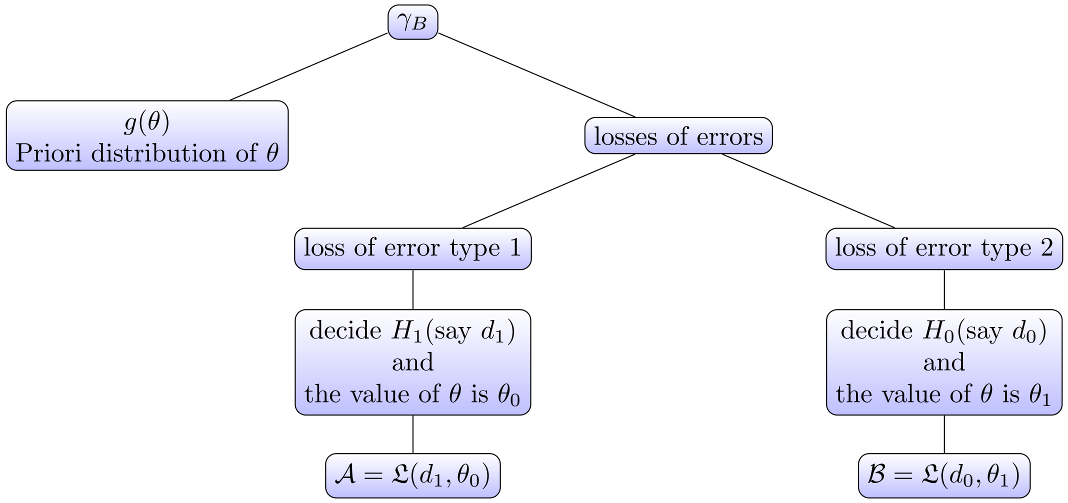
Chapter 6 :Bayes Test

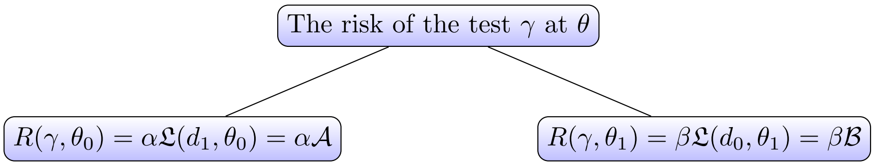
To study γ_B, we must have information about θ:



Theorem

The test γ_B for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ under the priori and the losses $g(\theta)$, $\mathcal{A} = \mathfrak{L}(d_1, \theta_0)$ and $\mathcal{B} = \mathfrak{L}(d_0, \theta_1)$ is LRT and rejects if:

$$\lambda = \frac{\ell(\underset{\sim}{X}; \theta_0)}{\ell(\underset{\sim}{X}; \theta_1)} < k = \frac{\mathfrak{L}(d_0, \theta_1)g(\theta_1)}{\mathfrak{L}(d_1, \theta_0)g(\theta_0)} = \frac{\mathcal{B}g(\theta_1)}{\mathcal{A}g(\theta_0)}.$$



Definitions

1. The Bayes risk of the test γ under the priori $g(\theta)$ is the expectation:

$$\begin{aligned} \mathfrak{B}(\gamma) &= \mathbb{E}_\theta\left(R(\gamma, \theta)\right) = R(\gamma, \theta_0)g(\theta_0) + R(\gamma, \theta_1)g(\theta_1) \\ &= \alpha\mathcal{A}g(\theta_0) + \beta\mathcal{B}g(\theta_1). \end{aligned}$$

2. The Bayes test γ_B minimizes the risk. More exactly,

$$\text{For all test } \gamma, \quad \mathfrak{B}(\gamma_B) < \mathfrak{B}(\gamma).$$

Example 1 :
Let X be normal random variable with distribution $N(\theta,1)$. Let $X_1, X_2, ..., X_{16}$ be 16 copies of X. Test the hypothesis $H_0 : \theta = 0$ vs $H_a : \theta = 1$ by γ_B. Consider the following priori and the losses functions :

$$g(\theta_0) = 0.7, g(\theta_1) = 0.3, \mathcal{A} = \mathfrak{L}(d_1, \theta_0) = 8, \mathcal{B} = \mathfrak{L}(d_0, \theta_1) = 3$$

1. Find the rejection condition of γ_B .
2. Compute α_B and β_B .
3. Find the risks $R(\gamma_B, \theta_0)$,$R(\gamma_B, \theta_1)$ and $\mathfrak{B}(\gamma_B)$.
4. Find the risks $R(\gamma_{MP}, \theta_0)$,$R(\gamma_{MP}, \theta_1)$ and $\mathfrak{B}(\gamma_{MP})$.
5. Compare :
   (a) α_{MP} and α_B.
   (b) β_{MP} and β_B .
   (c) $\mathfrak{B}(\gamma_B)$ and $\mathfrak{B}(\gamma_{MP})$.

