Geometry and Topology

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1 Connected spaces

1.1 Connected spaces

Definition 1.1 *Let (X, τ) be a topological space. A* ***separation*** *of (X, τ) is a pair $U, V \in \tau$ such that $U, V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = X$. The space (X, τ) is said to be* ***disconnected*** *if there exists a separation of (X, τ). It is said to be* ***connected*** *if there is no separation of (X, τ).*

Questions 1 *Which of the following topological spaces are connected:*

1. $\tau = \mathcal{P}(X)$ *(Power of X) (Discrete topology).*
2. $\tau = \{\emptyset, X\}$ *(Indiscrete topology or trivial topology).*
3. $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.
4. $X = [0, 1] \cup [2, 3]$, *τ the induced topology of $\mathbb{R}$.*
5. *X infinite set, $\tau = \{A \subseteq X | X \setminus A \ finite\} \cup \{\emptyset\}$ (finite complement topology).*

Lemma 1.2 *A topological space (X, τ) is connected if and only if the only subsets of X that are both open and closed are $\emptyset$ and X.*

Recall that if $A \subseteq X$, the boundary of A is the set $\partial A = \overline{A} \setminus A^{\circ}$.

Question 2 *Prove the following: A topological space (X, τ) is connected if and only if the boundary of any proper subset[1] of X is not empty.*

Questions 3 *If τ, τ' are two topologies on X such that (X, τ) is connected.*

1. *Is (X, τ') connected?*
2. *If τ' is finer[2] than τ, is (X, τ') connected?*
3. *If τ is finer than τ', is (X, τ') connected?*

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[1] Proper subset means neither equal to X nor equal to $\emptyset$.

[2] Recall that τ' is said to be **finer** than τ or τ is said to be **coarser** than τ' if $\tau \subseteq \tau'$.

1.2 Connected subspaces

Recall that $x \in X$ is called a **limit point** (or "cluster point" or "point of accumulation") of $A \subseteq X$ if $\forall U \in \tau$, if $x \in U$ then $U \cap A \setminus \{x\} \neq \emptyset$.

Questions 4 *Let us denote by A' the set of limit points of A.*

1. *What is A' for the following sets $A \subseteq \mathbb{R}$: $A = [0,1]$, $A = (0,1)$, $A = [0,1] \cup \{2\}$.*
2. *Prove that if (X,τ) is a topological space and $A \subseteq X$ then the closure of A satisfies $\overline{A} = A \cup A'$.*
3. *Prove that if (X,τ) is a topological space and $A \subseteq X$ then A is closed if and only if $A' \subseteq A$.*

Here is another formulation of the definition of separation for subspace Y of a topological space X (with the induced topology):

Lemma 1.3 *If $Y \subseteq X$, a separation of Y is a pair $A, B \subset Y$ such that $A, B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = Y$ and neither A contains limit points of B nor B contains limit points of A.*

Questions 5 *Are the following separations:*

1. *For $Y = [-1,1]$ with the induced topology of $\mathbb{R}$, $A = [-1,0]$, $B = (0,1]$.*
2. *For $Y = [-1,0) \cup (0,1]$ with the induced topology of $\mathbb{R}$, $A = [-1,0)$ and $B = (0,1]$.*
3. *For $Y = \{1,2\}$ with the induced finite complement topology of $\mathbb{N}$, $A = \{0\}$ and $B = \{2\}$.*
4. *For $Y = 2\mathbb{N} = \{2,4,6,\cdots\}$ with the induced finite complement topology of $\mathbb{N}$, $A = \{2\}$ and $B = \{4,6,8,\cdots\}$.*
5. *For $Y = A \cup B$ where $A = \mathbb{R} \times \{0\}$ and $B = \{(x,1/x) | x > 0\}$ with the induced topology of $\mathbb{R}^2$.*
6. *Is $\mathbb{Q}$ connected for the induced topology of $\mathbb{R}$?*

Lemma 1.4 *Let (X,τ) be a topological space and U, V be a separation of X. If $Y \subset X$ is connected then either $Y \subseteq U$ or $Y \subseteq V$.*

Theorem 1.5 *Let (X,τ) be a topological space and $\{Y_i\}_{i \in I}$ be a family of connected subspaces of X. If $\cap_{i \in I} Y_i \neq \emptyset$ then $\cup_{i \in I} Y_i$ is connected.*

Theorem 1.6 *Let (X,τ) be a topological space and $Y_1, Y_2, \cdots, Y_k, \cdots$ be a family of connected subspaces of X. If $Y_i \cap Y_{i+1} \neq \emptyset, \forall i = 1,2,\cdots$ then $Y_1 \cup Y_2 \cup \cdots \cup Y_k \cup \cdots$ is connected.*

