Chapter 6 :Minimax Test

Example 1 :
Let X be gamma random variable with distribution $Gamma(5,\theta)$. Let $X_1, X_2, ..., X_6$ be 6 copies of X. Test the hypothesis $H_0 : \theta = 1$ vs $H_a : \theta = \frac{1}{2}$ by γ_{MM}. Consider the following priori and the losses functions :

$$g(\theta_0) = 0.6, g(\theta_1) = 0.4, \mathcal{A} = \mathfrak{L}(d_1, \theta_0) = 9, \mathcal{B} = \mathfrak{L}(d_0, \theta_1) = 2$$

Find γ_{MM} and verify that $k = 43.631$.
Solution 1:
Probability distribution function of gamma :

$$\begin{aligned} f(x;\theta) &= \frac{\theta^5}{\Gamma(5)} x^{5-1} e^{-\theta x} \\ &= e^{5\log(\theta) - \log(\Gamma(5)) + 4\log(x) - \theta x} \end{aligned}$$

Hence

$$\begin{aligned} a(\theta) &= 5\log(\theta) \\ b(x) &= 4\log(x) - \log(\Gamma(5)) \\ c(\theta) &= -\theta \\ d(x) &= x \end{aligned}$$

$f(x;\theta)$ belongs to the class of exponential family .
Since $c(\theta)$ is an decreasing function , then γ_{MM} reject H_0 if $\sum d(x) > k$:

$$\Rightarrow \quad Reject \quad H_0 \quad if \sum x > k$$

where k is found by solving the equation:

$$\begin{aligned} \alpha_{MM}\mathcal{A} &= \beta_{MM}\mathcal{B} \\ 9 \times P(\sum x > k|\theta = 1) &= 2 \times P(\sum x < k|\theta = 0.5) \\ 9 \times P(S > k|\theta = 1) &= 2 \times P(S < k|\theta = 0.5) \\ 9 \times P(U > 2k) &= 2 \times P(U < k) \end{aligned}$$

Thus $k = 43.631$. Compute α_{MM} and β_{MM} :

$$\begin{aligned} \alpha_{MM} &= P(Type \quad I \quad Error) \\ &= P(Reject H_0|H_0 true) \\ &= P(\sum x > 43.631|\theta = 1) \\ &= P(S > 43.631|\theta = 1) \\ &= P(U > 2 \times 43.631) \\ &= P(U > 87.262) \\ &= \frac{0.025 + 0.01}{2} \\ &= 0.0175. \end{aligned}$$

$$
\begin{aligned}
\beta_{MM} &= P(Type \quad II \quad Error) \\
&= P(Accept H_0 | H_1 true) \\
&= P(\sum x < 43.631 | \theta = \frac{1}{2}) \\
&= P(S < 43.631 | \theta = \frac{1}{2}) \\
&= P(U < 43.631) \\
&= 1 - P(U > 43.631) \\
&= 1 - \frac{0.95 + 0.90}{2} \\
&= 0.075.
\end{aligned}
$$

Compare γ_{MM} and γ_{MP} :

$$R(\gamma_{MM}, \theta_0) = \alpha_{MM} \mathcal{A} = 0.15$$
$$R(\gamma_{MM}, \theta_1) = \beta_{MM} \mathcal{B} = 0.15$$

$$
\begin{aligned}
max(R(\gamma_{MM}, \theta_0), R(\gamma_{MM}, \theta_1)) &< max(R(\gamma_{MP}, \theta_0), R(\gamma_{MP}, \theta_1)) \\
max(0.15, 0.15) &< max(0.45, 0.06) \\
0.15 &< 0.45
\end{aligned}
$$

degrees of freedom	Area to the right of the Critical Value									
	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
21	8.034	8.897	10.283	11.591	13.240	29.615	32.671	35.479	38.932	41.401
22	8.643	9.452	10.982	12.338	14.042	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	37.196	41.337	44.461	48.278	50.993
29	13.121	14.257	16.047	17.708	19.768	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	40.256	43.773	46.979	50.892	53.672
40	20.707	22.164	24.433	26.509	29.051	51.805	55.758	59.342	63.691	66.766
50	27.991	29.707	32.357	34.764	37.689	63.167	67.505	71.420	76.154	79.490
60	35.534	37.485	40.482	43.188	46.459	74.397	79.082	83.298	88.379	91.952
70	43.275	45.442	48.758	51.739	55.329	85.527	90.531	95.023	100.43	104.21
80	51.172	53.540	57.153	60.391	64.278	96.578	101.88	106.63	112.33	116.32
90	59.196	61.754	65.647	69.126	73.291	107.57	113.15	118.14	124.12	128.30
100	67.328	70.065	74.222	77.929	82.358	118.50	124.34	129.56	135.81	140.17

Example 2 : **Homework**

Let X be gamma random variable with distribution $normal(\theta, 1)$. Let $X_1, X_2, ..., X_{16}$ be 16 copies of X. Test the hypothesis $H_0 : \theta = 0$ vs $H_a : \theta = 1$ by γ_{MM}. Consider the following priori and the losses functions :

$$g(\theta_0) = 0.7, g(\theta_1) = 0.3, \mathcal{A} = \mathfrak{L}(d_1, \theta_0) = 8, \mathcal{B} = \mathfrak{L}(d_0, \theta_1) = 3$$

$k = 0.5516$

$$\alpha_{MM} = P(\overline{X} > 0.5516 | \theta = 0)$$
$$\beta_{MM} = P(\overline{X} < 0.5516 | \theta = 1)$$