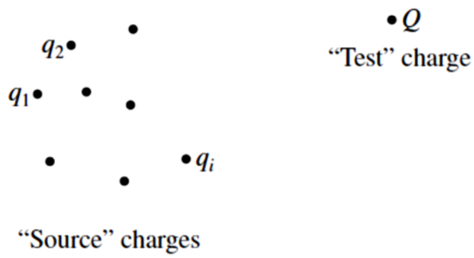
Chapter 2: Electrostatics

If we have some electric charges, q_1, q_2, q_3, . . . (called **source charges**), exerting electric forces on another charge, Q (called **test charge**), the interaction between any two charges is completely unaffected by the presence of others. This means that to determine the total electric force on Q, we can first calculate the force F_1, due to q_1 alone (ignoring all the others); then we calculate the force F_2, due to q_2 alone; and so on. Finally, we take the vector sum of all these individual forces: $F = F_1 + F_2 + F_3 + \ldots$ (**the principle of superposition**)



2.1 Coulomb’s Law

The force between two stationary electric point charges q and Q is found experimentally to be:

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}.}$$

The constant ε_o is called the **permittivity of free space**.
In SI units, where force is in newtons (N), distance in meters (m), and charge in coulombs (C),

$$\epsilon_0 = 8.85 \times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}.$$

The Coulomb constant k_e in SI units has the value $k_e = 8.9876 \times 10^9\ \mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}^2$

Where

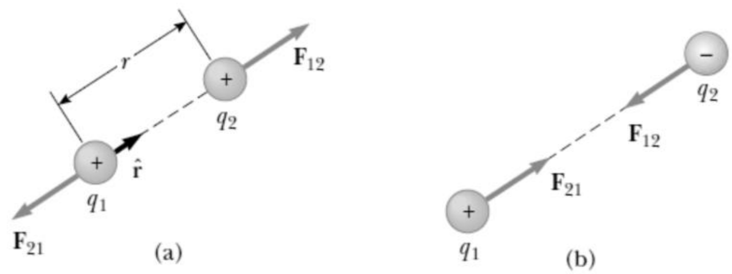
$$k_e = \frac{1}{4\pi\epsilon_0}$$

The charge on an electron (- e) or a proton (+ e), has a magnitude:

$$e = 1.602\ 18 \times 10^{-19}\ \text{C}$$

The electric force:

- acts along the line joining the two charges
- is proportional to the product of the charges and inversely proportional to the square of the separation distance



Example 2.1

The electron and proton of a hydrogen atom are separated by a distance of approximately 5.3×10^{-11} m. Find the ratio of the magnitude of the electric force to the magnitude of the gravitational force between the two particles.

Particle	Charge (C)	Mass (kg)
Electron (e)	$-1.602\ 176\ 5 \times 10^{-19}$	$9.109\ 4 \times 10^{-31}$
Proton (p)	$+1.602\ 176\ 5 \times 10^{-19}$	$1.672\ 62 \times 10^{-27}$

From Coulomb's law $|\vec{F}_e| = k_e \frac{|e||-e|}{r^2} = 8.2 \times 10^{-8}$ N

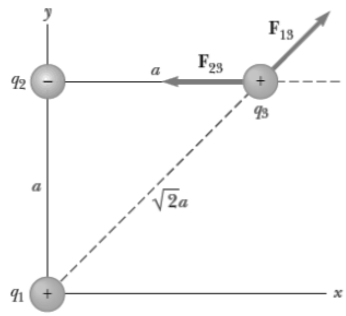
From Newton's law $|\vec{F}_g| = G\frac{m_e m_p}{r^2} = 6.67\times10^{-11} \frac{9.11\times10^{-31}\times1.67\times10^{-27}}{(5.3\times10^{-11})^2}$

$= 3.6 \times 10^{-47}$ N

$\Rightarrow \frac{F_e}{F_g} \simeq 2\times10^{39}$

Example 2.2

Consider three point charges located at the corners of a right triangle as shown in the following figure, where $q_1 = q_3 = 5\ \mu C$, $q_2 = -2\ \mu C$, and $a = 0.1$ m. Find the resultant force exerted on q_3 .



$$\vec{F}_{23} = -k_e \frac{(2\times10^{-6})(5\times10^{-6})}{(0.1)^2}\hat{x}$$

$$\Rightarrow \vec{F}_{23} = -8.99\,\hat{x} \quad N$$

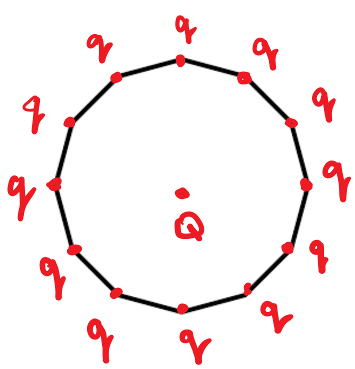
$$|\vec{F}_{13}| = \frac{k_e q_1 q_3}{r^2} = 8.99\times10^{9}\frac{(5\times10^{-6})(5\times10^{-6})}{(\sqrt{2}a)^2} = 11.2\,N$$

$$\Rightarrow \vec{F}_{13} = 11.2\cos 45^\circ\,\hat{x} + 11.2\sin 45^\circ\,\hat{y}$$

$$\vec{F}_3 = \vec{F}_{23} + \vec{F}_{13} = [-1.04\,\hat{x} + 7.94\,\hat{y}]\ N$$