Solution 1:
1- The test γ_B reject H_0 if :

$$\lambda = \frac{\ell(\underline{X};\theta_0)}{\ell(\underline{X};\theta_1)} < k = \frac{\mathcal{B}g(\theta_1)}{\mathcal{A}g(\theta_0)}$$

The ratio λ is equal to:

$$\begin{aligned}
\lambda &= \frac{\ell(\underline{X};\theta_0)}{\ell(\underline{X};\theta_1)} \\
&= \frac{(2\Pi)^{-\frac{n}{2}} e^{\frac{-1}{2}\sum(x_i-0)^2}}{(2\Pi)^{-\frac{n}{2}} e^{\frac{-1}{2}\sum(x_i-1)^2}} \\
&= \frac{e^{\frac{-1}{2}\sum(x_i)^2}}{e^{\frac{-1}{2}\sum(x_i-1)^2}} \\
&= e^{\frac{-1}{2}\sum(x_i)^2} e^{\frac{1}{2}\sum(x_i-1)^2} \\
&= e^{\frac{-1}{2}\sum[(x_i)^2-(x_i-1)^2]}
\end{aligned}$$

and

$$k = \frac{\mathcal{B}g(\theta_1)}{\mathcal{A}g(\theta_0)} = \frac{3\times 0.3}{8\times 0.7} = 0.1607$$

then

$$\begin{aligned}
\lambda &< k \\
e^{\frac{-1}{2}\sum[(x_i)^2-(x_i-1)^2]} &< 0.1607 \\
\frac{-1}{2}\sum[(x_i)^2-(x_i-1)^2] &< \ln(0.1607) \\
\frac{-1}{2}\sum[(x_i)^2-(x_i^2-2x_i+1)] &< -1.8282 \\
\frac{-1}{2}\sum(2x_i-1) &< -1.828 \\
\sum(x_i-\frac{1}{2}) &> 1.828 \\
\overline{X} &> 0.6143
\end{aligned}$$

2- Compute α_B and β_B :

$$\begin{aligned}
\alpha_B &= P(Type \quad I \quad Error) \\
&= P(Reject H_0 | H_0 true) \\
&= P(\overline{X} > 0.6143 | \theta = 0) \\
&= P(Z > \frac{0.6143 - 0}{1/\sqrt{16}}) \\
&= P(Z > 2.46) \\
&= 1 - P(Z < 2.46) \\
&= 1 - 099305 \\
&= 0.00695
\end{aligned}$$

$$\begin{aligned}
\beta_B &= P(Type \quad II \quad Error) \\
&= P(Accept H_0 | H_1 true) \\
&= P(\overline{X} < 0.6143 | \theta = 1) \\
&= P(Z < \frac{0.6143 - 1}{1/\sqrt{16}}) \\
&= P(Z < -1.54) \\
&= 0.06178
\end{aligned}$$

3- Find the risks $R(\gamma_B, \theta_0)$,$R(\gamma_B, \theta_1)$ and $\mathfrak{B}(\gamma_B)$:

$$R(\gamma_B, \theta_0) = \alpha_B \times \mathcal{A} = 0.00695 \times 8 = 0.0556.$$
$$R(\gamma_B, \theta_1) = \beta_B \times \mathcal{B} = 0.06178 \times 3 = 0.18534.$$

$$\begin{aligned}
\mathfrak{B}(\gamma_B) &= R(\gamma_B, \theta_0) g(\theta_0) + R(\gamma_B, \theta_1) g(\theta_1) \\
&= \alpha_B \times \mathcal{A} \times g(\theta_0) + \beta_B \times \mathcal{B} \times g(\theta_1) \\
&= 0.0556 \times 0.7 + 0.18534 \times 0.3 \\
&= 0.094522.
\end{aligned}$$