Theorem 1.7 *Let (X, τ) be a topological space and Y be a connected subspace of X. If $Y \subseteq Z \subseteq \overline{Y}$, then Z is connected.*

Question 6 *Let Y be a subset of X and Z be a connected subspace of X. Prove that if $Z \cap Y^\circ \neq \emptyset$ and $Z \cap (X - Y)^\circ \neq \emptyset$ then $Z \cap \partial Y \neq \emptyset$ where ∂Y is the boundary of Y.*

1.3 Totaly disconnected spaces

Definition 1.8 *A topological space (X, τ) is said to be totaly disconnected if the only connected subspaces are singletons.*

Questions 7

1. *Is $\mathbb{Q}$ totaly disconnected?*
2. *Is a discrete space totaly disconnected?*
3. *Is a totaly disconnected space discrete?*
4. *Is a finite totaly disconnected space discrete?*

1.4 Connectedness and continuity

Theorem 1.9 *The image of a connected space under a continuous map is connected.*

Question 8 *Let (X, τ) be a topological space and Y be a totaly disconnected space. Prove that X is connected if and only if any continuous map $f : X \to Y$ is constant.*

1.5 Connectedness and constructions

Questions 9

1. *Is the union of connected subspaces connected?*
2. *Is the intersection of connected subspaces connected?*
3. *Is the interior of a connected subspace connected?*
4. *Is the closure of a connected subspace connected?*
5. *Is the boundary of a connected subspace connected?*

Theorem 1.10 *The finite cartesian product of connected spaces is connected.*

Recall that for the infinite cartesian product $X = \prod_{i \in I} X_i$ we have two natural topologies:

- The **box topology** whose basis is $\left\{ \prod_{i \in I} U_i | U_i \in \tau_i, i \in I \right\}$.
- The **product topology** whose basis is

$$\left\{ \prod_{i \in I} U_i | \exists J \underset{finite}{\subset} I \text{ finite s.t. } U_i \in \tau_i \text{ if } i \in J \text{ and } U_i = X_i \text{ if not} \right\}.$$

It is clear that the box topology is finer than the product topology.

Theorem 1.11 *The infinite cartesian product of connected spaces is connected for the product topology.*

Sketch of the proof. Fix $\{a_i\}_{i\in I}$ an element of $\prod_{i\in I} X_i$. For $J \subset_f I$ a finite subset of I define $X_J = \left\{ \{x_i\}_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } x_i = a_i,\ \forall i \notin J \right\}$. The set X_J is homeomorphic to $\prod_{j\in J} X_j$ and hence is connected. The intersection $\bigcap_{J\subset_f I} X_J = \{\{a_i\}_{i\in I}\} \neq \emptyset$. Then $\bigcup_{J \subset_f I} X_J$ is connected.

The closure of $\bigcup_{J \subset_f I} X_J$ for the product topology is $\prod_{i\in I} X_i$ so $\prod_{i\in I} X_i$ is connected for the product topology.

Question 10 *Let $X = \mathbb{R}^{\mathbb{N}} = \prod_{n\in\mathbb{N}} \mathbb{R}$ be the set of real sequences. Let $Y \subset X$ be the set of bounded real sequences.*

1. *Prove that Y is an open and closed subset of X for the box topology.*
2. *Is X connected for the box topology?*

Question 11 *Show that if a cartesian product $\prod_{i\in I} X_i$ is connected then the spaces $X_i, i \in I$ are all connected.*

1.6 Connected subsets in $\mathbb{R}$

Recall that an interval $A \subseteq \mathbb{R}$ is a subset which satisfies $\forall a < b \in A$, if $a < c < b$ then $c \in A$.

Theorem 1.12 *A subset $A \subseteq \mathbb{R}$ is connected if and only if A is an interval.*

As consequences of this theorem:

Theorem 1.13 (The intermediate value) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $y_0 \in]f(a), f(b)[$ the non oriented open interval then $\exists c \in]a, b[$ such that $f(c) = y_0$.*

Theorem 1.14 *Any continuous function $f : [a, b] \to \mathbb{Z}$ is constant.*

To generalize this theorem, let us recall the following definitions:

Definition 1.15 *Let X be a set and $\mathcal{R} \subseteq X \times X$ be a binary relation on X.*

- *The relation $\mathcal{R}$ is called a* ***linear order*** *or a* ***simple order*** *if it satisfies the following axioms:*

  1. *For all $x, y \in X$ we have: $x \neq y$ if and only if either $x\mathcal{R}y$ or $y\mathcal{R}x$.*
  2. *For all $x, y, z \in X$, if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$.*