Example 2.3

Twelve equal charges, q, are situated at the corners of a regular 12-sided polygon.
(a) What is the net force on a test charge Q at the center?
(b) Suppose one of the 12 q's is removed (the one at "6 o'clock"). What is the force on Q?



a) zero

b) $\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{y}$

2.2 The Electric Field

If we have several point charges $q_1, q_2, \ldots, q_n$, at distances $r_1, r_2, \ldots, r_n$ from Q, the total force on Q is:

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \ldots\right)$$
$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \frac{q_3}{\mathscr{r}_3^2}\hat{\mathscr{r}}_3 + \ldots\right),$$

We can write the total force as:

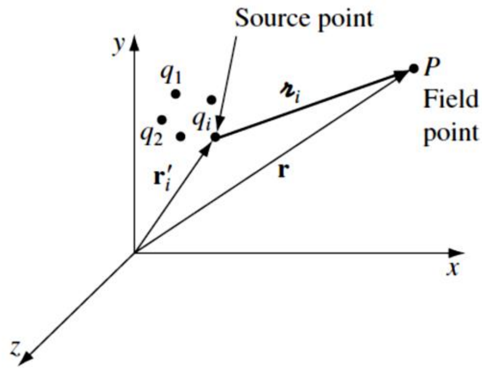
$$\boxed{\mathbf{F} = Q\mathbf{E},}$$

Where

$$\mathbf{E}(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i .$$

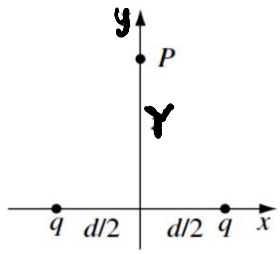
E(r) is called the electric field of the source charges. Notice that:

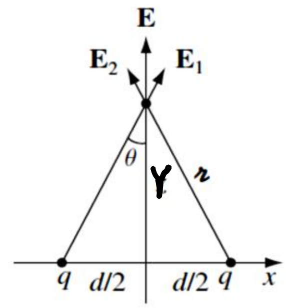
- It is a function of position (r), because the separation vectors $\mathbf{r}_\mathbf{i}$ depend on the location of the field point P
- It is a vector quantity that varies from point to point and is determined by the configuration of source charges.
- The electric field makes no reference to the test charge Q, but it is the force per unit charge that would be exerted on a test charge, if you were to place one at the field point P.



Example 2.4

Find the electric field a distance Y above the midpoint between two equal charges (q), a distance d apart.





$$\vec{E}_1 = E_{1x}\hat{x} + E_{1y}\hat{y}$$

$$\vec{E}_2 = E_{2x}\hat{x} + E_{2y}\hat{y}$$

$$\because \quad E_{1x}\hat{x} = -E_{2x}\hat{x} \Rightarrow E_{1x}\hat{x} + E_{2x}\hat{x} = 0$$

$$\because \quad E_{1y}\hat{y} = E_{2y}\hat{y}$$

$$\Rightarrow \quad \vec{E} = \vec{E}_1 + \vec{E}_2 = 2E_{1y}\hat{y}$$

$$\vec{E} = 2\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{Y^2+\left(\frac{d}{2}\right)^2}\cos\theta\right]\hat{y}$$

$$\boxed{\vec{E} = 2\left[\frac{1}{4\pi\varepsilon_0}\frac{qY}{\left(Y^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}\right]\hat{y}}$$

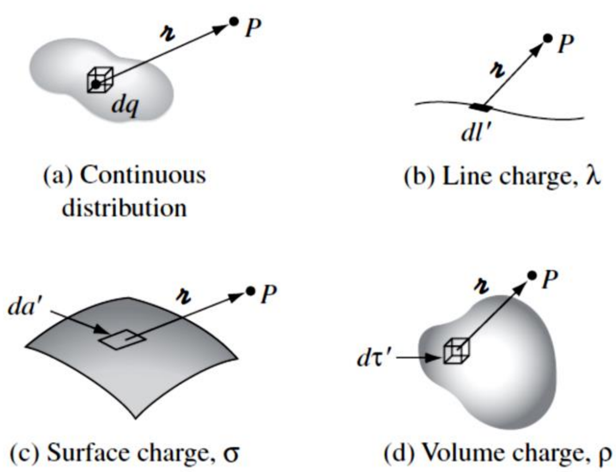
when $Y >> d \Rightarrow \vec{E} \approx \frac{1}{4\pi\varepsilon_0}\frac{2q}{Y^2}\hat{y}$

2.3 Continuous Charge Distributions

If the charge is distributed continuously over some region, the sum becomes an integral

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dq$$

- If the charge is spread out along a line (see the following figure b), with charge-per-unit-length λ, then dq = λ dl′ (where dl′ is an element of length along the line);
- If the charge is spread out over a surface (figure c), with charge per unit area σ, then dq = σ da′ (where da′ is an element of area on the surface).
- If the charge fills a volume (figure d), with charge per unit volume ρ, then dq = ρ dτ′ (where dτ′ is an element of volume).