4- Find the risks $R(\gamma_{MP}, \theta_0)$,$R(\gamma_{MP}, \theta_1)$ and $\mathfrak{B}(\gamma_{MP})$:

$$R(\gamma_{MP}, \theta_0) = \alpha_{MP} \times \mathcal{A} = 0.05 \times 8 = 0.4.$$
$$R(\gamma_{MP}, \theta_1) = \beta_{MP} \times \mathcal{B} = 0.00914 \times 3 = 0.0274.$$

$$\begin{aligned}
\mathfrak{B}(\gamma_{MP}) &= R(\gamma_{MP}, \theta_0) g(\theta_0) + R(\gamma_{MP}, \theta_1) g(\theta_1) \\
&= \alpha_{MP} \times \mathcal{A} \times g(\theta_0) + \beta_{MP} \times \mathcal{B} \times g(\theta_1) \\
&= 0.4 \times 0.7 + 0.0274 \times 0.3 \\
&= 0.2288.
\end{aligned}$$

5- Compare γ_B and γ_{MP} :
Optimality of γ_{MP} can be shown because:

$$\begin{aligned}
\alpha_B = 0.00695 &< \alpha_{MP} = 0.05 \\
\beta_{MP} = 0.00914 &< \beta_B = 0.06178
\end{aligned}$$

Optimality of γ_B :

$$\mathfrak{B}(\gamma_B) = 0.094522 < \mathfrak{B}(\gamma_{MP}) = 0.288$$

Example 2 :

Let X be gamma random variable with distribution $Gamma(5,\theta)$. Let $X_1, X_2, ..., X_6$ be 6 copies of X. Test the hypothesis $H_0 : \theta = 1$ vs $H_a : \theta = \frac{1}{2}$ by γ_B. Consider the following priori and the losses functions :

$$g(\theta_0) = 0.6, g(\theta_1) = 0.4, \mathcal{A} = \mathfrak{L}(d_1, \theta_0) = 9, \mathcal{B} = \mathfrak{L}(d_0, \theta_1) = 2$$

1. Find the rejection condition of γ_B .
2. Compute α_B and β_B .
3. Find the risks $R(\gamma_B, \theta_0)$,$R(\gamma_B, \theta_1)$ and $\mathfrak{B}(\gamma_B)$.
4. Find the risks $R(\gamma_{MP}, \theta_0)$,$R(\gamma_{MP}, \theta_1)$ and $\mathfrak{B}(\gamma_{MP})$.
5. Compare :
   (a) α_{MP} and α_B.
   (b) β_{MP} and β_B .
   (c) $\mathfrak{B}(\gamma_B$ and $\mathfrak{B}(\gamma_{MP})$.

Solution 2:

1- The test γ_B reject H_0 if :

$$\lambda = \frac{\ell(\underline{X}; \theta_0)}{\ell(\underline{X}; \theta_1)} \quad < \quad k = \frac{\mathcal{B}g(\theta_1)}{\mathcal{A}g(\theta_0)}$$

The ratio λ is equal to:

$$\begin{aligned}
\lambda &= \frac{\ell(\underline{X}; \theta_0)}{\ell(\underline{X}; \theta_1)} \\
&= \frac{(\frac{1^5}{\Gamma(5)})^n (\prod (x_i)^{5-1})(e^{-\sum x})}{(\frac{\frac{1}{2}^5}{\Gamma(5)})^n (\prod (x_i)^{5-1})(e^{-\frac{1}{2}\sum x})} \\
&= \frac{(1^5)^n e^{-\sum x}}{(\frac{1}{2}^5)^n e^{-\frac{1}{2}\sum x}} \\
&= 2^{5\times n} e^{-\sum x + \frac{1}{2}\sum x} \\
&= 2^{5\times 6} e^{-\frac{1}{2}\sum x}
\end{aligned}$$

and

$$k = \frac{\mathcal{B}g(\theta_1)}{\mathcal{A}g(\theta_0)} = \frac{2 \times 0.4}{9 \times 0.6} = 0.1481$$

then

$$\begin{aligned}
\lambda &< k \\
2^{30} e^{-\frac{1}{2}\sum x} &< 0.1481 \\
e^{-\frac{1}{2}\sum x} &< 1.3793 \times 10^{-10} \\
\frac{-1}{2}\sum x &< \ln(1.3793 \times 10^{-10}) \\
\frac{-1}{2}\sum x &< -22.7043 \\
\sum x &> 45.4086
\end{aligned}$$

