmatrix}=e^{-2x}\ln x$$

(2)

$u_1'=\dfrac{W_1}{W}=-x\ln x \Rightarrow u_1=-\int x\ln x\,dx=\boxed{-\dfrac{x^2}{2}\ln+\dfrac{x^2}{4}=u_1}$ (1)

$u_2'=\dfrac{W_2}{W}=\ln x \Rightarrow u_2=\int \ln x\,dx=\boxed{x\ln x-x=u_2}$ (1)

$y_p=y_1u_1+y_2u_2=e^{-x}\left[-\dfrac{x^2}{2}\ln x+\dfrac{x^2}{4}\right]+xe^{-x}[x\ln x-x]$

$$y_p = e^{-x}\left[-\frac{x^2}{2}\ln x + x^2\ln x + \frac{x^2}{4} - x^2\right]$$

$$y_p = e^{-x}\left[\frac{x^2}{2}\ln x - \frac{3}{4}x^2\right]$$

$$\boxed{y_p = \frac{1}{2}x^2 e^{-x}\left(\ln x - \frac{3}{2}\right)}$$

1

Hence the general solution of the D.E is

$$\boxed{y = y_c + y_p = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2}x^2 e^{-x}\left(\ln x - \frac{3}{2}\right)}$$

1

2