(a) Continuous distribution

(b) Line charge, λ

(c) Surface charge, σ

(d) Volume charge, ρ

The electric field of a line charge is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, dl'$$

for a surface charge

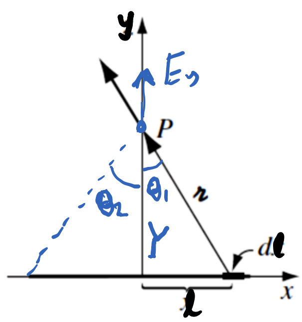
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,da'$$

and for a volume charge,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'$$

Example 2.5

Find the electric field a distance Y above the midpoint of a straight line segment of length L that carries a uniform line charge λ



$$\vec{E} = E_x\,\hat{x} + E_y\,\hat{y} \Rightarrow \vec{E} = E\sin\theta\,\hat{x} + E\cos\theta\,\hat{y}$$

$$dq = \lambda\, d\ell$$

$$r = Y\sec\theta$$

$$\ell = Y\tan\theta \Rightarrow d\ell = Y\sec^2\theta\, d\theta$$

$$\Rightarrow dq = \lambda Y \sec^2\theta\, d\theta$$

$$\vec{E} = \int_{\theta_2}^{\theta_1} \frac{1}{4\pi\varepsilon_0}\frac{\sin\theta\, dq}{r^2}\,\hat{x} + \int_{\theta_2}^{\theta_1}\frac{1}{4\pi\varepsilon_0}\frac{\cos\theta\, dq}{r^2}\,\hat{y}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{\theta_2}^{\theta_1}\frac{\sin\theta\,\lambda\,\cancel{Y}\,\cancel{\sec^2}\theta\, d\theta}{Y^{\cancel{2}}\,\cancel{\sec^2}\theta}\,\hat{x} + \frac{1}{4\pi\varepsilon_0}\int_{\theta_2}^{\theta_1}\frac{\cos\theta\,\lambda\,\cancel{Y}\,\cancel{\sec^2}\theta\, d\theta}{Y^{\cancel{2}}\,\cancel{\sec^2}\theta}\,\hat{y}$$

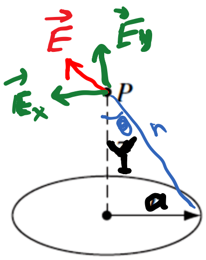
$$= \frac{\lambda}{4\pi\varepsilon_0 Y}\int_{\theta_2}^{\theta_1}\sin\theta\, d\theta\,\hat{x} + \frac{\lambda}{4\pi\varepsilon_0 Y}\int_{\theta_2}^{\theta_1}\cos\theta\, d\theta\,\hat{y}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 Y}\left[\cos\theta_2 - \cos\theta_1\right]\hat{x} + \frac{\lambda}{4\pi\varepsilon_0 Y}\left[\sin\theta_1 - \sin\theta_2\right]\hat{y}$$

if $L = \infty \Rightarrow \theta_2 = -\frac{\pi}{2}$ $\quad \theta_1 = \frac{\pi}{2}$ $\quad \Rightarrow \vec{E} \approx \frac{\lambda}{4\pi\varepsilon_0 Y}[2]\,\hat{y} \approx \frac{\lambda\,\hat{y}}{2\pi\varepsilon_0 Y}$

Example 2.6

Find the electric field a distance Y above the center of a circular loop of radius a that carries a uniform line charge λ.



$$\vec{E} = \vec{E}_y = \int_0^q \frac{\cos\theta}{4\pi\varepsilon_0}\frac{dq}{r^2}\,\hat{y}$$

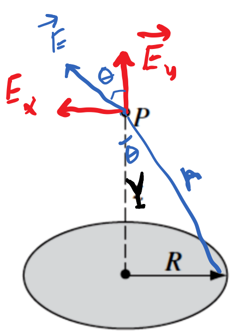
$$\because \cos\theta = \frac{Y}{r} = \frac{Y}{\sqrt{Y^2+a^2}}$$

$$\Rightarrow \vec{E} = \int_0^q \frac{1}{4\pi\varepsilon_0}\frac{Y}{(Y^2+a^2)^{3/2}}\,dq\,\hat{y} = \frac{qY}{4\pi\varepsilon_0 (Y^2+a^2)^{3/2}}\,\hat{y}$$

if $Y >> a \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{Y^2}\,\hat{y}$

Example 2.7

Find the electric field a distance Y above the center of a flat circular disk of radius R that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$?



$$\Rightarrow \vec{E} = E\hat{y} = \frac{1}{4\pi\varepsilon_0}\int \frac{\cos\theta\, dq}{r^2}\,\hat{y}$$

$$\because q = \sigma a \Rightarrow dq = \sigma\, da \Rightarrow dq = \sigma\, 2\pi r\, dr$$

$$\because r = \sqrt{Y^2+r^2} \qquad \cos\theta = \frac{Y}{\sqrt{Y^2+r^2}}$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{2\pi\sigma r Y}{(Y^2+r^2)^{3/2}}\,dr\,\hat{y}$$

$$= \frac{\sigma Y}{2\varepsilon_0}\int_0^R \frac{r}{(Y^2+r^2)^{3/2}}\,dr\,\hat{y} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{Y}{(Y^2+R^2)^{1/2}}\right]\hat{y}$$

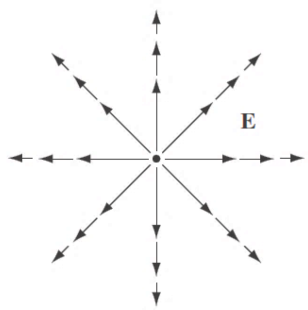
if $R \sim \infty \Rightarrow \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0}\,\hat{y}}$

2.4 Field Lines

From the previous discussion the electric field for a single point charge q can be calculated from the following formula:

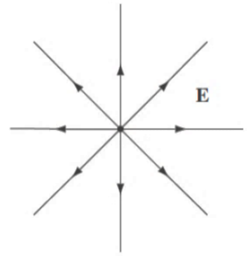
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

Representative vectors can be sketched, as shown in the following figure. The vectors point radially outward and the field falls off like $1/r^2$ (the vectors become shorter when go farther away from the origin)