note : if

$$x \sim Gamma(\alpha, \theta) \Rightarrow \sum x \sim Gamma(n\alpha, \theta)$$

2- Compute α_B and β_B :

$$\begin{aligned}
\alpha_B &= P(Type \quad I \quad Error) \\
&= P(Reject H_0|H_0 true) \\
&= P(\sum x > 45.4086|\theta = 1); \quad let \quad S = \sum x \quad where \quad S \sim Gamma(5 \times 6, \theta) \\
&= P(S > 45.4086|\theta = 1) \\
&= P(S \times 2(\theta) > 45.4086 \times 2(\theta)|\theta = 1)
\end{aligned}$$

note: If $S \sim Gamma(n = 30, \theta), then \quad U = 2\theta S \sim \chi^2_{2n}$.

then

$$\begin{aligned}
S \sim Gamma(n = 60, \theta = 1) \Rightarrow U &= 2\theta S \sim \chi^2_{2n} \\
U &= 2(1)S \sim \chi^2_{2(30)}
\end{aligned}$$

$$\begin{aligned}
&= P(U > 90.817) \\
&= \frac{0.01 + 0.005}{2}; \quad from \quad chi - squar \quad table \\
&= 0.0075
\end{aligned}$$

$$\begin{aligned}
\beta_B &= P(Type \quad II \quad Error) \\
&= P(Accept H_0|H_1 true) \\
&= P(\sum x < 45.4086|\theta = \frac{1}{2}) \\
&= P(S < 45.4086|\theta = \frac{1}{2}) \\
&= P(U < 45.4086) \\
&= 1 - P(U > 45.4086) \\
&= 1 - \frac{0.95 + 0.90}{2} \\
&= 0.075
\end{aligned}$$

3- Find the risks $R(\gamma_B, \theta_0)$,$R(\gamma_B, \theta_1)$ and $\mathfrak{B}(\gamma_B)$:

$$R(\gamma_B, \theta_0) = \alpha_B \times \mathcal{A} = 0.0075 \times 9 = 0.0675.$$

$$R(\gamma_B, \theta_1) = \beta_B \times \mathcal{B} = 0.075 \times 2 = 0.15.$$

$$\begin{aligned}
\mathfrak{B}(\gamma_B) &= R(\gamma_B, \theta_0)g(\theta_0) + R(\gamma_B, \theta_1)g(\theta_1) \\
&= \alpha_B \times \mathcal{A} \times g(\theta_0) + \beta_B \times \mathcal{B} \times g(\theta_1) \\
&= 0.0675 \times 0.6 + 0.15 \times 0.4 \\
&= 0.1015.
\end{aligned}$$

4- Find the risks $R(\gamma_{MP}, \theta_0)$,$R(\gamma_{MP}, \theta_1)$ and $\mathfrak{B}(\gamma_{MP})$:

$$R(\gamma_{MP}, \theta_0) = \alpha_{MP} \times \mathcal{A} = 0.05 \times 9 = 0.45.$$

$$R(\gamma_{MP}, \theta_1) = \beta_{MP} \times \mathcal{B} = 0.03 \times 2 = 0.06.$$

$$\begin{aligned}
\mathfrak{B}(\gamma_{MP}) &= R(\gamma_{MP}, \theta_0)g(\theta_0) + R(\gamma_{MP}, \theta_1)g(\theta_1) \\
&= \alpha_{MP} \times \mathcal{A} \times g(\theta_0) + \beta_{MP} \times \mathcal{B} \times g(\theta_1) \\
&= 0.45 \times 0.6 + 0.06 \times 0.4 \\
&= 0.294.
\end{aligned}$$

5- Compare γ_B and γ_{MP} :
Optimality of γ_{MP} can be shown because:

$$\begin{aligned} \alpha_B = 0.0075 &\quad < \quad \alpha_{MP} = 0.05 \\ \beta_{MP} = 0.03 &\quad < \quad \beta_B = 0.075 \end{aligned}$$

Optimality of γ_B :

$$\mathfrak{B}(\gamma_B) = 0.1005 < \mathfrak{B}(\gamma_{MP}) = 0.294.$$