- *If $\mathcal{R}$ is a linear order on X and $Y \subseteq X$ and $x \in X$ then Y is said to be* ***bounded above*** *by x (or x is said to be an* ***upper bound*** *of Y) if $\forall y \in Y$, $y\mathcal{R}x$ or $y = x$. The element $x \in X$ is said to be a* ***least upper bound*** *of Y if $\forall x' \in X$, if x' is an upper bound of Y then $x\mathcal{R}x'$ or $x = x'$.*

- *If $\mathcal{R}$ is a linear order on X then it is said to have the* ***least upper bound property*** *if $\forall Y \subseteq X$ if Y is bounded above then Y has a least upper bound.*

- *A* ***linear continuum*** *on X is a linear order $\mathcal{R}$ which has the least upper bound property and satisfies moreover the property: Let $x, y \in X$. If $x\mathcal{R}y$ then $\exists z \in X$ such that $x\mathcal{R}z$ and $z\mathcal{R}y$.*

- *If $\mathcal{R}$ is a linear order on X an* ***open interval*** *is a subset of the form*

$$]x, y[= \{z \in X, \text{ s.t. } x\mathcal{R}z \text{ and } z\mathcal{R}y\}$$

  for $x, y \in X$.

- *If $\mathcal{R}$ is a linear order on X the* ***order topology*** *on $(X.\mathcal{R})$ is the topology defined by the basis $\mathcal{B}$ whose elements are intervals $]x, y[$, $\{x_0\} \cup]x_0, y[$ and $]x, y_0[\cup \{y_0\}$ for $x, y \in X$ and x_0 a lower bound of X and y_0 an upper bound of X if they exist.*

Examples 12

1. *$(\mathbb{Q}, <)$ is a linear order but does not have the least upper bound property.*
2. *$[0, 1] \times \{0, 1\}$ with the dictionary order ($(a, b) <_{dic} (a', b')$ iff $a < a'$ or $a = a'$ and $b < b'$) is a linear order with the least upper bound property but is not a linear continuum.*
3. *$(\mathbb{R}, <)$ is a linear continuum and also $\mathbb{C} = \mathbb{R}^2$ with the dictionary order $<_{dic}$.*
4. *the order topology on $(\mathbb{R}, <)$ is the natural topology but the order topology on $\mathbb{C}$ with the dictionary order $<_{dic}$ is not the natural topology.*

Theorem 1.16 *Let $(X, <)$ be a linear continuum. The subset $Y \subseteq X$ is connected if and only if Y is an interval.*

1.7 Path connected spaces

Definition 1.17 *Let (X, τ) be a topological space.*

- *Let $x, y \in X$. A path from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$.*
- *The space (X, τ) is called path connected if for any $x, y \in X$ there is a path from x to y.*

Theorem 1.18 *Let (X, τ) be a topological space. If X is path connected then X is connected.*

Examples 13 *There are some connected spaces which are not path connected:*

1. *The deleted comb space:*

$$C = \{(0,1)\} \cup \{(x,0), s.t.\ x \in [0,1]\} \cup \left\{\left(\frac{1}{n}, y\right), s.t.\ n \in \mathbb{N}, y \in [0,1]\right\}.$$

2. *The topologist's sine curve:*

$$S = \{(0,y), s.t.\ y \in [-1,1]\} \cup \left\{\left(x, \sin\frac{1}{x}\right), s.t.\ x > 0\right\}.$$

Theorem 1.19 *Let $f : X \to Y$ be a continuous map. If X is path connected then $f(X)$ is path connected.*

Questions 14
1. *Let $U \subseteq \mathbb{R}$. Prove that if $U \cap [a, b]$ is open for the induced topology of $\mathbb{R}$ on $[a, b]$ and $U \cap [b, c]$ is open for the induced topology of $\mathbb{R}$ on $[b, c]$ then $U \cap [a, c]$ is open for the induced topology of $\mathbb{R}$ on $[a, c]$.*
2. *Deduce that if $f : [a, b] \to X$ and $g : [b, c] \to X$ are continuous and $f(b) = g(b)$ then the map $h = f * g : [a, c] \to X$ defined by $h(t) = f(t)$ if $a \le t \le b$ and $h(t) = g(t)$ if $b \le t \le c$ is continuous.*

Theorem 1.20 *Let (X, τ) be a topological space and $\{Y_i\}_{i\in I}$ be a family of path connected subspaces of X. If $\cap_{i\in I} Y_i \neq \emptyset$ then $\cup_{i\in I} Y_i$ is path connected.*

Theorem 1.21 *Let (X, τ) be a topological space and $Y_1, Y_2, \cdots, Y_k, \cdots$ be a family of path connected subspaces of X. If $Y_i \cap Y_{i+1} \neq \emptyset, \forall i = 1, 2, \cdots$ then $Y_1 \cup Y_2 \cup \cdots \cup Y_k \cup \cdots$ is path connected.*