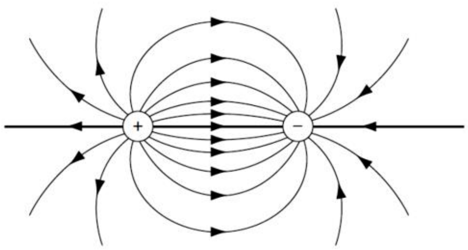


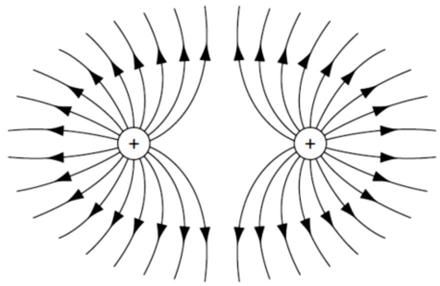
But it is nicer to connect up the arrows to represent field lines:

- The strength of the field, which was previously contained in the length of the vector line, can be indicated here by the density of the field lines. It is strong near the center where the field lines are close together and weak farther out where they are relatively far apart. Therefore, <u>the magnitude of the field is indicated by the density of the field lines</u>.
- The direction of the electric field vector **E** is tangent to the electric field line at each point.
- Field lines begin on positive charges and end on negative charges.
- Field lines cannot simply terminate in midair.
- Field lines can never cross.



For opposite charges

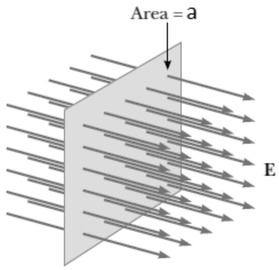




2.5 Electric Flux and Gauss's Law

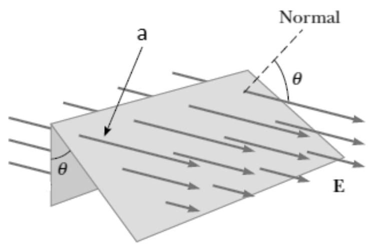
The total number of lines penetrating the surface (of area a) is proportional to the product $\mathbf{E}\cdot\mathbf{a}$

This product of the electric field **E** and surface area **a** perpendicular to the field is called the electric flux Φ_E

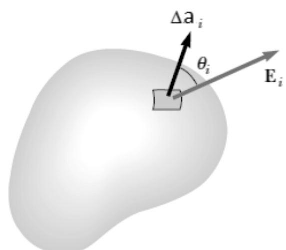


If the electric field is constant over the surface area then the electric flux becomes

$$\Phi_E = E \cdot a = E\, a \cos\theta$$



If the electric field is not constant over the surface area then the electric flux becomes:

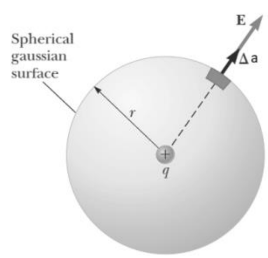


$$\Phi_E \approx \lim_{\Delta a \to 0} \sum E_i \cdot \Delta a_i$$

$$\Phi_E = \int_{surface} E \cdot da$$

The unite of electric flux in SI system is Nm^2 / C

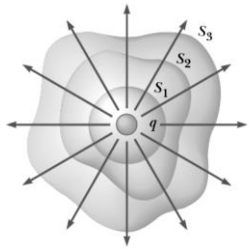
Let us consider a positive point charge q located at the center of a closed surface which is here a sphere of radius r and we call this surface (<u>Gaussian surface</u>)



The magnitude of the electric field everywhere on the surface of the sphere is constant and the field lines are directed radially outward and perpendicular to the surface at every point on the surface, Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{\mathbf{r}}\right) \cdot (r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}) = \frac{1}{\epsilon_0}q$$

The same number of field lines pass through any sphere centered at the origin (<u>regardless of its size and shape</u>) would be the same number of field lines.



Therefore, <u>the flux through any closed surface enclosing the charge is q/ϵ_0 .</u>

The net electric flux through a closed surface that surrounds no charge is zero

If we have a group of scattered charges, according to the principle of superposition, the total field is the (vector) sum of all the individual fields

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i .$$

The flux through a surface that encloses them all is

$$\oint \mathbf{E} \cdot d\mathbf{a} = \sum_{i=1}^{n} \left(\oint \mathbf{E}_i \cdot d\mathbf{a}\right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i\right)$$

For any closed surface

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}},}$$

Q_{enc} is the total charge enclosed within the surface and this is **Gauss's law as an integral equation**

Gauss's law relates the flux of E through a closed surface to the total charge enclosed within the surface.

By applying the divergence theorem

$$\oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{a} = \int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau.$$

And

$$Q_{enc} = \int_{\mathcal{V}} \rho\, d\tau$$

Gauss's law becomes

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{E})\, d\tau = \int_{\mathcal{V}} \left(\frac{\rho}{\epsilon_0}\right) d\tau$$

Gauss's law in differential form becomes

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho.}$$

2.5 Applications of Gauss's Law

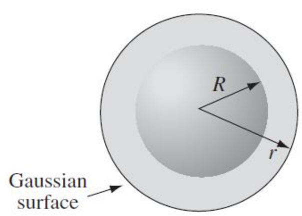
In the following examples we should always take advantage of the symmetry of the charge distribution so that we can remove E from the integral and solve it with the help of:

- The value of the electric field can be claimed to be constant over the surface by symmetry.
- The dot product in Gauss's law can be expressed as E da because E and da are parallel.
- The dot product in Gauss's law is zero if **E** and **da** are perpendicular.

Example 2.7

Find the field outside a uniformly charged solid sphere of radius R and total charge q.

Solution:



If Gaussian surface is a spherical surface at radius r where $r > R$ and $Q_{enc} = q$ then

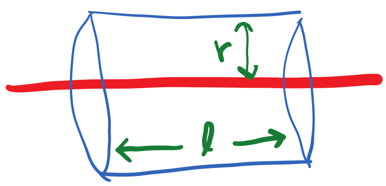
$$\oint_s \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

$$\oint_s E \, da = \frac{q}{\epsilon_0}$$

$$E \oint_s da = \frac{q}{\epsilon_0} \Rightarrow E(4\pi r^2) = \frac{q}{\epsilon_0} \Rightarrow \boxed{E = \frac{q}{4\pi\epsilon_0 r^2}}$$

Example 2.8

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ



$$\oint_s \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

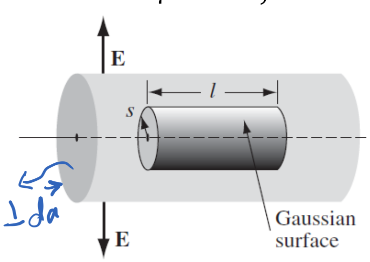
$$E\int_0^{2\pi r l} da = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow E = \frac{1}{2\pi r \cancel{l}}\,\frac{\lambda \cancel{l}}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{1}{2\pi\epsilon_0}\,\frac{\lambda}{r}\,\hat{r}}$$

Example 2.9

A long cylinder carries a charge density that is proportional to the distance from the axis: ρ = ks, for some constant k. Find the electric field inside this cylinder.



$\vec{E} \perp d\vec{a}$

$$\oint_s \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \int_v \rho\, d\tau = \int^{s} ks\; s\,ds\,d\phi\,dz$$

$$\Rightarrow Q_{enc} = \int_0^s\int_0^{2\pi}\int_0^{l} ks^2\,ds\,d\phi\,dz$$

$$= 2\pi k l\left[\frac{s^3}{3}\right]_0^s$$

$$\Rightarrow Q_{enc} = \frac{2}{3}\pi k l s^3$$

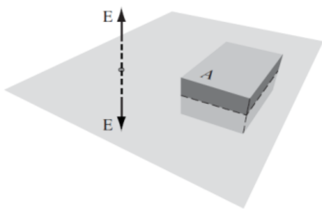
$$\Rightarrow \int \vec{E}\cdot d\vec{a} = \frac{2}{3}\,\frac{\pi k l s^3}{\epsilon_0}$$

$$E\int_0^{2\pi l s} da = \frac{2}{3}\,\frac{\pi k l s^3}{\epsilon_0}$$

$$E = \frac{2}{3}\,\frac{\pi k l s^3}{\epsilon_0}\,\frac{1}{2\pi l s} \Rightarrow \boxed{\vec{E} = \frac{1}{3}\,\frac{k s^2}{\epsilon_0}\,\hat{s}}$$

Example 2.10

An infinite plane carries a uniform surface charge σ. Find its electric field.



$$\oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

$$\Rightarrow 2AE = \frac{\sigma A}{\epsilon_0}$$

$$\Rightarrow E = \frac{\sigma}{2\epsilon_0} \qquad \Rightarrow \boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}}$$

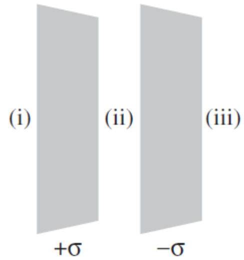
Example 2.11

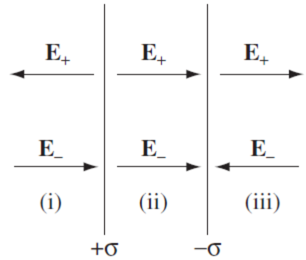
Two infinite parallel planes carry equal, but opposite uniform charge densities $\pm\sigma$. Find the field in each of the three regions:

(i) to the left of both,

(ii) between them,

(iii) (iii) to the right of both.





i) $\vec{E}_+ + \vec{E}_- = 0$

ii) $\vec{E}_+ + \vec{E}_- = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$

iii) $\vec{E}_+ + \vec{E}_- = 0$

Example 2.12

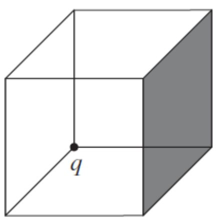
Suppose the electric field in some region is found to be $E = kr^3\,\hat{r}$, in spherical coordinates (k is constant). Find the total charge contained in a sphere of radius R centered at the origin.

$$\oint_s \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0} \Rightarrow Q_{enc} = \epsilon_0 \oint_s \vec{E}\cdot d\vec{a}$$

$$Q_{enc} = \epsilon_0\, k R^3 (4\pi R^2) = 4\pi\epsilon_0 k R^5$$

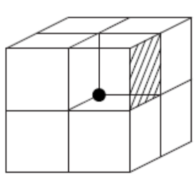
Example 2.13

A charge q sits at the back corner of a cube, as shown in following figure.
What is the flux of E through the shaded side with area a ?