Questions 15 *Let (X, τ) be a topological space and Y be a path connected subspace of X.*

1. *If $Y \subseteq Z \subseteq \overline{Y}$, is Z path connected?*
2. *Is $\overline{Y}$ path connected?*

3. *If* $Z \subseteq X$ *such that* $Y \cap Z^\circ \neq \emptyset$ *and* $Y \cap (X - Z)^\circ \neq \emptyset$ *is* $Y \cap \partial Z \neq \emptyset$*?*

4. *Is the union of path connected sets path connected?*

5. *Is the intersection of path connected sets path connected?*

6. *Is the interior of a path connected set path connected?*

Theorem 1.22 *The finite cartesian product of path connected sets is path connected.*

Questions 16 1. *Is the infinite cartesian product of path connected sets path connected for the box topology?*

2. *Is the infinite cartesian product of path connected sets path connected for the product topology?*

1.8 Components and path components

Definition 1.23 *Let* (X, τ) *be a topological space. We define on* X *two relations. If* $x, y \in X$ *then*

1. $x \sim y$ *iff* $\exists Y \subseteq X$ *connected such that* $x, y \in Y$.

2. $x \sim_p y$ *iff* $\exists Y \subseteq X$ *path connected such that* $x, y \in Y$.

Lemma 1.24 *The relations* $\sim$ *and* $\sim_p$ *are equivalence relations.*

Definition 1.25 *The equivalence classes of* $\sim$ *are called* ***components*** *of* X *or* ***connected components*** *of* X.
The equivalence classes of $\sim_p$ *are called* ***path components*** *of* X *or* ***path connected components*** *of* X.

Theorem 1.26 *Let* X *be a topological space. The components of* X *have the following properties:*

1. *They form a partition of* X.

2. *Any connected subset of* X *is contained in one of these components.*

3. *These components are connected.*

4. *These components are closed.*

Theorem 1.27 *Let* X *be a topological space. The path components of* X *have the following properties:*

1. *They form a partition of* X.

2. *Any path connected subset of* X *is contained in one of these path components.*

3. *These path components are path connected.*

Questions 17 1. *What are components of* $\mathbb{Q}$*?*

2. *What are components of a totally disconnected space?*

3. *Are components open?*

4. *Are path components open?*

5. *What are the components of the topologist's sine curve?*

6. *What are the path components of the topologist's sine curve?*

7. *Are path components closed?*

8. *What are the components and the path components of the deleted comb space?*

1.9 Local connectedness

Definition 1.28 *Let* X *be a topological space. Let* $x \in X$*. We say that* X *is* ***locally (path) connected at*** x *if for any* U *open subset of* X *containing* x *there exists* V *a (path) connected neighbourhood of* x *contained in* U *(*$x \in V \subseteq U$*). We say that* X *is* ***locally (path) connected*** *if it is locally (path) connected at all* $x \in X$*.*

Remark 1.29 *If* $\mathcal{B}$ *is a basis of the topology of* X *then* X *is locally (path) connected at* $x \in X$ *iff* $\forall U \in \mathcal{B}$*, if* $x \in U$*,* $\exists V \subseteq X$ *(path) connected,* $\exists U' \in \mathcal{B}$ *such that* $x \in U' \subseteq V \subseteq U$*.*

Remark 1.30 *If* X *is locally path connected at* $x \in X$ *then it is locally connected at* x*.*

Questions 18 1. *Is* $\mathbb{R}$ *locally (path) connected?*

2. *Is* $[0,1] \cup [2,3]$ *locally (path) connected?*

3. *Is a discrete space locally connected? locally path connected?*

4. *Is* $\mathbb{N}$ *with the complement finite topology locally connected? path connected? locally path connected?*

5. *The deleted comb space is connected but not locally connected.*

6. *What about the topologist's sine curve?*

7. *If* $X = C \cup \{(0, 1/n), n \in \mathbb{N}\}$*, where* C *is the deleted comb space then* X *is locally connected at* $(0,0)$ *but not locally path connected at* $(0,0)$*.*

Theorem 1.31 *Let* X *be a topological space.* X *is locally connected if and only if the components of any open subset of* X *are open.*

Corollary 1.32 *If* X *is locally connected then the components of* X *are open.*

Example 19 *The space* $\mathbb{Q}$ *is not locally connected and any totally disconnected space which is not discrete is not locally connected.*

Theorem 1.33 *Let* X *be a topological space.* X *is locally path connected if and only if the path components of any open subset of* X *are open.*

Theorem 1.34 *Let* X *be a topological space. Any path component of* X *lies in a component of* X*. Moreover, if* X *is locally path connected then the components of* X *are precisely the path components of* X*.*