Solution:

This cube can be one of 8 cubes surrounding the charge q (which sits in the center of the 8 cubes shape) and their total area A



$$\Phi_E = \oint_s \vec{E}\cdot d\vec{a} = \frac{1}{24}\oint_s \vec{E}\cdot d\vec{A}$$

$$\because \oint \vec{E}\cdot d\vec{A} = \frac{q}{\epsilon_0} \Rightarrow \boxed{\Phi_E = \oint_s \vec{E}\cdot d\vec{a} = \frac{q}{24\epsilon_0}}$$

2.6 Electric Potential

If the curl of a vector field (such as E) vanishes (everywhere), then E can be written as the gradient of a scalar potential (V):

$$\nabla \times \mathrm{E} = 0 \;\Leftrightarrow\; \mathrm{E} = -\nabla \mathrm{V}$$

Consider a test point charge q_o that can be moved in an electric field. The work done within the charge–field system by the electric field on this test charge to move it from a point "a" to a point "b" along a given path is:

$$dW = F \cdot dl = q_o E \cdot dl$$

As the relation between the work done within the system by a conservative force and the change in its potential energy ΔU is:

$$W = -\Delta U$$

$$dU = -q_o E \cdot dl$$

$$\Delta U = -q_o \int_a^b E \cdot dl$$

The electric potential difference (or simply the potential difference) between two points "a" and "b" is given by:

$$\Delta V = \frac{\Delta U}{q_o} = -\int_a^b E \cdot dl$$

The potential difference between two points is a measure of potential energy difference per unit charge.

In general,

$$\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.}$$

Here *O* is a reference point

The potential difference between two points “a” and “b” is

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} + \int_{\mathcal{O}}^{\mathbf{a}} \mathbf{E} \cdot d\mathbf{l}$$
$$= -\int_{\mathcal{O}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} - \int_{\mathbf{a}}^{\mathcal{O}} \mathbf{E} \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

Now, the fundamental theorem for gradients states that

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l},$$

$$\int_{\mathbf{a}}^{\mathbf{b}} (\nabla V) \cdot d\mathbf{l} = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}.$$

$$\boxed{\mathbf{E} = -\nabla V.}$$

This is the same conclusion we have when we discuss the curl of E

The SI unit of both electric potential and potential difference (which are **scalar quantities**) is joules per coulomb, which is defined as a volt (V):

$$1 \text{ V} = 1 \text{ J/C}$$

This means the unit of the electric field can be 1 N/C or 1 V/m

A unit of energy commonly used is the electron volt (eV), which is defined as the energy a charge–field system gains or loses when a charge of magnitude **e** (that is, an electron or a proton) is moved through a potential difference of 1 V.

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ C} \cdot \text{V} = 1.60 \times 10^{-19} \text{ J}$$

2.7 Electric Potential and Potential Energy Due to Point Charges

An isolated positive point charge q produces an electric field directed radially outward from the charge and the electric potential difference can be calculated from the equation mentioned in the previous section (The Curl of E):

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l} = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{a}}^{\mathbf{b}} \frac{q}{r^2} dr = \frac{-1}{4\pi\epsilon_0} \frac{q}{r} \bigg|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_a} - \frac{q}{r_b} \right),$$

Therefore, for a single charge (q) the electric potential due to this point charge at any distance r from the charge is:

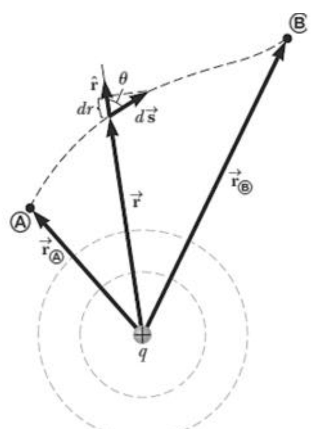
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\mathscr{r}}$$

Also we can get the same result using:

$$\boxed{V(\mathbf{r}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}.}$$

When dl= dr and O = infinity

- The potential can be taken to be zero at infinity (as a reference point)
- An equipotential surface is a surface over which the potential is constant (all points are at the same electric potential). Equipotential surfaces are perpendicular to electric field lines. For example, the surface of any charged conductor is equipotential surface.



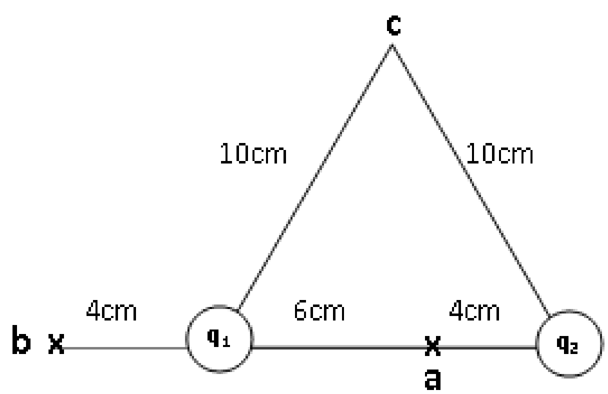
For a group of point charges, we can write the total electric potential at P as

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i}$$

and $\mathscr{r}_i$ is the distance from the point P to the charge q_i.

Example 2.14

As shown in the following figure, if $q_1 = 12\text{x}10^{-9}$ C and $q_2 = -12\text{x}10^{-9}$ C



Find:

a) The total electric potential due to these two charges at points a, b and c
b) Find the change in potential energy of the system of these two charges and a third charge $4\text{x}10^{-9}$ C as the latter charge moves from infinity to points a, b and c
c) The potential difference V_{ab} ,V_{ba} and V_{bc}

a) $V(r) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i}$

$V_a = \frac{1}{4\pi\epsilon_0}\left[\frac{-12\times10^{-9}}{4\times10^{-2}} + \frac{12\times10^{-9}}{6\times10^{-2}}\right] \simeq -900\text{ V}$

$V_b = \frac{1}{4\pi\epsilon_0}\left[\frac{12\times10^{-9}}{4\times10^{-2}} - \frac{12\times10^{-9}}{14\times10^{-2}}\right] \simeq 1930\text{ V}$

$V_c = 0$

b) $U_a = q_0 V_a = 4 \times 10^{-9} \times (-900) = -36 \times 10^{-7}\ J$

$U_b = 77 \times 10^{-7}\ J$ $\qquad U_c = 0$

c) $V_{ab} = V_a - V_b = -900 - 1930 = -2830\ V$

$V_{ba} = 2830\ V$

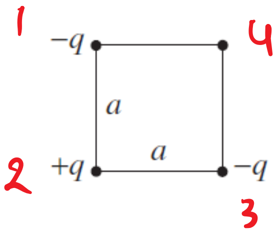
$V_{bc} = V_b - V_c = 1930 - 0 = 1930\ V$

Example 2.15

Three charges are situated at the corners of a square (side a), as shown in the following figure.

(a) How much work (done by external force) does it take to bring in another charge, +q, from far away and place it in the fourth corner?

(b) How much work does it take to assemble the whole configuration of four charges?



a) $V = \frac{1}{4\pi\epsilon_0}\left[-\frac{q}{a} + \frac{q}{\sqrt{2}\,a} - \frac{q}{a}\right] = \frac{q}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$

$\Rightarrow W_4 = U = qV = \frac{q^2}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$

b) $W_1 = 0$

$W_2 = q\left(\frac{1}{4\pi\epsilon_0}\frac{(-q)}{a}\right) = -\frac{q^2}{4\pi\epsilon_0 a}$

$W_3 = \frac{-q}{4\pi\epsilon_0}\left(\frac{-q}{\sqrt{2}\,a} + \frac{q}{a}\right) = \frac{q^2}{4\pi\epsilon_0 a}\left(\frac{1}{\sqrt{2}} - 1\right)$

$W_4 = \frac{q^2}{4\pi\epsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$

$\Rightarrow W_{total} = \frac{q^2}{4\pi\epsilon_0 a}\left(-4 + \frac{2}{\sqrt{2}}\right)$

2.8 Electric Potential Due to Continuous Charge Distributions

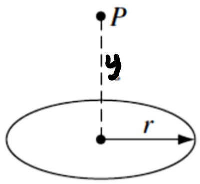
The electric potential due to continuous charge distributions can be calculated by:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\imath}\,dq$$

Example 2.16

(a) Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius r and total charge q.

(b) Find an expression for the magnitude of the electric field at point P.



a)

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\imath} \qquad \because\ \imath = \sqrt{y^2+r^2}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\sqrt{y^2+r^2}}$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{q}{\sqrt{y^2+r^2}}$$

b)

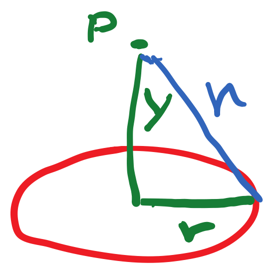
$$\vec{E} = -\vec{\nabla} V = -\left[\cancel{\frac{\partial V}{\partial x}}\,\hat{x} + \frac{\partial V}{\partial y}\,\hat{y} + \cancel{\frac{\partial V}{\partial z}}\,\hat{z}\right]$$

$$= -\frac{\partial}{\partial y}\left[\frac{1}{4\pi\epsilon_0}\,\frac{q}{\sqrt{y^2+r^2}}\right]\hat{y}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\,\frac{q\,y}{(y^2+r^2)^{3/2}}\,\hat{y}}$$

Example 2.17

A uniformly charged disk has radius R and surface charge density σ.
(a) Find the electric potential at a point P along the perpendicular central axis of the disk.
(b) Find the y component of the electric field at a point P along the perpendicular central axis of the disk.



a) $$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{n}$$

$$dq = \sigma\, da = \sigma(2\pi r\, dr)$$

$$\Rightarrow V = \frac{1}{4\pi\epsilon_0}\int_0^r \frac{2\pi\sigma r\, dr}{\sqrt{y^2+r^2}}$$

$$= \frac{\sigma}{4\epsilon_0}\int_0^r \frac{2r\, dr}{\sqrt{y^2+r^2}}$$

use $u = y^2 + r^2$
$du = 2r\,dr$

$$= \frac{\sigma}{4\epsilon_0}\left[2\sqrt{y^2+r^2} - 2y\right]$$

$$\boxed{V = \frac{\sigma}{2\epsilon_0}\left[\sqrt{y^2+r^2} - y\right]}$$

b) $$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial y}\hat{y} = -\left[\frac{\sigma}{2\epsilon_0}\left(\frac{y}{\sqrt{y^2+r^2}} - 1\right)\right]\hat{y}$$

$$\Rightarrow \boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{y}{\sqrt{y^2+r^2}}\right)\hat{y}}$$

Example 2.18

Find the potential inside and outside a spherical shell of radius R that carries a uniform surface charge. Set the reference point at infinity.

when $r > R$

$$V = -\int_{\infty}^{r} \vec{E}\, d\vec{l} = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{q}{r^2}\, dr$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

inside the spherical shell $r < R$

from Gauss's law $E = 0$

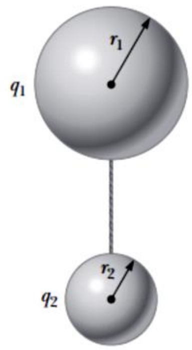
$$\Rightarrow V = -\frac{1}{4\pi\epsilon_0}\int_{\infty}^{R} \frac{q}{r^2}\, dr - \cancel{\int_{R}^{r} 0\, dr} = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$

V is constant inside, but it is not zero

Example 2.19

Two spherical conductors of radii r_1 and r_2 are separated by a distance much greater than the radius of either sphere. The spheres are connected by a conducting wire as shown in the following figure. The charges on the spheres in equilibrium are q_1 and q_2, respectively, and they are uniformly charged.

Find the ratio of the magnitudes of the electric fields at the surfaces of the spheres.



$$V = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1} = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2}$$

$$\Rightarrow \frac{q_1}{r_1} = \frac{q_2}{r_2}$$

$$\boxed{\frac{q_1}{q_2} = \frac{r_1}{r_2}} \quad - \ 1$$

$$E_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r_1^2} \qquad E_2 = \frac{1}{4\pi\epsilon_0}\frac{q_2}{r_2^2}$$

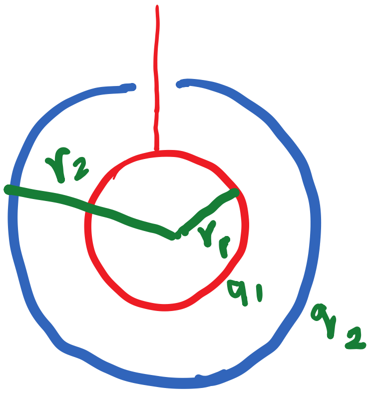
$$\frac{E_1}{E_2} = \frac{q_1}{q_2}\frac{r_2^2}{r_1^2}$$

from eq 1 $$\frac{E_1}{E_2} = \frac{r_1}{r_2}\frac{r_2^{\cancel{2}}}{r_1^{2}} \Rightarrow \boxed{\frac{E_1}{E_2} = \frac{r_2}{r_1}}$$

The electric field is very large at sharp points

Example 2.20

A small conducting sphere of radius r_1 and charge q_1 is surrounded by a spherical conducting shell of radius r_2 and charge q_2. Find the potential difference between them.



$$V_1 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

$$V_2 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_2} + \frac{q_2}{r_2}\right)$$

$$\Rightarrow V_1 - V_2 = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1}{r_1} - \frac{q_1}{r_2}\right]$$

$$\boxed{V_1 - V_2 = \frac{q_1}{4\pi\epsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]}$$

if q is positive $\Rightarrow V_1 > V_2$

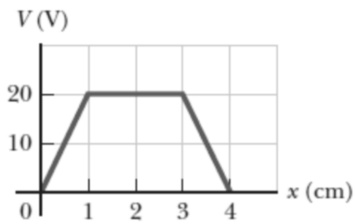
when they are connected $V_1 - V_2 = 0 \Rightarrow \underline{\underline{q_1 = 0}}$

q_1 will necessarily flow from the sphere to the shell

Example 2.21

This figure represents a graph of the electric potential in a region of space versus position x, where the electric field is parallel to the x axis.

Draw a graph of the x component of the electric field versus x in this region.



$$\vec{E} = -\vec{\nabla} V$$

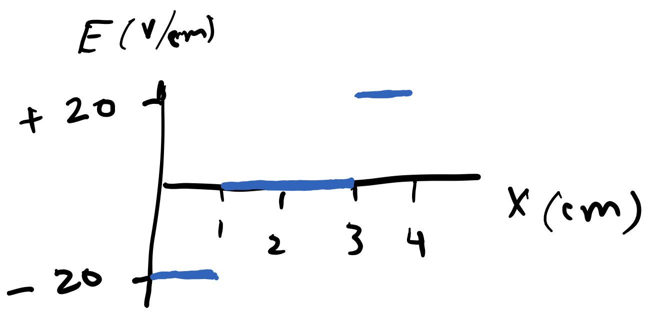
$$\Rightarrow E_x = -\frac{\partial V}{\partial x}$$

from the figure $E = -\frac{\Delta V}{\Delta x} = -$ slope of line

$x = 0 \rightarrow x = 1\,\text{cm}$ $\quad E_x = -\frac{(20-0)}{1} = -20\ \text{V/cm}$

$x = 1\,\text{cm} \rightarrow x = 3\,\text{cm}$ $\quad E_x = -\frac{0}{2} = 0\ \text{V/cm}$

$x = 3\,\text{cm} \rightarrow x = 4\,\text{cm}$ $\quad E = -\frac{(0-20)}{4-3} = 20\ \text{V/cm}$



Example 2.22

Over a certain region of space, the electric potential is $V= 5x -3x^2y +2yz^2$.

(a) Find the expressions for the x, y, and z components of the electric field over this region.

(b) What is the magnitude of the field at the point P that has coordinates (1, 0, -2) m?

a) $\vec{E} = -\vec{\nabla} V \Rightarrow \vec{E} = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z}$

$$\vec{E} = -(5-6xy)\hat{x} - (-3x^2+2z^2)\hat{y} - (4yz)\hat{z}$$

$$\Rightarrow \boxed{\vec{E} = (-5+6xy)\hat{x} + (3x^2-2z^2)\hat{y} - 4yz\,\hat{z}}$$

b) $\vec{E} = -5\hat{x} + \left[3-2(-2)^2\right]\hat{y} - 0\,\hat{z}$

$$\vec{E} = (-5\hat{x} - 5\hat{y})\ \frac{V}{m}$$

$$\vec{E} = \sqrt{(-5)^2+(-5)^2} = \sqrt{50}\ \frac{V}{m}$$